

Study of excited D and D_s mesons in a relativized quark model*

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Abstract: We use a modified relativistic quark model to study the properties of excited charmed and charmed strange mesons. We calculate the masses and wave functions of conventional charmed and charmed strange mesons incorporating both spin and S - D mixing effects and fit parameters of the potential model with known experimental states using differential evolution technique. Using Leading Born-Oppenheimer expansion, we also compute the spectrum and wave functions of first gluonic excited state of charmed and charmed strange mesons. We examine the effects of gluonic excitation on the spectrum of resultant hybrid mesons. By using our calculated spectrum and wave functions, we determine the radiative transitions of the conventional and hybrid open charm mesons. We compare our calculations with experimental data and other works. We expect our results will be beneficial in the detection of the charmed and charmed strange conventional and hybrids mesons.

Keywords: Charmed and Charmed Strange Mesons, Hybrid Charmed and Charmed Strange Mesons, Phenomenological potential model of mesons, Spectroscopy and Radiative transitions of Charmed and Charmed Strange mesons, Spectroscopy and Radiative transitions of Hybrid Charmed and Charmed Strange mesons

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I. INTRODUCTION

Quantum Chromodynamics (QCD) is experimentally well established theory of strong interaction of quarks and gluons. However, describing rich spectrum of hadrons from the first principle is still challenging due to non-perturbative nature of QCD at low energy. Considering this difficulty, many models and theoretical approaches have been adopted to calculate hadron properties. The quark model, where applicable, is considered to be a convenient tool. According to the quark model, a conventional meson is a bound state of a quark and an antiquark. The model predicts that a meson can have only certain specific values of J^{PC} . The mesons with J^{PC} differing from quark-model values are called exotic mesons. In 1993 an exotic meson in the light quark sector having $J^{PC} = 1^{-+}$, was observed by the VES Collaboration [1]. After that, many exotic mesons have been observed. Theoretically, exotic mesons could be glueball, tetra quark or hybrid mesons. It is noted that J^{PC} value of these mesons can be allowed or forbidden by the quark model. Lattice QCD successfully describes these exotics [2–6]. Phenomenological models also provide good description of the properties of exotic mesons [7–14]. In open flavour

mesons where quark and antiquark have different flavours, C parity no longer remains a good quantum number. In this case states are identified by their J^P values only. The states having the same J^P value are mixed. Indeed, all J^P states are mixed except 0^+ and 0^- . Mixing is either the spin mixing or the S - D mixing. We will discuss it in detail in Sec. III. Incorporating mixing, where needed, we study the properties of hybrid as well as conventional mesons in open charm meson sector specifically charmed (D) mesons and charmed strange (D_s) mesons.

Gluonic field plays a dominant role in binding quarks. In conventional mesons, the gluonic field is in the ground state while for hybrid mesons, it is in the excited state. We are particularly interested in studying the Π_u state, where gluonic field is in its first excited state. We use the leading Born-Oppenheimer approximation (LBO) for calculating the spectrum of hybrid mesons. In this approximation, a hybrid meson is treated as a diatomic molecule in which quarks and gluons are considered as nuclei and electron respectively. There are two steps involved in applying LBO approximation. In the first step the static potential energy of gluons is determined by treating quark

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antiquark spatially fixed, and in the second step the motion of quarks is restored through using this potential in Schrödinger wave equation. Parity of hybrid meson is $\varepsilon(-1)^{L+\Lambda+1}$, where Λ is magnitude of the projection of total angular momentum of gluonic field onto the molecular axis. For Π_u potential, in which gluonic field is in its first excited state, $\Lambda = 1$ and $\varepsilon = \pm 1$ [2, 3].

