

Predictions for the isospin-violating decays of $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ *

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Abstract: In this work we study the isospin-violating decays of $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$, which may provide additional information for the determination of the properties of the first orbital excitation states of $B_c(1P)^+$. By assuming a dual relation between the U(1) anomaly soft-gluon coupling for $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ and the intermediate meson loop transitions, we can quantify the isospin-violating decay effects for these four P -wave states. We find that the partial decay width of $B_c^{*+} \rightarrow B_c^+\pi^0$ is about three orders of magnitude larger than that for $B_c^{*+} \rightarrow B_c^+\pi^0$. It implies that B_c^{*+} can be established in the $B_c^+\pi^0$ decay channel as a single state. Meanwhile, the two axial-vector states $B_{c1}^+/B_{c1}'^+$ can be possibly identified in $B_{c1}^+/B_{c1}'^+ \rightarrow B_c^{*+}\pi^0$ with comparable strengths. Although these isospin-violating decays turn out to be small, the theoretical predictions should be useful for guiding future experimental efforts.

Keywords: isospin-violating decay, heavy flavor meson, long-distance mechanism

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I. INTRODUCTION

As the only meson containing two different heavy quarks, the B_c mesons family provides a unique perspective for studying the properties of the non-perturbative regime of quantum chromodynamics (QCD). The mass spectrum of B_c mesons has been predicted by many theoretical approaches [1–13]. Experimentally, however, only the ground state B_c^+ [14] and the first radial excitation $B_c(2S)^+$ [15–18] have been observed so far. In addition, the mass and decay modes of the B_c^+ have been measured with high precision [19–26]. However, for excited B_c states beyond the ground state, experimental information on their decay modes remains scarce.

Recently, the first observation of the $B_c(1P)^+$, the first orbital excitation of the B_c meson, was reported by the LHCb Collaboration in the invariant mass spectrum of the $B_c^+\gamma$ final state [27, 28]. The $B_c(1P)^+$ multiplet consists of four physical states: $B_{c0}^{*+}(1^3P_0)$, $B_{c1}^+(1P_1)$, $B_{c1}'^+(1P_1')$, and $B_{c2}^{*+}(1^3P_2)$, where $1P_1$ and $1P_1'$ are mixtures of the 1^1P_1 and 1^3P_1 states [8]. All these four states can decay into $B_c^{*+}\gamma$ with $B_c^{*+} \rightarrow B_c^+\gamma$ to be detected in the final state. Meanwhile, the $1P_1$ and $1P_1'$ states can reach the $B_c^+\gamma$ fi-

nal state directly through an S -wave transition. However, due to the limited statistics and energy resolution, only one photon associated with the observed two peaks in the $B_c^+\gamma$ spectrum are actually effective. Namely, LHCb only measured a B_c^+ plus a photon in the final state. Thus, it is likely that the data actually contain contributions from all these four states.

In addition to the radiative decays, by analogy with the isospin-violating decays of D_s^{*+} [29, 30], the $B_c(1P)^+$ states can also decay to the final states $B_c^{(*)+}\pi^0$ via isospin-violating processes ($B_{c0}^*/B_{c2}^* \rightarrow B_c^+\pi^0$, $B_{c1}^+/B_{c1}'^+ \rightarrow B_c^{*+}\pi^0$). Although the B_c^{*+} has not yet been observed experimentally, it is rather reliable that the B_c^{*+} mass is below the $B_c^+\pi^0$ threshold through theoretical studies [13]. Therefore, the decays of $B_{c0}^*/B_{c2}^* \rightarrow B_c^+\pi^0$ and $B_{c1}^+/B_{c1}'^+ \rightarrow B_c^{*+}\pi^0$ can provide more information about these four P -wave states.

In this work, we employ the effective Lagrangian approach to calculate the decay widths of the isospin-violating processes $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$. At the quark level the isospin-violating decay $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ can be described by the soft-gluon coupling to the light $u\bar{u}$ and $d\bar{d}$ via the U(1) anomaly term. Meanwhile, since π^0 is an

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isovector with the flavor wavefunction $(u\bar{u} - d\bar{d})/\sqrt{2}$, the two terms will cancel each other. Taking into account the property of the U(1) anomaly coupling which is proportional to the quark mass and only partially conserves the axial current, the isospin-violating coupling is proportional to a term of $m_d - m_u$, which will be convoluted by the transition form factor for $B_c(1P)^+ \rightarrow B_c^{(*)+}$. According to the concept of quark-hadron duality [31], this process can be interpreted as the $B_c(1P)^+$ meson producing intermediate $\mathcal{B}^{(*)}$ and $\mathcal{D}^{(*)}$ states, which then rescatterings via the exchange of a $\mathcal{D}^{(*)}$ meson to produce the $B_c^{(*)+}\pi^0$ final state. The two loop amplitudes involving the intermediate mesons with u and d can have constructive phase to produce η meson or destructive phase to produce π^0 due to isospin conservation. This mechanism can be quantified by the effective Lagrangian approach, where the vertex couplings can be determined by the 3P_0 model [32–34]. This method has been broadly applied to the studies of various Okubo-Zweig-Iizuka (OZI) rule [35] evading processes [36–38], isospin symmetry breaking processes [29, 38–42] and the helicity selection rule violation processes [36, 43, 44] in the literature.

As follows, We first introduce our formalism in Sec. II. The numerical results and discussions are presented in Sec. III. A brief summary and conclusion are given in Sec. IV.

II. FORMALISM

In this section we first present the effective Lagrangians, and derive the loop amplitudes for $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$. We then determine the coupling constants appearing in the effective Lagrangians within the framework of the 3P_0 model [32].

A. Effective Lagrangians

The effective Lagrangians for the B_{cJ} couplings to $\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ are as follows [45, 46]:

$$\mathcal{L}_{B_c} = ig_{B_c\mathcal{B}^*\mathcal{D}} B_c \mathcal{B}_\mu^{*i} \partial^\mu \mathcal{D}_i + ig_{B_c\mathcal{B}\mathcal{D}^*} B_c \partial^\mu \mathcal{B}_i \mathcal{D}_\mu^{*i} + g_{B_c\mathcal{B}^*\mathcal{D}^*} \varepsilon_{\mu\nu\rho\sigma} B_c \partial^\mu \mathcal{B}^{*i\nu} \partial^\rho \mathcal{D}_i^{*\sigma} + h.c., \quad (1)$$

$$\mathcal{L}_{B_{c0}^*} = g_{B_{c0}^*\mathcal{B}\mathcal{D}} B_{c0}^* \mathcal{B}^i \mathcal{D}_i + g_{B_{c0}^*\mathcal{B}^*\mathcal{D}^*} B_{c0}^* \mathcal{B}_\mu^{*i} \mathcal{D}_i^{*\mu} + h.c., \quad (2)$$

$$\mathcal{L}_{B_{c2}^*} = g_{B_{c2}^*\mathcal{B}\mathcal{D}} \partial_\mu B_{c2}^* \partial^\mu \mathcal{B}_i \mathcal{D}_i + g_{B_{c2}^*\mathcal{B}^*\mathcal{D}^*} \partial_\mu B_{c2}^* \partial^\mu \mathcal{B}_i^{*\nu} \mathcal{D}_\nu^{*i} + h.c., \quad (3)$$

$$\mathcal{L}_{B_c^*} = ig_{B_c^*\mathcal{B}\mathcal{D}} B_c^* \partial_\mu \mathcal{B}_i \mathcal{D}^i + g_{B_c^*\mathcal{B}^*\mathcal{D}^*} \varepsilon_{\mu\nu\rho\sigma} \partial^\mu B_c^* \partial^\nu \mathcal{B}_i^{*\rho} \mathcal{D}_i^{*\sigma} + g_{B_c^*\mathcal{B}\mathcal{D}^*} \varepsilon_{\mu\nu\rho\sigma} \partial^\mu B_c^* \partial^\nu \mathcal{D}_i^{*\rho} \mathcal{B}_i^{*\sigma} + ig_{B_c^*\mathcal{B}^*\mathcal{D}^*} (\partial^\mu B_c^* - \partial^\nu B_c^*) \mathcal{B}_\mu^{*i} \mathcal{D}_\nu^{*i} + h.c., \quad (4)$$

$$\mathcal{L}_{B_{c1}} = g_{B_{c1}\mathcal{B}\mathcal{D}^*} B_{c1}^\mu \mathcal{D}_\mu^{*i} \mathcal{B}_i + g_{B_{c1}\mathcal{B}^*\mathcal{D}} B_{c1}^\mu \mathcal{B}_\mu^{*i} \mathcal{D}_i + ig_{B_{c1}\mathcal{B}^*\mathcal{D}^*} \varepsilon_{\mu\nu\rho\sigma} \partial^\mu B_{c1}^\nu \mathcal{B}_i^{*\rho} \mathcal{D}^{*i\sigma} + h.c., \quad (5)$$

$$\mathcal{L}_{B_{c1}'} = g_{B_{c1}'\mathcal{B}\mathcal{D}^*} B_{c1}'^\mu \mathcal{D}_\mu^{*i} \mathcal{B}_i + g_{B_{c1}'\mathcal{B}^*\mathcal{D}} B_{c1}'^\mu \mathcal{B}_\mu^{*i} \mathcal{D}_i + ig_{B_{c1}'\mathcal{B}^*\mathcal{D}^*} \varepsilon_{\mu\nu\rho\sigma} \partial^\mu B_{c1}'^\nu \mathcal{B}_i^{*\rho} \mathcal{D}^{*i\sigma} + h.c., \quad (6)$$

where $\mathcal{B} = (B^+, B^0, B_s^0)$ and $\mathcal{D} = (D^0, D^+, D_s^+)$. The coupling constants for the $B_{c(0,1,2)}^{(*)}\mathcal{B}^{(*)}\mathcal{D}$ vertices will be calculated using the 3P_0 model [32, 47].