Experimentally immense data is available in D and D_s meson domain. First two charmed mesons (D^0 and $D^\pm$) were discovered in Mark I experiments in 1976 [15, 16]. Their J^P is 0^- . D^0 and $D^\pm$ have masses 1864.83 ± 0.05 MeV and 1869.58 ± 0.09 MeV respectively. One year later, first charmed strange meson ($D_s^\pm$) of $J^P = 0^-$, was observed in DASP Collaboration [17]. Mass of $D_s^\pm$ is 1968.34 ± 0.07 MeV [18]. In the same experiment $D_s^{*\pm}$ meson was also discovered having mass 2112.1 ± 0.4 MeV and $J^P = 1^-$. Now we have data for more than twenty well recognized states. Information of their masses, J^P values and dozens of their decay modes are available in the Particle Data Group [18]. Side by side, a lot of work has been done to calculate their masses and other properties. By using Schrödinger-like wave equation in relativistic dynamics, spectrum and wave functions of D and D_s mesons are calculated in Ref. [19–22]. In Ref. [20], strong and radiative transitions are also reported. Ref. [23, 24] implemented Dirac formalism to calculate spectrum, radiative transitions, decay constant and leptonic decay width of D_s and D meson system respectively. In Ref. [25], spectra and wave functions of D and D_s mesons are obtained by using Bethe-Salpeter equation. Ref. [26] computed the spectra of heavy-light meson with the help of QCD motivated relativistic quark model. They investigated Regge trajectories too. By using the Dirac Hamiltonian, Ref. [27] computed the mass spectra, wave functions and hadronic transitions of heavy-light mesons. They also added leading order corrections in $1/m_{c,b}$. In Ref. [28], spectrum of D and D_s mesons is calculated by using Wilson twisted mass lattice QCD. Ref. [21, 22] discretized the relativistic wave equation using Cornell potential (Coulomb plus linear potential). After calculating the masses and wavefunctions of mesons by diagonalizing the resultant Hamiltonian matrix, the spin dependent part of the Hamiltonian is added perturbatively. Since states with different principal quantum numbers can also have same J^P . Thus several principal quantum numbers collectively contribute in one mixed state (either spin mixed or S - D mixed). In Ref. [21, 22] the contribution of two to three principal quantum numbers is included to describe a mixed state. Another limitation of their work is that the perturbative effects of spin dependent potential are included in the masses but not in the wave function. This is understandable because first order perturbative correction to wave function involves a sum over all possible internal states which is extremely

difficult to calculate. We adopted the same approach to solve the relativistic wave equation as did in Ref. [21, 22] with the improvements that address the limitations of approach of Ref. [21, 22]. Instead of including spin dependent part of the Hamiltonian perturbatively, we also include spin dependent terms in the Hamiltonian matrix. For this purpose we developed a system of coupled equations for mixed states. Since we are solving the coupled equations nonperturbatively, we are not restricted to include the contribution of only two to three principal quantum numbers to describe a mixed state. Two charmed strange mesons named $D_{s0}^*(2317)^\pm$ and $D_{s1}(2460)^\pm$ attract special attention as the observed values of their masses are much lower than the quark model predictions. Attempts have been made to describe them through alternative models [29–32]. A lot of work is done in literature to identify the nature of these mesons. In some literature [33, 34] these states are interpreted as tetraquarks, while in some places $D_{s0}^*(2317)^\pm$ and $D_{s1}(2460)^\pm$ are taken as DK and DK^* molecule states respectively [35, 36]. In Refs. [37, 38], these states are well described by including coupled-channel effects. Ref. [37] suggests that these states are the mixtures of bare $c\bar{s}$ core and $D^{(*)}K$ component. In Refs. [20–22, 39], these states have been studied as conventional mesons. The exact structure of these particles is still an open question in hadron physics.

In this work, we calculate masses, wave functions and radiative transitions of conventional as well as hybrid D and D_s mesons incorporating spin and S - D mixing effects on spectrum and wave function. Unlike previous works, we solve Schrödinger-type equation using full semi-relativistic Hamiltonian with spin-dependent part. This allows us to study both spin and S - D mixing without using perturbation theory.

Roadmap of our work is the following: In Sec. II, we describe our potential model for conventional and hybrid mesons. We use an extended version of the famous GI model. For the hybrid mesons, we add the potential model of gluonic excitation which we suggested in our previous work. In this paper we improve the parameter fitting of our proposed gluonic excited potential model by considering the latest lattice data [6]. In Sec. III, the phenomena of state mixing are examined. We distinguish two categories of mixed states. We also discuss the effects of spin dependent part of the potential model on the spectrum. In Sec. IV, we describe our relativistic wave equation. We explicitly describe how we calculate masses and wave functions of mesons using relativistic wave equation. State mixing effects in the Hamiltonian matrix are discussed too. Radiative transitions of charmed and charmed strange mesons are reviewed in Sec. V. In the last section we report our results and concluding remarks based on our calculations.

II. EFFECTIVE POTENTIAL MODEL OF MESONS

Phenomenological potential models are found very successful in describing various properties of heavy quark mesons especially mass [19–25, 40–43]. Among the suggested models, the GI model [19] is considered to be the most famous one. In this work, we have used it with some modifications.

A. Potential model of conventional mesons

Conventional D and D_s meson is a bound state of a heavy quark ($Q = c$) and a light antiquark ($\bar{q} = \bar{u}/\bar{d}, \bar{s}$). Potential model for a conventional meson $V_{Q\bar{q}}(r)$ is

$$V_{Q\bar{q}}(r) = -\frac{4\alpha_s(r)}{3r} + br + c + V_{spin-dep}(r), \quad (1)$$

where r is inter-quark distance. The first term of the potential model, called Coulomb term, is calculated by one gluon exchange diagram in perturbation theory. It determines the behaviour of potential at small distances. $\alpha_s(r)$ is a strong running coupling constant of QCD. We use its following parameterization [19],