The effective Lagrangian for the $\mathcal{D}^{(*)}\mathcal{D}^{(*)}\mathcal{P}$ coupling is given by [48, 49]

$$\mathcal{L}_{\mathcal{D}^{(*)}\mathcal{D}^{(*)}\mathcal{P}} = -ig_{\mathcal{D}^*\mathcal{D}^*\mathcal{P}} (\mathcal{D}^i \partial^\mu \mathcal{P}_{ij} \mathcal{D}_\mu^{*j} - \mathcal{D}_\mu^{*i} \partial^\mu \mathcal{P}_{ij} \mathcal{D}^{*j}) + \frac{1}{2} g_{\mathcal{D}^*\mathcal{D}^*\mathcal{P}} \varepsilon_{\mu\nu\alpha\beta} \mathcal{D}_i^{*\mu} \partial^\nu \mathcal{P}^{ij} \overleftrightarrow{\partial}^\alpha \mathcal{D}_j^{*\beta} + h.c., \quad (7)$$

where

$$\mathcal{P} \equiv \begin{pmatrix} \frac{\sin \alpha_P \eta' + \cos \alpha_P \eta + \pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\sin \alpha_P \eta' + \cos \alpha_P \eta - \pi^0}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & \cos \alpha_P \eta' - \sin \alpha_P \eta \end{pmatrix}, \quad (8)$$

with $\alpha_P = 40.6^\circ$ adopted for the $\eta - \eta'$ mixing angle in the SU(3) flavor basis, and the value is an average from the Particle Data Group [50].

The coupling constants for the effective interaction vertices $\mathcal{D}^{(*)}\mathcal{D}^{(*)}\pi^0$ can be obtained from heavy quark effective theory [41, 48, 49, 51]:

$$g_{\mathcal{D}^*\mathcal{D}^*\pi} = \frac{g_{\mathcal{D}^*\mathcal{D}\pi}}{\sqrt{m_{\mathcal{D}}m_{\mathcal{D}^*}}} = \frac{2}{f_\pi} g, \quad g_{D_s^{(*)}D_s^{(*)}\pi} = \sqrt{\frac{m_{D_s^{(*)}}m_{D_s^{(*)}}}{m_{D^{(*)}}m_{D^{(*)}}}} g_{\mathcal{D}^{(*)}\mathcal{D}^{(*)}\pi} \quad (9)$$

where $g = 0.59$, and $f_\pi = 132 \text{ MeV}$ is the π decay constant.

Due to the $\eta - \pi^0$ mixing effect, we need to consider the contribution from the isospin-conserving transition $\mathcal{D}^{(*)} \rightarrow \mathcal{D}^{(*)}\eta$ followed by $\eta \rightarrow \pi^0$ to violate the isospin symmetry. For the $\eta \rightarrow \pi^0$ process, we use the $\eta - \pi^0$ mixing angle $\theta_{\eta\pi^0}$ which is given by the leading order chiral expansion [52]

$$\tan(2\theta_{\eta\pi^0}) = \frac{\sqrt{3} m_d - m_u}{2 m_s - \hat{m}}. \quad (10)$$

where $\hat{m} = (m_u + m_d)/2$. The values of m_u , m_d , and m_s are taken from the PDG [50]. Since $\theta_{\eta\pi^0}$ is very small, we take

$$\theta_{\eta\pi^0} \simeq \frac{\sqrt{3}}{4} \frac{m_d - m_u}{m_s - \hat{m}}, \quad (11)$$

as broadly adopted in the literature.

B. Loop transition amplitudes of $B_c(1P)^+ \rightarrow B_c^{(*)+} \pi^0$

Based on the effective Lagrangians given above, we can derive the corresponding loop amplitudes.

For the loop-level amplitudes without the $\eta - \pi^0$ mixing corresponding to Fig. 1, we denote the amplitude as $i\mathcal{M}_k(P_1, P_2, P_3)$, where $k = 0, 2, 1$ and $1'$ corresponds to $B_{c0}^{*+}, B_{c2}^{*+}, B_{c1}^{*+}$ and B_{c1}^{*+} , respectively. Here, P_i represents the intermediate meson with momentum q_i and $B_{\mu\nu}$ denotes the polarization tensor of the initial B_{c2}^{*+} . ε_μ^i denotes the polarization vector of the initial $B_{c1}^{(*)+}$ and ε_μ^f denotes the polarization vector of the final B_c^{*+} . The amplitudes corresponding to Fig. 1 can thus be expressed as

$$i\mathcal{M}_0(\mathcal{B}, \mathcal{D}, \mathcal{D}^*) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c0}^* \mathcal{B} \mathcal{D}} g_{B_c \mathcal{B} \mathcal{D}^*} g_{\mathcal{D}^* \mathcal{D} \mathcal{P}} q_1^\mu \left(g_{\mu\nu} - \frac{q_{3\mu} q_{3\nu}}{m_3^2} \right) p_3^\nu}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \mathcal{F}(q_i^2), \quad (12)$$

$$i\mathcal{M}_0(\mathcal{B}^*, \mathcal{D}^*, \mathcal{D}) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c0}^* \mathcal{B}^* \mathcal{D}^*} g_{B_c \mathcal{B}^* \mathcal{D}^*} g_{\mathcal{D}^* \mathcal{D} \mathcal{P}} \left(g^{\mu\nu} - \frac{q_1^\mu q_1^\nu}{m_1^2} \right) \left(g_{\mu\alpha} - \frac{q_{2\mu} q_{2\alpha}}{m_2^2} \right) q_{3\nu} p_3^\alpha}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \mathcal{F}(q_i^2), \quad (13)$$

$$i\mathcal{M}_0(\mathcal{B}^*, \mathcal{D}^*, \mathcal{D}^*) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c0}^* \mathcal{B}^* \mathcal{D}^*} g_{B_c \mathcal{B}^* \mathcal{D}^*} g_{\mathcal{D}^* \mathcal{D}^* \mathcal{P}} \left(g^{\mu\nu} - \frac{q_1^\mu q_1^\nu}{m_1^2} \right) g_{\mu\alpha} \left(g^{\alpha\beta} - \frac{q_2^\alpha q_2^\beta}{m_2^2} \right)}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \\ \times \left(g^{\rho\sigma} - \frac{q_3^\rho q_3^\sigma}{m_3^2} \right) \varepsilon_{\nu\delta\rho\kappa} q_1^\delta q_3^\kappa \varepsilon_{\beta\lambda\sigma\tau} q_3^\lambda q_2^\tau \mathcal{F}(q_i^2), \quad (14)$$

$$i\mathcal{M}_2(\mathcal{B}, \mathcal{D}, \mathcal{D}^*) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c2}^* \mathcal{B} \mathcal{D}} g_{B_c \mathcal{B} \mathcal{D}^*} g_{\mathcal{D}^* \mathcal{D} \mathcal{P}} B_{\alpha\beta} q_1^\alpha q_2^\beta \left(g_{\mu\nu} - \frac{q_{3\mu} q_{3\nu}}{m_3^2} \right) q_1^\mu p_3^\nu}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \mathcal{F}(q_i^2), \quad (15)$$

$$i\mathcal{M}_2(\mathcal{B}^*, \mathcal{D}^*, \mathcal{D}) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c2}^* \mathcal{B}^* \mathcal{D}^*} g_{B_c \mathcal{B}^* \mathcal{D}^*} g_{\mathcal{D}^* \mathcal{D} \mathcal{P}} B_{\mu\alpha} \left(g^{\mu\nu} - \frac{q_1^\mu q_1^\nu}{m_1^2} \right) \left(g^{\alpha\beta} - \frac{q_2^\alpha q_2^\beta}{m_2^2} \right) q_{3\nu} p_{3\beta}}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \mathcal{F}(q_i^2), \quad (16)$$

$$i\mathcal{M}_2(\mathcal{B}^*, \mathcal{D}^*, \mathcal{D}^*) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c2}^* \mathcal{B}^* \mathcal{D}^*} g_{B_c \mathcal{B}^* \mathcal{D}^*} g_{\mathcal{D}^* \mathcal{D}^* \mathcal{P}} B_{\mu\alpha} \left(g^{\mu\nu} - \frac{q_1^\mu q_1^\nu}{m_1^2} \right) \left(g^{\alpha\beta} - \frac{q_2^\alpha q_2^\beta}{m_2^2} \right)}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \\ \times \left(g^{\rho\sigma} - \frac{q_3^\rho q_3^\sigma}{m_3^2} \right) \varepsilon_{\nu\delta\rho\kappa} q_1^\delta q_3^\kappa \varepsilon_{\beta\lambda\sigma\tau} q_3^\lambda q_2^\tau \mathcal{F}(q_i^2), \quad (17)$$

$$i\mathcal{M}_{1(\prime)}(\mathcal{B}, \mathcal{D}^*, \mathcal{D}) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c1}^{(\prime)} \mathcal{B} \mathcal{D}^*} g_{B_c \mathcal{B} \mathcal{D}} g_{\mathcal{D}^* \mathcal{D} \mathcal{P}} \varepsilon_\mu^i \left(g^{\mu\nu} - \frac{q_2^\mu q_2^\nu}{m_2^2} \right) \varepsilon_\rho^f q_1^\rho p_{3\nu}}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \mathcal{F}(q_i^2), \quad (18)$$