$$\alpha_s(r) = \sum_{k=1}^3 \alpha_k \frac{2}{\sqrt{\pi}} \int_0^{\gamma_k r} e^{-x^2} dx, \quad (2)$$

where α_k 's and γ_k 's are free parameters. In this work we take $\gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{\sqrt{10}}{2}$ and $\gamma_3 = \frac{\sqrt{1000}}{2}$ from Ref. [19]. With this choice of γ 's, α_2 and α_3 control the shape of running coupling at small distance (large Q^2), where it is already fixed by pQCD. This allows us to fix $\alpha_2 = 0.16$ and $\alpha_3 = 0.20$ by fitting with 2-loop running coupling. Consequently, parameter α_1 (or $\alpha_s^c \equiv \alpha_1 + \alpha_2 + \alpha_3$), which control the shape of running coupling in infrared region, is treated as a free parameter which is to be fixed by spectroscopy [19, 44]. Comparison of $\alpha_s(Q^2)$ of the parametrized model with our fitted value of $\alpha_s^c = 0.24$ and pQCD 2-loop result is shown in Fig. 2.

The second term of the potential is the confinement term dominating at large distances and b is the string tension, determined by fitting the spectrum. The third term c of the potential model is also taken as a free parameter. The last part of the potential model, the spin dependent term $V_{spin-dep}(r)$, is defined as following

$$V_{spin-dep}(r) = V_{hyp}(r) + V_T(r) + V_{so}(r), \quad (3)$$

where $V_{hyp}(r)$ is spin-spin contact hyperfine interaction [43]. It is defined as

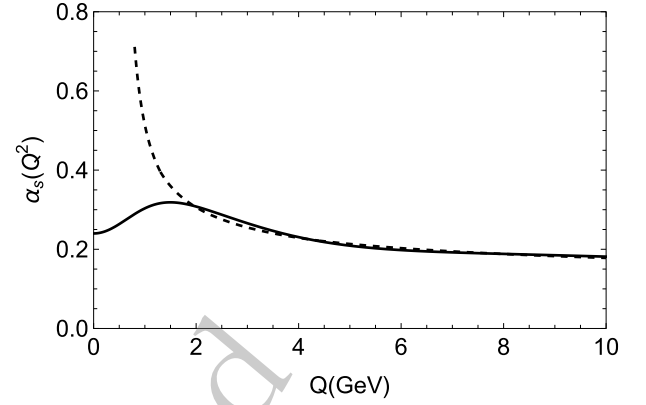


Fig. 1. Strong running coupling constant in momentum space. Solid curve represent data taken from parametrized model and dotted curve represent data from QCD perturbative calculations up to two loop corrections.

$$V_{hyp}(r) = \frac{32\pi}{9m_Q\tilde{m}_{q1}} \alpha_s(r) \left(\frac{\sigma}{\sqrt{\pi}} \right)^3 e^{-\sigma^2 r^2} \mathbf{S}_Q \cdot \mathbf{S}_{\bar{q}}, \quad (4)$$

where $\langle \mathbf{S}_Q \cdot \mathbf{S}_{\bar{q}} \rangle = \frac{S(S+1)}{2} - \frac{3}{4}$. In Ref. [19] parameter σ is

$$\sigma = \sqrt{\sigma_0^2 \left(\frac{1}{2} + \frac{1}{2} \left(\frac{4m_Q m_{\bar{q}}}{(m_Q + m_{\bar{q}})^2} \right)^4 \right) + s_0^2 \left(\frac{2m_Q m_{\bar{q}}}{m_Q + m_{\bar{q}}} \right)^2}.$$

Numerical values of parameters σ_0 and s_0 are determined by fitting the spectrum. In Eq. (3), $V_T(r)$ is tensor term defined as

$$V_T(r) = \frac{4\alpha_s(r)}{m_Q\tilde{m}_{q2}} \frac{T}{r^3}, \quad (5)$$

where T is tensor operator,

$$T = \mathbf{S}_Q \cdot \hat{r} \mathbf{S}_{\bar{q}} \cdot \hat{r} - \frac{1}{3} \mathbf{S}_Q \cdot \mathbf{S}_{\bar{q}}. \quad (6)$$

The tensor operator has nonvanishing matrix elements only between $L > 0$ and spin triplet states. Diagonal Hamiltonian matrix entries of tensor operator are [43]

$$\langle {}^3L_J | T | {}^3L_J \rangle = \begin{cases} -\frac{L}{6(2L+3)} & J = L+1 \\ \frac{1}{6} & J = L \\ -\frac{(L+1)}{6(2L-1)} & J = L-1. \end{cases} \quad (7)$$

To study S - D mixed states we also need following off diagonal matrix element of the tensor operator

$$\langle {}^3L_J | T | {}^3L'_J \rangle = \frac{\sqrt{(L+1)(L+2)}}{2(2L+3)}, \quad (8)$$

where $L' = L + 2$. In Eq. (3), $V_{so}(r)$ is spin-orbit interaction term. It consists of two parts, such that

$$V_{so}(r) = V_{sov}(r) + V_{sos}(r). \quad (9)$$

Here $V_{sov}(r)$ is spin-orbit vector part

$$V_{sov}(r) = \frac{4}{3} \frac{\alpha_s(r)}{r^3} \left[\left(\frac{1}{2m_Q^2} + \frac{1}{m_Q \tilde{m}_{q3}} \right) \mathbf{L} \cdot \mathbf{S}_Q + \left(\frac{1}{2\tilde{m}_{q3}^2} + \frac{1}{m_Q \tilde{m}_{q3}} \right) \mathbf{L} \cdot \mathbf{S}_{\bar{q}} \right], \quad (10)$$

and $V_{sos}(r)$ is spin-orbit scalar part of spin-orbit interaction term

$$V_{sos}(r) = -\frac{b}{r} \left(\frac{\mathbf{L} \cdot \mathbf{S}_Q}{2m_Q^2} + \frac{\mathbf{L} \cdot \mathbf{S}_{\bar{q}}}{2\tilde{m}_{q4}^2} \right). \quad (11)$$

For diagonal Hamiltonian matrices, $\langle \mathbf{L} \cdot \mathbf{S}_Q \rangle = \langle \mathbf{L} \cdot \mathbf{S}_{\bar{q}} \rangle = \frac{\langle \mathbf{L} \cdot \mathbf{S} \rangle}{2}$, where $\langle \mathbf{L} \cdot \mathbf{S} \rangle$ is

$$\begin{aligned} \langle {}^1L_L | \mathbf{L} \cdot \mathbf{S} | {}^1L_L \rangle &= \langle {}^3L_L | \mathbf{L} \cdot \mathbf{S} | {}^3L_L \rangle \\ &= \frac{J(J+1) - L(L+1) - S(S+1)}{2}. \end{aligned} \quad (12)$$

To study spin mixed states, we need following matrix element of the $\langle \mathbf{L} \cdot \mathbf{S}_Q \rangle$ and $\langle \mathbf{L} \cdot \mathbf{S}_{\bar{q}} \rangle$ operators:

$$\langle {}^1L_L | \mathbf{L} \cdot \mathbf{S}_Q | {}^3L_L \rangle = -\frac{\sqrt{L(L+1)}}{2}, \quad (13)$$

$$\langle {}^1L_L | \mathbf{L} \cdot \mathbf{S}_{\bar{q}} | {}^3L_L \rangle = \frac{\sqrt{L(L+1)}}{2}. \quad (14)$$

The form of the spin dependent part of the potential model is extracted from the leading order perturbation theory in the non-relativistic approximation. However, this is not good approximation, especially for light quark in heavy light meson. Due to the presence of light quark in charmed and charmed strange meson, it is intimately important to incorporate relativistic effects. These effects modify the potential in two different ways [19]: First, quark antiquark separation (r) become smeared over a distance of order of inverse quark mass. By introducing the smearing function [19], smearing over $1/r^3$ terms in spin dependent part of the potential model is incorporated. Smearing also eliminates the instability in the solu-

tion of the relativistic wave equation which occurs at short distances due to $1/r^3$ term in spin dependent potential model, and makes the solution stable even at short distances. The second way to incorporate relativistic effects is to make potential model momentum dependent. Momentum dependence in a spin dependent term can effectively be incorporated by introducing multiple constant factors. This is done only for light quark, for heavy quark momenta drop out. Since momentum dependence modifies each spin interaction term in a different way, so we are required to take multiple constant (ϵ_i) interaction dependent, where $i=1, 2, 3$ and 4 is for hyperfine term, tensor term, spin orbit vector term and spin orbit scalar term respectively. We combine these constants into light mass and define interaction dependent mass as [21, 22], $\tilde{m}_{qi} = \tilde{m}_q \epsilon_i$. Numerical values of these parameters are obtained by fitting the spectrum. It noted that since we are treating m_q as a free parameter, therefore, we must take of $\epsilon_1 = 1$.

B. Potential model of hybrid mesons

In hybrid mesons, the gluonic field is in its excited state. In this case, quark antiquark potential can be written as following

$$V_{hybrid}(r) = V_{Q\bar{q}}(r) + V_g(r), \quad (15)$$

where $V_{Q\bar{q}}(r)$ is the potential for conventional mesons appearing in Eq. (1). $V_g(r)$ is the potential energy difference between the ground and the first excited state of gluonic field. Our proposed model of $V_g(r)$ is [45],

$$V_g(r) = A e^{-Br^\nu} + \frac{\mu}{r}. \quad (16)$$

Numerical values of parameters A , B , ν and μ are determined by fitting the model with recent lattice data taken from Fig. 2 of Ref. [6]. The parameters are estimated by minimizing the χ^2 function using a global numerical op-