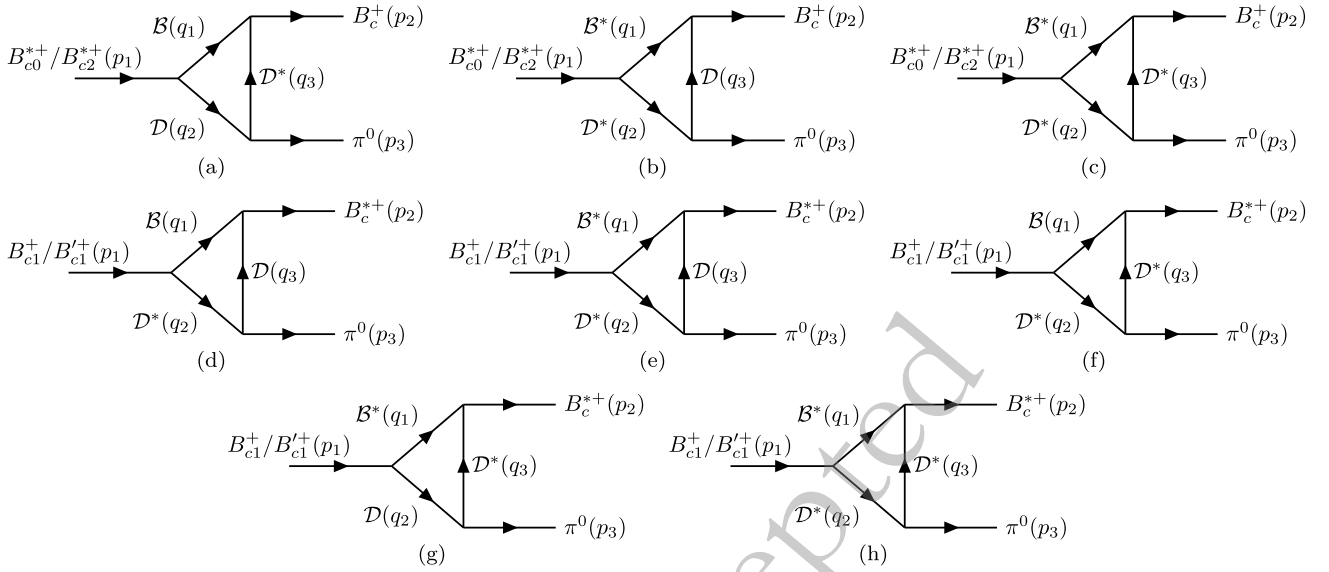


Fig. 1. Schematic diagrams of the decay $B_c(1P)^+ \rightarrow B_c^{(*)+} \pi^0$ via intermediate meson loops without $\eta - \pi^0$ mixing, where $\mathcal{B} = (B^+, B^0)$ and $\mathcal{D} = (D^0, D^+)$. (a)–(h) illustrate different intermediate meson loops.

$$i\mathcal{M}_{1(\phi)}(\mathcal{B}^*, \mathcal{D}^*, \mathcal{D}) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c1}^{(*)} \mathcal{B}^* \mathcal{D}^*} g_{B_c^* \mathcal{B}^* \mathcal{D}^*} g_{\mathcal{D}^* \mathcal{D}^* \mathcal{D}^*} \varepsilon_{\mu\nu\rho\sigma} p_1^\mu \varepsilon_i^\nu \left(g^{\rho\alpha} - \frac{q_1^\rho q_1^\alpha}{m_1^2} \right)}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \times (g^{\sigma\beta} - \frac{q_2^\sigma q_2^\beta}{m_2^2}) \varepsilon_{\delta\lambda\kappa\alpha} p_2^\delta \varepsilon_f^\lambda q_1^\kappa p_{3\beta} \mathcal{F}(q_i^2), \quad (19)$$

$$i\mathcal{M}_{1(\phi)}(\mathcal{B}, \mathcal{D}^*, \mathcal{D}^*) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c1}^{(*)} \mathcal{B} \mathcal{D}^*} g_{B_c^* \mathcal{B} \mathcal{D}^*} g_{\mathcal{D}^* \mathcal{D}^* \mathcal{D}^*} \varepsilon_\mu^i \left(g^{\mu\nu} - \frac{q_2^\mu q_2^\nu}{m_2^2} \right) \varepsilon_{\rho\sigma\alpha\beta} p_2^\rho \varepsilon_f^\sigma p_3^\alpha \left(g^{\beta\delta} - \frac{q_3^\beta q_3^\delta}{m_3^2} \right) \varepsilon_{\kappa\lambda\delta\nu} q_3^\kappa p_3^\lambda}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \mathcal{F}(q_i^2), \quad (20)$$

$$i\mathcal{M}_{1(\phi)}(\mathcal{B}^*, \mathcal{D}, \mathcal{D}^*) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c1}^{(*)} \mathcal{B}^* \mathcal{D}} g_{B_c^* \mathcal{B}^* \mathcal{D}^*} g_{\mathcal{D}^* \mathcal{D}^* \mathcal{D}^*} \varepsilon_\mu^i \left(g^{\mu\nu} - \frac{q_1^\mu q_1^\nu}{m_1^2} \right) (p_{2\nu} \varepsilon_\rho^f - p_{2\rho} \varepsilon_\nu^f) \left(g^{\rho\sigma} - \frac{q_3^\rho q_3^\sigma}{m_3^2} \right) p_{3\sigma}}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \mathcal{F}(q_i^2), \quad (21)$$

$$i\mathcal{M}_{1(\phi)}(\mathcal{B}^*, \mathcal{D}^*, \mathcal{D}^*) = \int \frac{d^4 q_3}{(2\pi)^4} \frac{g_{B_{c1}^{(*)} \mathcal{B}^* \mathcal{D}^*} g_{B_c^* \mathcal{B}^* \mathcal{D}^*} g_{\mathcal{D}^* \mathcal{D}^* \mathcal{D}^*} \varepsilon_{\mu\nu\rho\sigma} p_1^\mu \varepsilon_i^\nu \left(g^{\rho\alpha} - \frac{q_1^\rho q_1^\alpha}{m_1^2} \right) \left(g^{\sigma\beta} - \frac{q_2^\sigma q_2^\beta}{m_2^2} \right)}{(q_1^2 - m_1^2)(q_2^2 - m_2^2)(q_3^2 - m_3^2)} \times (p_{2\alpha} \varepsilon_\delta^f - p_{2\delta} \varepsilon_\alpha^f) (g^{\delta\lambda} - \frac{q_3^\delta q_3^\lambda}{m_3^2}) \varepsilon_{\kappa\tau\beta\lambda} p_3^\kappa q_3^\tau \mathcal{F}(q_i^2). \quad (22)$$

In the above equations $\mathcal{F}(q_i^2)$ is a form factor adopted for cutting off the ultraviolet divergence in the loop integrals, and has the following form

$$\mathcal{F}(q_i^2) = \prod_i \left(\frac{\Lambda_i^2 - m_i^2}{\Lambda_i^2 - q_i^2} \right), \quad (23)$$

where $\Lambda_i \equiv m_i + \alpha \Lambda_{\text{QCD}}$ with the m_i the mass of the i -th internal particle, and the QCD energy scale $\Lambda_{\text{QCD}} = 220 \text{ MeV}$ with $\alpha = 1 \sim 2$ as the cutoff parameter [29, 38, 53, 54].