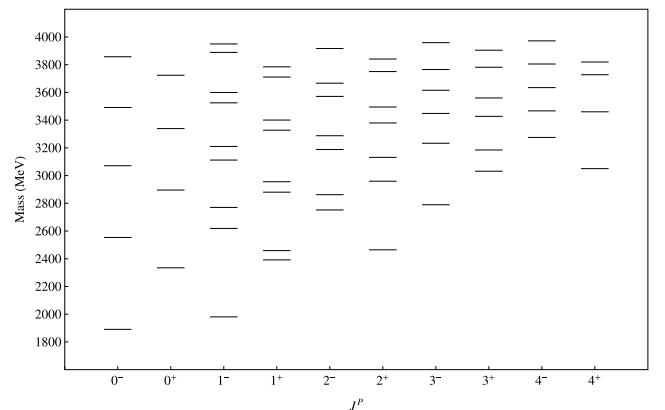


Fig. 2. Conventional charmed mesons spectrum.

timization algorithm. Our improved fitted values of A , B , ν and μ are 3.0552 GeV, 0.9369 GeV, 0.3881 and -0.0025 respectively. There is no physical motivation for using above model for fitting gluon potential except that it provides a nice fit to the lattice data.

III. MIXED STATES

In D and D_s meson, charge conjugation is not a conserved quantum number, and states are described only by their J^P values. States having the same value of J^P can mix. Except 0^- and 0^+ all possible values of J^P have mixed states. State mixing is categorized into two types, S - D mixing and spin mixing.

A. S - D mixing

It is mixing of spin triplet states ($^3L_J \leftrightarrow ^3L_{J'}$) having different values of orbital angular momentum (L, L') and same value of total angular momentum $J=L+1=L'-1$, so that $L'=L+2$. 3S_1 and 3D_1 states have the same values of J^P , therefore, they are mixed. Similarly $^3P_2 \leftrightarrow ^3F_2$, $^3D_3 \leftrightarrow ^3G_3$, $^3F_4 \leftrightarrow ^3H_4$, $^3G_5 \leftrightarrow ^3I_5$ and so on, all are referred as S - D mixed states. S - D mixing occurs due to the tensor operator in the potential model, as this operator does not conserve orbital angular momentum.

B. Spin mixing

The spin-orbit interaction term in the potential model does not conserve total spin of quark and antiquark. It causes mixing between spin triplet and spin singlet states ($^1L_L \leftrightarrow ^3L_L$), for which $J=L$. In this case, possible combinations of mixed states are $^1P_1 \leftrightarrow ^3P_1$, $^1D_2 \leftrightarrow ^3D_2$, $^1F_3 \leftrightarrow ^3F_3$, $^1G_4 \leftrightarrow ^3G_4$, $^1H_5 \leftrightarrow ^3H_5$ and so on.

IV. RELATIVISTIC WAVE EQUATION

In studying the charmed and charmed strange mesons relativistic effects should be incorporated due to the presence of light quark. We use the relativistic wave equation to study the bound states of charmed and charmed strange mesons [19]. The Schrödinger-type equation in relativistic dynamics is

$$H\psi(\mathbf{r}) = E\psi(\mathbf{r}), \quad (17)$$

where $\psi(\mathbf{r})$ and E are wave function and energy of meson respectively. H is effective Hamiltonian of meson:

$$H = \sqrt{\mathbf{p}^2 + m_Q^2} + \sqrt{\mathbf{p}^2 + m_{\bar{q}}^2} + V(\mathbf{r}), \quad (18)$$

where $\mathbf{p}$ is center-of-mass momentum of quark such that $\mathbf{p} = \mathbf{p}_Q = -\mathbf{p}_{\bar{q}}$. $\mathbf{p}_Q$ and $\mathbf{p}_{\bar{q}}$ are momenta of quark and antiquark. m_Q and $m_{\bar{q}}$ are mass of heavy quark (*i.e.*, charm

quark) and light antiquark respectively. $V(\mathbf{r})$ is the effective potential of bound state of quark antiquark system, discussed in Sec. II. We assumed our potential is central potential, *i.e.*, $V(\mathbf{r}) \equiv V(r)$. For conventional mesons $V(r) = V_{Q\bar{q}}(r)$ (appearing in Eq. (1)) and for hybrid mesons $V(r) = V_{\text{hybrid}}(r)$ (appearing in Eq. (15)).

A conventional meson is a bound state of a quark and an antiquark. When the distance between them is large enough, the wave function vanishes. We assume the distance at which the wave function vanishes is R . In order to solve Eq. (17), we find matrix representation of relativistic Hamiltonian H using spherical Bessel function as basis. The diagonalization of the Hamiltonian matrix yields $\psi(\mathbf{r})$ and E . For unmixed state (*i.e.*, 0^- , 0^+) radial wave function is

$$R_l(r) = \sum_{i=1}^N c_i \frac{a_i}{R} j_l\left(\frac{a_i r}{R}\right), \quad (19)$$

where c_i s are expansion coefficients, j_l is the spherical Bessel function and a_i is i^{th} root of the spherical Bessel function such that $j_l(a_i) = 0$. In this basis the Eq. (17) reduces to Ref. [22]

$$\begin{aligned} & \frac{2\Delta a_i a_i^2}{\pi R^3} N_{li}^2 \left(\sqrt{\left(\frac{a_i}{R}\right)^2 + m_Q^2} + \sqrt{\left(\frac{a_i}{R}\right)^2 + m_{\bar{q}}^2} \right) c_i \\ & + \sum_{j=1}^N \frac{a_j}{N_{li}^2 a_i} \int_0^R dr V(r) r^2 j_l\left(\frac{a_i r}{R}\right) j_l\left(\frac{a_j r}{R}\right) c_j \\ & = E c_i, \end{aligned} \quad (20)$$

where N_{li}^2 is radial integral of the spherical Bessel function give by

$$N_{li}^2 = \int_0^R dr' r'^2 j_l\left(\frac{a_i r'}{R}\right)^2. \quad (21)$$

In Eq. (20), the Hamiltonian matrix has the order $N \times N$. By diagonalising the Hamiltonian matrix, we get N eigenvalues and eigenfunctions. Each eigenvalue and eigenfunction is associated with a particular value of the radial quantum number. For each eigen vector the resultant radial wave function is obtained from Eq. (19). In case of mixed states, $\psi(\mathbf{r})$ is combination of two states. For spin mixed states, $\psi(\mathbf{r})$ is

$$\psi(\mathbf{r}) = \sum_{i=1}^N c_i^{(1)} \frac{a_i}{R} j_l\left(\frac{a_i r}{R}\right) |^1 L_L\rangle_{\hat{r}} + \sum_{i=1}^N c_i^{(2)} \frac{a_i}{R} j_l\left(\frac{a_i r}{R}\right) |^3 L_L\rangle_{\hat{r}}, \quad (22)$$

where a_i is zeroth roots of the spherical Bessel function corresponding to L orbital angular momentum. $c_i^{(1)}$ and

$c_i^{(2)}$ are expansion coefficients associated to $|^1L_L\rangle_{\hat{r}}$ state and $|^3L_L\rangle_{\hat{r}}$ state respectively, which are defined as:

$$|^1L_L\rangle_{\hat{r}} = |0,0\rangle_s Y_{L,L}(\hat{r}), \quad (23)$$

$$|^3L_L\rangle_{\hat{r}} = \sum_{m_s} C_{m_s} |1, m_s\rangle_s Y_{L,L-m_s}(\hat{r}). \quad (24)$$

Where C_{m_s} is Clebsch-Gorden coefficient. In mixed states the Eq. (17) is reduced to two coupled equations. For spin mixed states the equations are:

$$\begin{aligned} & \frac{2}{\pi R^3} \Delta a_i a_i^2 N_{li}^2 \left(\sqrt{\left(\frac{a_i}{R}\right)^2 + m_Q^2} + \sqrt{\left(\frac{a_i}{R}\right)^2 + m_q^2} \right) c_i^{(1)} \\ & + \sum_{j=1}^N \frac{a_j}{N_{li}^2 a_i} \int_0^R dr (\langle V \rangle_{11} c_j^{(1)} + \langle V \rangle_{12} c_j^{(2)}) \\ & r^2 j_l \left(\frac{a_i r}{R} \right) j_l \left(\frac{a_j r}{R} \right) = E c_i^{(1)}, \end{aligned} \quad (25)$$

$$\begin{aligned} & \frac{2}{\pi R^3} \Delta a_i a_i^2 N_{li}^2 \left(\sqrt{\left(\frac{a_i}{R}\right)^2 + m_Q^2} + \sqrt{\left(\frac{a_i}{R}\right)^2 + m_q^2} \right) c_i^{(2)} \\ & + \sum_{j=1}^N \frac{a_j}{N_{li}^2 a_i} \int_0^R dr (\langle V \rangle_{21} c_j^{(1)} + \langle V \rangle_{22} c_j^{(2)}) \\ & r^2 j_l \left(\frac{a_i r}{R} \right) j_l \left(\frac{a_j r}{R} \right) = E c_i^{(2)}. \end{aligned} \quad (26)$$

In this case the Hamiltonian matrix H can be expressed through four $N \times N$ matrices H_{11} , H_{12} , H_{21} , and H_{22} as following

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}. \quad (27)$$

H_{11} and H_{12} are read off from Eq. (25), and H_{21} and H_{22} from Eq. (26). The kinetic energy of mesons contributes only to diagonal Hamiltonian matrices (H_{11}, H_{22}). $\langle V \rangle_{11}$, $\langle V \rangle_{12}$, $\langle V \rangle_{21}$ and $\langle V \rangle_{22}$ are potential energy matrix elements of H_{11} , H_{12} , H_{21} and H_{22} respectively. The definition of the potential energy matrix elements is mentioned in Table 1. It is noted that mixing of the states occurs due to off-diagonal Hamiltonian matrices (H_{12}, H_{21}). For spin mixed state, contribution to $\langle V \rangle_{12}$ and $\langle V \rangle_{21}$ come through spin-orbit terms of the potential energy.