For the loop diagrams involving the $\eta - \pi^0$ mixing corresponding to Fig. 2, the difference from the loop diagrams of Fig. 1 lies in the coupling constants associated

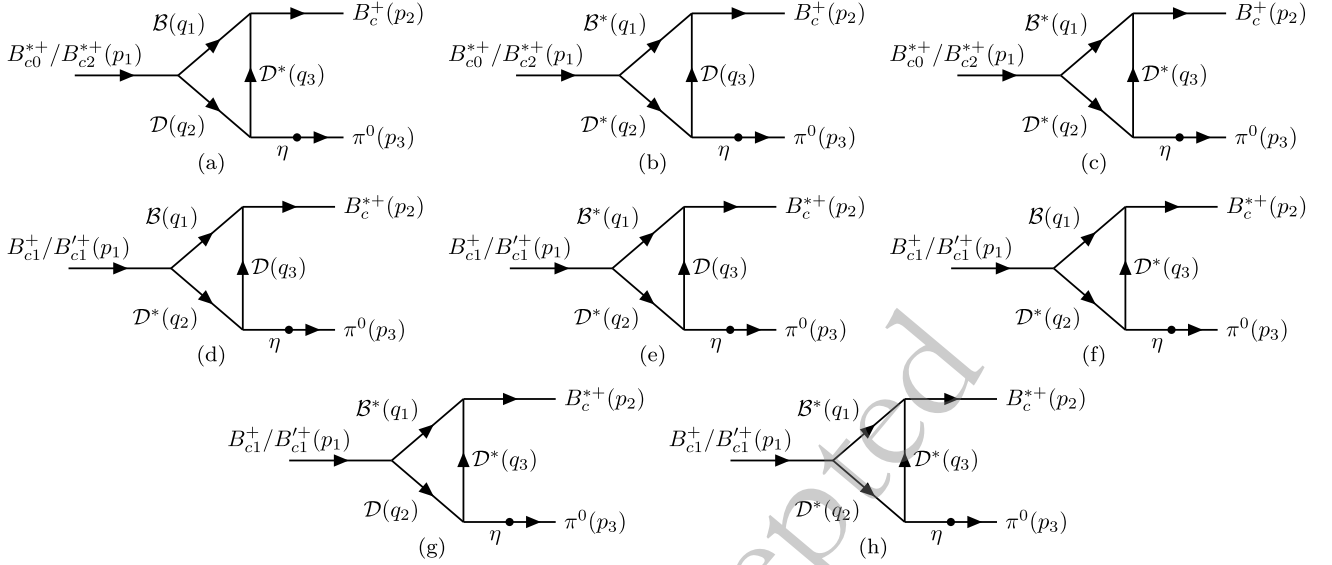


Fig. 2. Schematic diagrams of the decay $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ via intermediate meson loops and $\eta-\pi^0$ mixing, where $\mathcal{B} = (B^+, B^0, B_s^0)$ and $\mathcal{D} = (D^0, D^+, D_s^+)$. (a)–(h) illustrate different intermediate meson loops.

with the η and π^0 mesons. Specifically, for these two isospin-related channels, their amplitudes will interfere constructively in the $B_c(1P)^+ \rightarrow B_c^{(*)+}\eta$ transition, but destructively in the $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ transition. In addition, for $B_c(1P)^+ \rightarrow B_c^{(*)+}\eta$ we also need to consider the contributions from the $(B_s^{(*)}D_s^{(*)}D_s^{(*)})$ loop diagrams. This is because η can be produced via its $s\bar{s}$ component. Therefore, it can be accessed by the $(B_s^{(*)}D_s^{(*)}D_s^{(*)})$ loop transitions. In contrast, the loop transition for $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ contains the coupling vertex of $D_s^* \rightarrow D_s\pi^0$, which is isospin-violating and will be suppressed at the one-loop level. Thus, we only include the $(B_s^{(*)}D_s^{(*)}D_s^{(*)})$ loop diagrams in those processes where the η meson is produced in the final state. They will then contribute to $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ via the $\eta-\pi^0$ mixing. We denote the amplitude as $i\mathcal{M}_{0,2,1,1'}(P_1, P_2, P_3, \eta)$, and for each process in Fig. 2, their explicit expressions are given by

$$\begin{aligned}
 & i\mathcal{M}_k(\mathcal{B}^{(*)}, \mathcal{D}^{(*)}, \mathcal{D}^{(*)}, \eta) \\
 &= i\mathcal{M}_k(\mathcal{B}^{(*)}, \mathcal{D}^{(*)}, \mathcal{D}^{(*)}) \frac{g_{\mathcal{D}^{(*)}\mathcal{D}^{(*)}\eta}\theta_{\eta\pi^0}}{g_{\mathcal{D}^{(*)}\mathcal{D}^{(*)}\pi}}, \\
 & k = 0, 2, 1, 1'.
 \end{aligned} \quad (24)$$

From the above amplitudes, we can obtain the total loop amplitude by summing up the contributions from the different loop diagrams. The decay $B_{c0}^{*+} \rightarrow B_c^+\pi^0$ is a SPP type decay process, the decay $B_{c2}^{*+} \rightarrow B_c^+\pi^0$ is a TPP type decay process, and the decay $B_{c1}^{(\prime)+} \rightarrow B_c^{*+}\pi^0$ is a AVP type decay process so we can always parametrize the total amplitude as

$$\begin{aligned}
 i\mathcal{M}_{B_{c0}^{*+}} &= ig_{B_{c0}^{*+}B_c^+\pi^0}m_{B_{c0}^{*+}}, \quad i\mathcal{M}_{B_{c2}^{*+}} = i\frac{g_{B_{c2}^{*+}B_c^+\pi^0}}{m_{B_{c2}^{*+}}}B_{\mu\nu}p_2^\mu p_3^\nu, \\
 i\mathcal{M}_{B_{c1}^{(\prime)+}} &= ig_{B_{c1}^{(\prime)+}B_c^{*+}\pi^0}m_{B_{c1}^{(\prime)+}}\epsilon_\mu^i\epsilon_\mu^j.
 \end{aligned} \quad (25)$$

Then, the corresponding partial decay width is

$$\begin{aligned}
 \Gamma(B_{c0}^{*+} \rightarrow B_c^+\pi^0) &= \frac{g_{B_{c0}^{*+}B_c^+\pi^0}^2|\mathbf{p}_\pi|}{8\pi}, \\
 \Gamma(B_{c2}^{*+} \rightarrow B_c^+\pi^0) &= \frac{g_{B_{c2}^{*+}B_c^+\pi^0}^2|\mathbf{p}_\pi|^5}{60\pi m_{B_{c2}^{*+}}^4}, \\
 \Gamma(B_{c1}^{(\prime)+} \rightarrow B_c^{*+}\pi^0) &= \left(3 + \frac{|\mathbf{p}_\pi|^2}{m_{B_{c1}^{(\prime)+}}^2}\right) \frac{g_{B_{c1}^{(\prime)+}B_c^{*+}\pi^0}^2|\mathbf{p}_\pi|}{24\pi},
 \end{aligned} \quad (26)$$

where $|\mathbf{p}_\pi|$ is the momentum of the π^0 in the rest frame of the initial particle.

C. Coupling constants of $B_{c(0,1,2)}^{(*)}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ vertices

We calculate the amplitudes for the processes $B_{c(0,1,2)}^{(*)} \rightarrow \mathcal{B}^{(*)}\mathcal{D}^{(*)}$ using the 3P_0 model, and match them with those obtained from the effective Lagrangian approach. In this way, the coupling constants in the effective Lagrangian can be determined.

The corresponding transition operator in the 3P_0 model is given by[32–34, 47, 55]

$$\begin{aligned}
 T &= -3\gamma \sum_m \langle 1, m; 1, -m | 0, 0 \rangle \int d^3\mathbf{p}_3 d^3\mathbf{p}_4 \delta^3(\mathbf{p}_3 + \mathbf{p}_4) \\
 &\times \mathcal{Y}_1^m\left(\frac{\mathbf{p}_3 - \mathbf{p}_4}{2}\right) \chi_{1,-m}^{34} \phi_0^{34} \omega_0^{34} b_{3i}^\dagger(\mathbf{p}_3) d_{4j}^\dagger(\mathbf{p}_4),
 \end{aligned} \quad (27)$$

where i and j are color indices, $\chi_{1,-m}^{34}$ is the spin wave

function, ϕ_0^{34} is the flavor singlet wave function, $\omega_0^{34} = \delta_{ij}/\sqrt{3}$ is the color singlet wave function, and $\mathcal{Y}_1^m(\mathbf{p}) = |\mathbf{p}|Y_1^m(\theta, \phi)$.

The meson wave function is given by [56]

$$\begin{aligned} |M(\mathbf{P}, J, J_z)\rangle &= \sum_{S_z, L_z, c_i} \langle L, L_z; S, S_z | J, J_z \rangle \\ &\times \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) \psi_{N, L, L_z}(\mathbf{p}_1, \mathbf{p}_2) \\ &\times \frac{\delta_{c_1 c_2}}{\sqrt{3}} \phi_{f_1, f_2} \chi_{s_1, s_2}^{S, S_z} b_{c_1, f_1, s_1, \mathbf{p}_1}^\dagger d_{c_2, f_2, s_2, \mathbf{p}_2}^\dagger |0\rangle, \end{aligned} \quad (28)$$

where c_j , s_j , and f_j ($j = 1, 2$) denote the color, spin, and flavor indices of the quark, respectively. $b^\dagger$ and $d^\dagger$ are the creation operators for quarks and antiquarks, and ψ_{N, L, L_z} is the spatial wave function in the harmonic oscillator basis.

The spatial wave functions for S -wave and P -wave mesons are given by

$$\begin{aligned} \psi_{0,0,0}(\mathbf{p}_1, \mathbf{p}_2) &= \frac{1}{\pi^{3/4} R^{3/2}} \exp\left(-\frac{(\mathbf{p}_1 - \mathbf{p}_2)^2}{8R^2}\right), \\ \psi_{0,1,m}(\mathbf{p}_1, \mathbf{p}_2) &= \sqrt{\frac{2}{3}} \frac{|\mathbf{p}_1 - \mathbf{p}_2|}{\pi^{1/4} R^{5/2}} Y_1^m(\theta, \phi) \exp\left(-\frac{(\mathbf{p}_1 - \mathbf{p}_2)^2}{8R^2}\right), \end{aligned} \quad (29)$$

where Y_1^m is the spherical harmonic function and R is the parameter of the meson wave function.

The $B_c(1P)^+$ multiplet which contains four physical states: $B_{c0}^{*+}(1^3P_0)$, $B_{c2}^{*+}(1^3P_2)$, $B_{c1}^+(1P_1)$, and $B_{c1}^{*+}(1P_1')$. The B_{c1}^+ and B_{c1}^{*+} states are mixtures of the 1^1P_1 and 1^3P_1 configurations, with the mixing defined as

$$\begin{pmatrix} B_{c1}^{*+} \\ B_{c1}^+ \end{pmatrix} = \begin{pmatrix} \cos\theta_{1P} & \sin\theta_{1P} \\ -\sin\theta_{1P} & \cos\theta_{1P} \end{pmatrix} \begin{pmatrix} 1^1P_1 \\ 1^3P_1 \end{pmatrix}, \quad (30)$$

where $\theta_{1P} = 22.4^\circ$. Therefore, for $B_{c1}^{(*)+}$, the mixing of the wave functions must be taken into account.

The transition amplitude for $A \rightarrow BC$ in the 3P_0 model is given by

$$\mathcal{M}_q = \langle BC | T | A \rangle, \quad (31)$$

and the amplitude obtained from the effective Lagrangian approach is denoted as $\mathcal{M}_h$. The relation between the two amplitudes is $\mathcal{M}_h = 8\pi^{3/2} \sqrt{m_A m_B m_C} \mathcal{M}_q$, from which the coupling constants in the effective Lagrangian can be determined. By substituting the wave functions from Eq. (32) into Eq. (35), and comparing the resulting amplitude

with that obtained from the effective Lagrangian in Eq. (5), the coupling constants for the $B_c^{(*)}$ meson vertices can be extracted. The explicit expressions for the coupling constants are given in the Appendix A.

III. NUMERICAL RESULTS AND DISCUSSIONS

Proceeding to the numerical calculations of the loop contributions for $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$, the relevant coupling constants can be calculated using Eqs. (A1), (A4) and (9). The parameters used in the calculations are summarized in Table 1 and the 3P_0 model coupling constant $\gamma_{c\bar{b}} = 2.145$ [57]¹⁾. The masses of the relevant particles are taken from the PDG [50], the masses of B_{c0}^{*+} and B_{c2}^{*+} are given by [27, 28]

$$\begin{aligned} m_{B_{c0}^{*+}} &= 6704.8 \pm 5.5 \pm 2.8 \pm 0.3 \text{ MeV}, \\ m_{B_{c2}^{*+}} &= 6752.4 \pm 9.5 \pm 3.1 \pm 0.3 \text{ MeV}, \end{aligned} \quad (32)$$

and the masses of B_{c1}^+ , B_{c1}^{*+} and B_c^{*+} are adopted from Ref. [13]: $m_{B_{c1}^+} = 6741 \text{ MeV}$, $m_{B_{c1}^{*+}} = 6750 \text{ MeV}$, and $m_{B_c^{*+}} = 6338 \text{ MeV}$.

The explicit values of the coupling constants are given in Tables 2 to 6.

With the above coupling constants, we can calculate the partial decay widths of $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$. In Table 7 we list the calculation results of the partial decay width of $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ with five typical cutoff parameter values $\alpha = 1.0, 1.35, 1.5, 1.65$ and 2.0 . For the typical value $\alpha = 1.5 \pm 0.15$ [29, 53, 54], we present the predicted partial decay widths of $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ in Table 8. It shows that the partial width $\Gamma(B_{c0}^{*+} \rightarrow B_c^+\pi^0)$ is nearly three orders of magnitude larger than $\Gamma(B_{c2}^{*+} \rightarrow B_c^+\pi^0)$. It indicates that the $B_c^+\pi^0$ decay channel can be used to distinguish B_{c0}^{*+} and B_{c2}^{*+} from each other given sufficient data samples from experiment. Meanwhile, if B_{c1}^+ and B_{c1}^{*+} can be distinguished by the measurement of $B_{c1}^{*+} \rightarrow B_c^+\gamma$, it seems that these four states will then be separated. It is likely that in the $B_c^+\pi^0$ channel, one will only see one state, i.e. B_{c0}^{*+} , while in the $B_{c2}^{*+}\pi^0$ channel one may be able to see two peaks for B_{c1}^+ and B_{c1}^{*+} . However, their structures as the mixture of the 1^1P_1 and 1^3P_1 configurations would rely on the detailed mixing dynamics. Notice that the isospin-violating decay process has indeed been significantly suppressed. There is no doubt that such a measurement would be extremely challenging. But for future establishment of these four states, such an effort should still be pursued.

To see more clearly the role played by the intermediate loop transitions, we plot the dependence of the decay width of $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ on the cut-off parameter α in Fig. 3. The behavior of the decay widths as a function of

1) Our value of γ is higher than that used in [57] by a factor of $\sqrt{24\pi}$ due to the different operator convention and the different normalization of the wave function.

Table 1. Harmonic oscillator strengths for meson wavefunctions involved in the $B_{c(0,1,2)}^{(*)}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ couplings are adopted from potential quark model studies [12, 13, 58–61].

HO Strength	R_D	R_{D^*}	R_{D_s}	$R_{D_s^*}$	R_B	R_{B^*}	R_{B_s}	$R_{B_s^*}$	R_{B_c}	$R_{B_{c0}^*}$	$R_{B_{c2}^*}$	$R_{B_{c1}^1(1P_1)}$	$R_{B_{c1}^3(3P_1)}$
Value (MeV)	601	516	651	562	580	542	636	595	1010	760	670	700	710

Table 2. Values of the $\mathcal{D}^{(*)}\mathcal{D}^{(*)}\mathcal{P}$ coupling constants.

Coupling constant	$g_{D^{*0}D^{*0}\pi^0}$	$g_{D^{*+}D^{*+}\pi^0}$	$g_{D^0D^{*0}\pi^0}$	$g_{D^+D^{*+}\pi^0}$	$g_{D^{*0}D^{*0}\eta}$
Value	6.32GeV ⁻¹	-6.32GeV ⁻¹	12.23	-12.23	6.16GeV ⁻¹
Coupling constant	$g_{D^{*+}D^{*+}\eta}$	$g_{D^0D^{*0}\eta}$	$g_{D^+D^{*+}\eta}$	$g_{D_s^{*+}D_s^{*+}\eta}$	$g_{D_s^{*+}D_s^{*+}\eta}$
Value	6.16GeV ⁻¹	11.94	11.94	-1.49GeV ⁻¹	-3.03

Table 3. Values of the $B_c^{*+}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ coupling constants.

Coupling constant	$g_{B_c^{*+}B^+D^0}$	$g_{B_c^{*+}B^{*+}D^0}$	$g_{B_c^{*+}B^+D^{*0}}$	$g_{B_c^{*+}B^{*+}D^{*0}}$	$g_{B_c^{*+}B_s^0D_s^+}$	$g_{B_c^{*+}B_s^{*0}D_s^+}$	$g_{B_c^{*+}B_s^0D_s^{*+}}$	$g_{B_c^{*+}B_s^{*0}D_s^{*+}}$
Value	27.1	4.31GeV ⁻¹	4.49GeV ⁻¹	14.5	27.4	4.37GeV ⁻¹	4.54GeV ⁻¹	14.7

Table 4. Values of the $B_{c(0,2)}^{(*)}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ coupling constants.

Coupling constant	$g_{B_c^{*+}B^+D^{*0}}$	$g_{B_c^{*+}B^{*+}D^0}$	$g_{B_c^{*+}B^+D^{*0}}$	$g_{B_c^{*+}B_s^0D_s^{*+}}$	$g_{B_c^{*+}B_s^{*0}D_s^+}$	$g_{B_c^{*+}B_s^0D_s^{*+}}$	$g_{B_c^{*+}B_s^{*0}D_s^{*+}}$
Value	24.1	23.18	1.82GeV ⁻¹	24.4	23.6	1.85GeV ⁻¹	7.51GeV
Coupling constant	$g_{B_{c0}^{*+}B^{*+}D^{*0}}$	$g_{B_{c0}^{*+}B_s^0D_s^+}$	$g_{B_{c0}^{*+}B_s^{*0}D_s^{*+}}$	$g_{B_{c2}^{*+}B^+D^0}$	$g_{B_{c2}^{*+}B^{*+}D^{*0}}$	$g_{B_{c2}^{*+}B_s^0D_s^+}$	$g_{B_{c2}^{*+}B_s^{*0}D_s^{*+}}$
Value	4.50GeV	8.64GeV	5.23GeV	3.58GeV ⁻¹	0.365GeV ⁻¹	3.53GeV ⁻¹	0.398GeV ⁻¹

Table 5. Values of the $B_{c1}^{*+}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ coupling constants.