In S - D mixed states, $\psi(\mathbf{r})$ is

Table 1. The definition of potential energy matrix elements for spin and S - D mixed states.

Potential energy matrix elements	Spin mixed state	S - D mixed state
$\langle V \rangle_{11}$	$\langle ^1L_L V ^1L_L \rangle$	$\langle ^3L_J V ^3L_J \rangle$
$\langle V \rangle_{12}$	$\langle ^1L_L V_{so} ^3L_L \rangle$	$\langle ^3L_J V_T ^3L_J' \rangle$
$\langle V \rangle_{21}$	$\langle ^3L_L V_{so} ^1L_L \rangle$	$\langle ^3L_J' V_T ^3L_J \rangle$
$\langle V \rangle_{22}$	$\langle ^3L_L V ^3L_L \rangle$	$\langle ^3L_J' V ^3L_J' \rangle$

$$\psi(\mathbf{r}) = \sum_{i=1}^N c_i^{(1)} \frac{a_i}{R} j_l \left(\frac{a_i r}{R} \right) |^3L_J\rangle_{\hat{r}} + \sum_{i=1}^N c_i^{(2)} \frac{a_i'}{R} j_{l'} \left(\frac{a_i' r}{R} \right) |^3L_J'\rangle_{\hat{r}}, \quad (28)$$

where a_i and a_i' are zeroth roots of the spherical Bessel function corresponding to L and L' orbital angular momentum respectively. In S - D mixed states, $c_i^{(1)}$ and $c_i^{(2)}$ are expansion coefficients associated to $|^3L_J\rangle_{\hat{r}}$ state and $|^3L_J'\rangle_{\hat{r}}$ state respectively, which are defined as:

$$|^3L_J\rangle_{\hat{r}} = |1, 1\rangle_s Y_{L,L}(\hat{r}), \quad (29)$$

$$|^3L_J'\rangle_{\hat{r}} = \sum_{m_s} C_{m_s} |1, m_s\rangle_s Y_{L+2, L+1-m_s}(\hat{r}). \quad (30)$$

For S - D mixed states, we have following set of equations:

$$\begin{aligned} & \frac{2}{\pi R^3} \Delta a_i a_i^2 N_{li}^2 \left(\sqrt{\left(\frac{a_i}{R}\right)^2 + m_Q^2} + \sqrt{\left(\frac{a_i}{R}\right)^2 + m_q^2} \right) c_i^{(1)} \\ & + \sum_{j=1}^N \frac{a_j}{N_{li}^2 a_i} \int_0^R dr \langle V \rangle_{11} r^2 j_l \left(\frac{a_i r}{R} \right) j_l \left(\frac{a_j r}{R} \right) c_j^{(1)} \\ & + \sum_{j=1}^N \frac{a_j'}{N_{li}^2 a_i} \int_0^R dr \langle V \rangle_{12} r^2 j_l \left(\frac{a_i r}{R} \right) j_{l'} \left(\frac{a_j' r}{R} \right) c_j^{(2)} = E c_i^{(1)}, \end{aligned} \quad (31)$$

$$\begin{aligned} & \frac{2}{\pi R^3} \Delta a_i' a_i'^2 N_{li'}^2 \left(\sqrt{\left(\frac{a_i'}{R}\right)^2 + m_Q^2} + \sqrt{\left(\frac{a_i'}{R}\right)^2 + m_q^2} \right) c_i^{(2)} \\ & + \sum_{j=1}^N \frac{a_j}{N_{li'}^2 a_i'} \int_0^R dr \langle V \rangle_{21} r^2 j_{l'} \left(\frac{a_i' r}{R} \right) j_l \left(\frac{a_j r}{R} \right) c_j^{(1)} \\ & + \sum_{j=1}^N \frac{a_j'}{N_{li'}^2 a_i'} \int_0^R dr \langle V \rangle_{22} r^2 j_{l'} \left(\frac{a_i' r}{R} \right) j_{l'} \left(\frac{a_j' r}{R} \right) c_j^{(2)} = E c_i^{(2)}. \end{aligned} \quad (32)$$

Once again these equations can be used to define Hamiltonian matrix H in terms of four $N \times N$ matrices

H_{11} , H_{12} , H_{21} and H_{22} . It is noted that for S - D mixed states, contribution to $\langle V \rangle_{12}$ and $\langle V \rangle_{21}$ comes through tensor operator of the potential.