Coupling constant	$g_{B_{c1}^{*+}B^+D^{*0}}$	$g_{B_{c1}^{*+}B^{*+}D^{*0}}$	$g_{B_{c1}^{*+}B^+D^0}$	$g_{B_{c1}^{*+}B_s^0D_s^{*+}}$	$g_{B_{c1}^{*+}B_s^{*0}D_s^+}$	$g_{B_{c1}^{*+}B_s^0D_s^{*+}}$
Value	9.13GeV	1.33	9.18GeV	10.1GeV	1.49	10.1GeV

Table 6. Values of the $B_{c1}^{*+}\mathcal{B}^{(*)}\mathcal{D}^{(*)}$ coupling constants.

Coupling constant	$g_{B_{c1}^{*+}B^+D^{*0}}$	$g_{B_{c1}^{*+}B^{*+}D^{*0}}$	$g_{B_{c1}^{*+}B^+D^0}$	$g_{B_{c1}^{*+}B_s^0D_s^{*+}}$	$g_{B_{c1}^{*+}B_s^{*0}D_s^+}$	$g_{B_{c1}^{*+}B_s^0D_s^{*+}}$
Value	8.50GeV	1.24	8.19GeV	9.39GeV	1.38	9.40GeV

Table 7. The partial decay widths of $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ with $\alpha = 1.0, 1.35, 1.5, 1.65$ and 2.0 in unit of eV.

α	1.0	1.35	1.5	1.65	2.0
$\Gamma(B_{c0}^{*+} \rightarrow B_c^{*+}\pi^0)$	0.40	1.42	2.16	3.12	6.35
$\Gamma(B_{c2}^{*+} \rightarrow B_c^{*+}\pi^0)(\times 10^{-3})$	0.95	3.79	6.06	9.17	20.7
$\Gamma(B_{c1}^{*+} \rightarrow B_c^{*+}\pi^0)$	0.64	2.35	3.62	5.29	11.1
$\Gamma(B_{c1}^{*+} \rightarrow B_c^{*+}\pi^0)$	0.60	2.17	3.34	4.88	10.2

Table 8. Predicted partial decay widths of $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$ for $\alpha = 1.5 \pm 0.15$.

Decay channel	$B_{c0}^{*+} \rightarrow B_c^{*+}\pi^0$	$B_{c2}^{*+} \rightarrow B_c^{*+}\pi^0$	$B_{c1}^{*+} \rightarrow B_c^{*+}\pi^0$	$B_{c1}^{*+} \rightarrow B_c^{*+}\pi^0$
Partial decay width	$2.2^{+1.0}_{-0.7}$ eV	$(6.1^{+3.1}_{-2.3}) \times 10^{-3}$ eV	$3.6^{+1.7}_{-1.3}$ eV	$3.3^{+1.5}_{-1.2}$ eV

the cutoff parameter α is found to be consistent with $D_s^{*+} \rightarrow D_s^+\pi^0$ [29]. To further clarify the contributions from different loop diagrams in the processes $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$, we present in Fig. 4 the dependence of the different loop contributions to $g_{B_c(1P)^+B_c^{(*)+}\pi^0}$ on the

cutoff parameter α . It is worth noting that for all four processes of $B_c(1P)^+ \rightarrow B_c^{(*)+}\pi^0$, the loop diagrams involving the exchange of a $\mathcal{D}^*$ meson and containing a $\mathcal{D}$ meson as an internal line provide the dominant contribution, which is more than one order of magnitude larger than

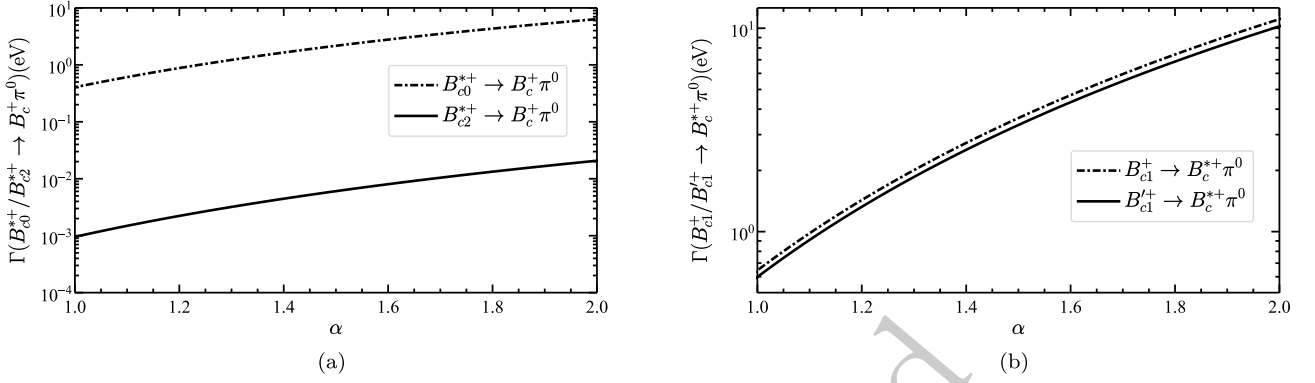


Fig. 3. The partial decay width of $B_c(1P)^+ \rightarrow B_c^{(*)+} \pi^0$ as a function of the cutoff parameter α .

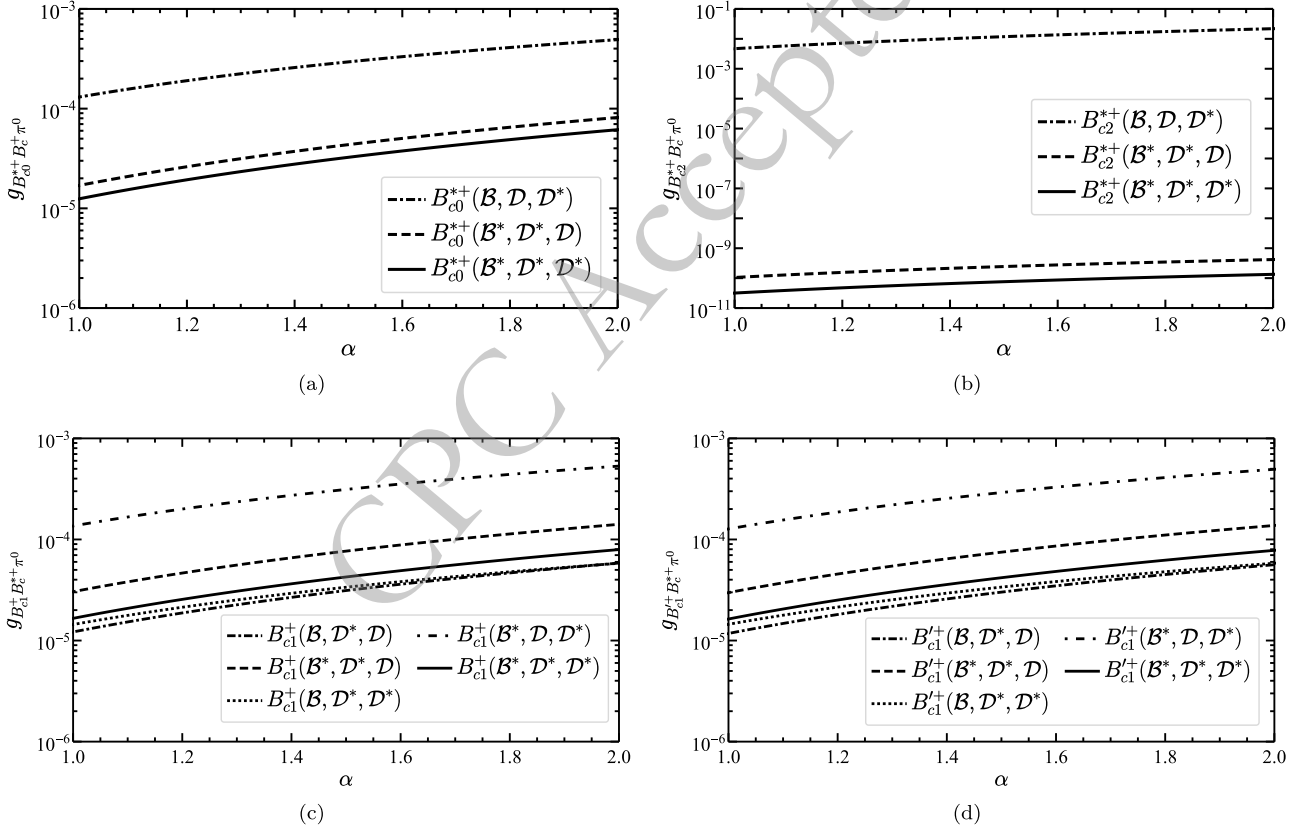


Fig. 4. Dependence of the contributions from different loop diagrams to $g_{B_c(1P)^+ B_c^{(*)+} \pi^0}$ on the cutoff parameter α . (a)–(d) correspond to different $B_c(1P)^+$ states.

the other loop contributions. This is especially pronounced in the case of $B_{c2}^{*+} \rightarrow B_c^+ \pi^0$.

IV. SUMMARY

In this work we calculate the partial decay widths for the four isospin-violating decay channels $B_c(1P)^+ \rightarrow B_c^{(*)+} \pi^0$ using the effective Lagrangian approach. We assume that the isospin-violating π^0 production via the U(1) anomaly term is dual to the intermediate meson loops involving $\mathcal{B}^{(*)}$ and $\mathcal{D}^{(*)}$ rescatterings. The isospin-violating transition is a consequence of the

cancellations between the two kinds of loops involving the intermediate heavy mesons containing $u\bar{u}$ or $d\bar{d}$. In particular, we find that the loop diagrams with $\mathcal{B}^{(*)}\mathcal{D}$ rescatterings by exchanging a $\mathcal{D}^*$ provide the dominant contributions. Our results show that the partial decay width of $B_{c0}^{*+} \rightarrow B_c^+ \pi^0$ is about three orders of magnitude larger than that for $B_{c2}^{*+} \rightarrow B_c^+ \pi^0$. This significant difference can be exploited experimentally to distinguish between the B_{c0}^{*+} and B_{c2}^{*+} states in the final state of $B_c^+ \pi^0$. Meanwhile, the two axial-vector states $B_{c1}^+ / B_{c1}^{'+}$ can be possibly identified in $B_{c1}^+ / B_{c1}^{'+} \rightarrow B_c^+ \pi^0$ though their structures as the mixture of the 1^1P_1 and 1^3P_1 states will rely on the de-

tailed mixing dynamics. Undoubtedly these measurements would be challenging for the present and even future experiment. But we emphasize that they are crucial for finally establishing these states in experiment.

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ledged.

APPENDIX A: COUPLING CONSTANTS OF BOTTOM-CHARM MESONS

Explicit expressions for the $B_{c(0,1,2)}^{(*)} \mathcal{B}^{(*)} \mathcal{D}^{(*)}$ couplings, which are calculated in the 3P_0 model, are listed below. Namely, for the B_c^+ couplings to $\mathcal{B}^{(*)} \mathcal{D}^{(*)}$, we have

$$\begin{aligned} g_{B_c \mathcal{B}^* \mathcal{D}} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}} \sqrt{m_{B_c} m_{\mathcal{B}^*} m_{\mathcal{D}}} \gamma_{c\bar{b}} (R_{B_c} R_{\mathcal{B}^*} R_{\mathcal{D}})^{3/2} (R_{B_c}^2 (R_{\mathcal{B}^*}^2 + R_{\mathcal{D}}^2) + 2R_{\mathcal{B}^*}^2 R_{\mathcal{D}}^2)}{\sqrt{3} (R_{B_c}^2 (R_{\mathcal{B}^*}^2 + R_{\mathcal{D}}^2) + R_{\mathcal{B}^*}^2 R_{\mathcal{D}}^2)^{5/2}}, \\ g_{B_c \mathcal{B} \mathcal{D}^*} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}} \sqrt{m_{B_c} m_{\mathcal{B}} m_{\mathcal{D}^*}} \gamma_{c\bar{b}} (R_{B_c} R_{\mathcal{B}} R_{\mathcal{D}^*})^{3/2} (R_{B_c}^2 (R_{\mathcal{B}}^2 + R_{\mathcal{D}^*}^2) + 2R_{\mathcal{B}}^2 R_{\mathcal{D}^*}^2)}{\sqrt{3} (R_{B_c}^2 (R_{\mathcal{B}}^2 + R_{\mathcal{D}^*}^2) + R_{\mathcal{B}}^2 R_{\mathcal{D}^*}^2)^{5/2}}, \\ g_{B_c \mathcal{B}^* \mathcal{D}^*} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}} \sqrt{m_{B_c} m_{\mathcal{B}^*} m_{\mathcal{D}^*}} \gamma_{c\bar{b}} (R_{B_c} R_{\mathcal{B}^*} R_{\mathcal{D}^*})^{3/2} (R_{B_c}^2 (R_{\mathcal{B}^*}^2 + R_{\mathcal{D}^*}^2) + 2R_{\mathcal{B}^*}^2 R_{\mathcal{D}^*}^2)}{\sqrt{3m_{B_c}} (R_{B_c}^2 (R_{\mathcal{B}^*}^2 + R_{\mathcal{D}^*}^2) + R_{\mathcal{B}^*}^2 R_{\mathcal{D}^*}^2)^{5/2}}. \end{aligned} \quad (\text{A1})$$

For the B_c^{*+} couplings to $\mathcal{B}^{(*)} \mathcal{D}^{(*)}$, we have

$$\begin{aligned} g_{B_c^* \mathcal{B} \mathcal{D}} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}} \sqrt{m_{B_c^*} m_{\mathcal{B}} m_{\mathcal{D}}} \gamma_{c\bar{b}} (R_{B_c^*} R_{\mathcal{B}} R_{\mathcal{D}})^{3/2} (R_{B_c^*}^2 (R_{\mathcal{B}}^2 + R_{\mathcal{D}}^2) + 2R_{\mathcal{B}}^2 R_{\mathcal{D}}^2)}{\sqrt{3} (R_{B_c^*}^2 (R_{\mathcal{B}}^2 + R_{\mathcal{D}}^2) + R_{\mathcal{B}}^2 R_{\mathcal{D}}^2)^{5/2}}, \\ g_{B_c^* \mathcal{B}^* \mathcal{D}} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}} \sqrt{m_{B_c^*} m_{\mathcal{B}^*} m_{\mathcal{D}}} \gamma_{c\bar{b}} (R_{B_c^*} R_{\mathcal{B}^*} R_{\mathcal{D}})^{3/2} (R_{B_c^*}^2 (R_{\mathcal{B}^*}^2 + R_{\mathcal{D}}^2) + 2R_{\mathcal{B}^*}^2 R_{\mathcal{D}}^2)}{\sqrt{3m_{B_c^*}} (R_{B_c^*}^2 (R_{\mathcal{B}^*}^2 + R_{\mathcal{D}}^2) + R_{\mathcal{B}^*}^2 R_{\mathcal{D}}^2)^{5/2}}, \\ g_{B_c^* \mathcal{B} \mathcal{D}^*} &= \frac{4\sqrt{2}\pi^{\frac{1}{4}} \sqrt{m_{B_c^*} m_{\mathcal{B}} m_{\mathcal{D}^*}} \gamma_{c\bar{b}} (R_{B_c^*} R_{\mathcal{B}} R_{\mathcal{D}^*})^{3/2} (R_{B_c^*}^2 (R_{\mathcal{B}}^2 + R_{\mathcal{D}^*}^2) + 2R_{\mathcal{B}}^2 R_{\mathcal{D}^*}^2)}{\sqrt{3m_{B_c^*}} (R_{B_c^*}^2 (R_{\mathcal{B}}^2 + R_{\mathcal{D}^*}^2) + R_{\mathcal{B}}^2 R_{\mathcal{D}^*}^2)^{5/2}}, \\ g_{B_c^* \mathcal{B}^* \mathcal{D}^*} &= \frac{2\sqrt{2}\pi^{\frac{1}{4}} \sqrt{m_{B_c^*} m_{\mathcal{B}^*} m_{\mathcal{D}^*}} \gamma_{c\bar{b}} (R_{B_c^*} R_{\mathcal{B}^*} R_{\mathcal{D}^*})^{3/2} (R_{B_c^*}^2 (R_{\mathcal{B}^*}^2 + R_{\mathcal{D}^*}^2) + 2R_{\mathcal{B}^*}^2 R_{\mathcal{D}^*}^2)}{\sqrt{3} (R_{B_c^*}^2 (R_{\mathcal{B}^*}^2 + R_{\mathcal{D}^*}^2) + R_{\mathcal{B}^*}^2 R_{\mathcal{D}^*}^2)^{5/2}}. \end{aligned} \quad (\text{A2})$$

For the B_{c0}^{*+} couplings to $\mathcal{B}^{(*)} \mathcal{D}^{(*)}$, we have

$$\begin{aligned} g_{B_{c0}^* \mathcal{B} \mathcal{D}} &= \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B_{c0}^*} m_{\mathcal{B}} m_{\mathcal{D}}} \gamma_{c\bar{b}} R_{B_{c0}^*} R_{\mathcal{B}} R_{\mathcal{D}} (R_{B_{c0}^*} R_{\mathcal{B}} R_{\mathcal{D}})^{5/2}}{(R_{B_{c0}^*}^2 (R_{\mathcal{B}}^2 + R_{\mathcal{D}}^2) + R_{\mathcal{B}}^2 R_{\mathcal{D}}^2)^{\frac{5}{2}}}, \\ g_{B_{c0}^* \mathcal{B}^* \mathcal{D}^*} &= \frac{16\pi^{\frac{1}{4}} \sqrt{m_{B_{c0}^*} m_{\mathcal{B}^*} m_{\mathcal{D}^*}} \gamma_{c\bar{b}} R_{B_{c0}^*} R_{\mathcal{B}^*} R_{\mathcal{D}^*} (R_{B_{c0}^*} R_{\mathcal{B}^*} R_{\mathcal{D}^*})^{5/2}}{3(R_{B_{c0}^*}^2 (R_{\mathcal{B}^*}^2 + R_{\mathcal{D}^*}^2) + R_{\mathcal{B}^*}^2 R_{\mathcal{D}^*}^2)^{\frac{5}{2}}}. \end{aligned} \quad (\text{A3})$$

For the B_{c2}^{*+} couplings to $\mathcal{B}^{(*)} \mathcal{D}^{(*)}$, we have

$$\begin{aligned} g_{B_{c2}^* \mathcal{B} \mathcal{D}} &= \frac{\pi^{\frac{1}{4}} \sqrt{m_{B_{c2}^*} m_{\mathcal{B}} m_{\mathcal{D}}} \gamma_{c\bar{b}} R_{B_{c2}^*}^{5/2} (R_{\mathcal{B}} R_{\mathcal{D}})^{\frac{3}{2}} (R_{\mathcal{B}}^2 + R_{\mathcal{D}}^2) (R_{B_{c2}^*}^2 (R_{\mathcal{B}}^2 + R_{\mathcal{D}}^2) + 2R_{\mathcal{B}}^2 R_{\mathcal{D}}^2)}{\sqrt{3} (R_{B_{c2}^*}^2 (R_{\mathcal{B}}^2 + R_{\mathcal{D}}^2) + R_{\mathcal{B}}^2 R_{\mathcal{D}}^2)^{7/2}}, \\ g_{B_{c2}^* \mathcal{B}^* \mathcal{D}^*} &= \frac{16\pi^{\frac{1}{4}} \sqrt{m_{B_{c2}^*} m_{\mathcal{B}^*} m_{\mathcal{D}^*}} \gamma_{c\bar{b}} R_{B_{c2}^*} R_{\mathcal{B}^*} R_{\mathcal{D}^*} (R_{B_{c2}^*} R_{\mathcal{B}^*} R_{\mathcal{D}^*})^{5/2}}{\sqrt{3} (R_{B_{c2}^*}^2 (R_{\mathcal{B}^*}^2 + R_{\mathcal{D}^*}^2) + R_{\mathcal{B}^*}^2 R_{\mathcal{D}^*}^2)^{\frac{5}{2}}}. \end{aligned} \quad (\text{A4})$$

For the $B_{c1}^{(\prime)+}$ couplings to $\mathcal{B}^{(*)} \mathcal{D}^{(*)}$, taking into account the mixing between the 1^1P_1 and 1^3P_1 we have

$$\begin{aligned}
g_{B'_{c1}B\mathcal{D}^*} &= \cos\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B'_{c1}} m_B m_{\mathcal{D}^*}} \gamma_{c\bar{b}} R_B R_{\mathcal{D}^*} (R_{B'_{c1}} R_B R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3}(R_{B'_{c1}}^2 (R_B^2 + R_{\mathcal{D}^*}^2) + R_B^2 R_{\mathcal{D}^*}^2)^{5/2}} + \sqrt{2} \sin\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B'_{c1}} m_B m_{\mathcal{D}^*}} \gamma_{c\bar{b}} R_B R_{\mathcal{D}^*} (R_{B'_{c1}} R_B R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3}(R_{B'_{c1}}^2 (R_B^2 + R_{\mathcal{D}^*}^2) + R_B^2 R_{\mathcal{D}^*}^2)^{5/2}}, \\
g_{B_{c1}B\mathcal{D}^*} &= -\sin\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B_{c1}} m_B m_{\mathcal{D}^*}} \gamma_{c\bar{b}} R_B R_{\mathcal{D}^*} (R_{B_{c1}} R_B R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3}(R_{B_{c1}}^2 (R_B^2 + R_{\mathcal{D}^*}^2) + R_B^2 R_{\mathcal{D}^*}^2)^{5/2}} + \sqrt{2} \cos\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B_{c1}} m_B m_{\mathcal{D}^*}} \gamma_{c\bar{b}} R_B R_{\mathcal{D}^*} (R_{B_{c1}} R_B R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3}(R_{B_{c1}}^2 (R_B^2 + R_{\mathcal{D}^*}^2) + R_B^2 R_{\mathcal{D}^*}^2)^{5/2}},
\end{aligned} \tag{A5}$$

$$\begin{aligned}
g_{B'_{c1}B^*\mathcal{D}} &= \cos\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B'_{c1}} m_{B^*} m_{\mathcal{D}}} \gamma_{c\bar{b}} R_{B^*} R_{\mathcal{D}} (R_{B'_{c1}} R_{B^*} R_{\mathcal{D}})^{\frac{5}{2}}}{\sqrt{3}(R_{B'_{c1}}^2 (R_{B^*}^2 + R_{\mathcal{D}}^2) + R_{B^*}^2 R_{\mathcal{D}}^2)^{5/2}} + \sqrt{2} \sin\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B'_{c1}} m_{B^*} m_{\mathcal{D}}} \gamma_{c\bar{b}} R_{B^*} R_{\mathcal{D}} (R_{B'_{c1}} R_{B^*} R_{\mathcal{D}})^{\frac{5}{2}}}{\sqrt{3}(R_{B'_{c1}}^2 (R_{B^*}^2 + R_{\mathcal{D}}^2) + R_{B^*}^2 R_{\mathcal{D}}^2)^{5/2}}, \\
g_{B_{c1}B^*\mathcal{D}} &= -\sin\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B_{c1}} m_{B^*} m_{\mathcal{D}}} \gamma_{c\bar{b}} R_{B^*} R_{\mathcal{D}} (R_{B_{c1}} R_{B^*} R_{\mathcal{D}})^{\frac{5}{2}}}{\sqrt{3}(R_{B_{c1}}^2 (R_{B^*}^2 + R_{\mathcal{D}}^2) + R_{B^*}^2 R_{\mathcal{D}}^2)^{5/2}} + \sqrt{2} \cos\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B_{c1}} m_{B^*} m_{\mathcal{D}}} \gamma_{c\bar{b}} R_{B^*} R_{\mathcal{D}} (R_{B_{c1}} R_{B^*} R_{\mathcal{D}})^{\frac{5}{2}}}{\sqrt{3}(R_{B_{c1}}^2 (R_{B^*}^2 + R_{\mathcal{D}}^2) + R_{B^*}^2 R_{\mathcal{D}}^2)^{5/2}},
\end{aligned} \tag{A6}$$

$$\begin{aligned}
g_{B'_{c1}B^*\mathcal{D}^*} &= \cos\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B'_{c1}} m_{B^*} m_{\mathcal{D}^*}} \gamma_{c\bar{b}} R_{B^*} R_{\mathcal{D}^*} (R_{B'_{c1}} R_{B^*} R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3m_{B'_{c1}}}(R_{B'_{c1}}^2 (R_{B^*}^2 + R_{\mathcal{D}^*}^2) + R_{B^*}^2 R_{\mathcal{D}^*}^2)^{5/2}} + \sqrt{2} \sin\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B'_{c1}} m_{B^*} m_{\mathcal{D}^*}} \gamma_{c\bar{b}} R_{B^*} R_{\mathcal{D}^*} (R_{B'_{c1}} R_{B^*} R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3m_{B'_{c1}}}(R_{B'_{c1}}^2 (R_{B^*}^2 + R_{\mathcal{D}^*}^2) + R_{B^*}^2 R_{\mathcal{D}^*}^2)^{5/2}}, \\
g_{B_{c1}B^*\mathcal{D}^*} &= -\sin\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B_{c1}} m_{B^*} m_{\mathcal{D}^*}} \gamma_{c\bar{b}} R_{B^*} R_{\mathcal{D}^*} (R_{B_{c1}} R_{B^*} R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3m_{B_{c1}}}(R_{B_{c1}}^2 (R_{B^*}^2 + R_{\mathcal{D}^*}^2) + R_{B^*}^2 R_{\mathcal{D}^*}^2)^{5/2}} + \sqrt{2} \cos\theta_{1P} \frac{8\pi^{\frac{1}{4}} \sqrt{m_{B_{c1}} m_{B^*} m_{\mathcal{D}^*}} \gamma_{c\bar{b}} R_{B^*} R_{\mathcal{D}^*} (R_{B_{c1}} R_{B^*} R_{\mathcal{D}^*})^{\frac{5}{2}}}{\sqrt{3m_{B_{c1}}}(R_{B_{c1}}^2 (R_{B^*}^2 + R_{\mathcal{D}^*}^2) + R_{B^*}^2 R_{\mathcal{D}^*}^2)^{5/2}}.
\end{aligned} \tag{A7}$$

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