Our numerical results also depend upon the order of the matrix N . We observe that for a large value of N , dependency is negligible. In our calculations we take $N = 50$ and $R = 25 \text{ GeV}^{-1}$. Numerical values of quark masses and other parameters of the conventional potential model are extracted from fitting the spectrum to experimentally known states. We used 14 experimentally well-established states for fitting. The states that we used for fitting parameter values are mentioned in Table 2. For fitting the spectrum, we used the differential evolution method [46]. Differential evolution (DE) is a stochastic global optimization algorithm. In consecutive independent trials, DE successfully converged to the global minimum of the function. By minimizing the χ^2 function using the DE algorithm, we find out the best fit values of parameters. Our fitted values of parameters are: $m_c = 1.47 \pm 0.01 \text{ GeV}$, $m_{u/d} = 0.020 \pm 0.001 \text{ GeV}$, $\tilde{m}_{u/d} = 0.430 \pm 0.001 \text{ GeV}$, $m_s = 0.336 \pm 0.007 \text{ GeV}$, $\tilde{m}_s = 0.5967 \pm 0.00046 \text{ GeV}$, $\alpha_s^c = 0.240 \pm 0.025$, $b = 0.211 \pm 0.002 \text{ GeV}^2$, $c = -0.4321 \pm 0.0041 \text{ GeV}$, $\sigma_0 = 0.5221 \pm 0.013 \text{ GeV}$, $s_0 = 0.91 \pm 0.15$, $e_2 = 1.5 \pm 0.59$, $e_3 = 1.5 \pm 0.1$, $e_4 = 1.31 \pm 0.14$. Our predicted spectra of conventional charmed and charmed strange mesons is given in Fig. 2 and Fig. 3. In Fig. 4 and Fig. 5, we present our calculated spectrum of hybrid charmed mesons and hybrid charmed strange mesons. Comparison of our calculated spectrum of conventional D and D_s mesons with other works is given in Tables 2 and 3.

V. RADIATIVE TRANSITIONS

Radiative transitions play a vital role in the identification of meson states. $E1$ radiative partial width between initial and final mesons in non-relativistic quark model is [56]

$$\Gamma(n^{2S+1}L_J \rightarrow n'^{2S'+1}L'_{J'} + \gamma) = \frac{4}{3} \langle e_Q \rangle^2 \alpha \omega^3 C_{fi} \delta_{SS'} \delta_{L,L' \pm 1} |\langle n'^{2S'+1}L'_{J'} | r | n^{2S+1}L_J \rangle|^2 \frac{E_f}{M_i}, \quad (33)$$

where $\alpha = \frac{1}{137}$ is electromagnetic fine structure constant, ω is photon energy, $|n^{2S+1}L_J\rangle$ is initial meson state, $|n'^{2S'+1}L'_{J'}\rangle$ is final meson state, E_f is the energy of final meson state and M_i is mass of initial meson. C_{fi} is angular momentum matrix element defined as

$$C_{fi} = \max(L, L')(2J' + 1) \begin{Bmatrix} L' & J' & S \\ J & L & 1 \end{Bmatrix}^2,$$

where $\begin{Bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{Bmatrix}$ is a 6- j symbol. In Eq. (33), $\langle e_Q \rangle$ is

$$\langle e_Q \rangle = \frac{m_c e_q - m_q e_{\bar{c}}}{m_q + m_c}. \quad (34)$$

Here m_c and m_q are constituent mass of charm quark and constituent mass of light (q) quark respectively. For

Table 2. Comparison of our calculated spectrum (in MeV) of conventional D and D_s mesons with other works.

Meson	J^P	Our Mass Results	GI Model[20]	[22]	[39]	PDG[18]
Charmed strange mesons:						
$D_s^\pm$	0^-	2001.36	1979	1963	1964	1968.35 ± 0.07
$D_s^{*\pm}$	1^-	2079.63	2129	2103	2102	2112.2 ± 0.4
$D_{s1}(2536)^\pm$	1^+	2535.92	2556	2533	2510	2535.11 ± 0.06
$D_{s2}^*(2573)$	2^+	2569.18	2592	2573	2548	2569.1 ± 0.8
$D_{s1}^*(2700)^\pm$	1^-	2706.22	2732	2709	2680	2714 ± 5
$D_{s1}^*(2860)^\pm$	1^-	2820.59	2899	2803	2771	2859 ± 27 [48, 49]
$D_{s3}^*(2860)^\pm$	3^-	2869.84	2917	2853	2816	2860 ± 7
Charmed mesons:						
D^0	0^-	1890.66	1877	1867		1864.83 ± 0.05
$D^*(2007)^0$	1^-	1980.63	2041	2004		2006.85 ± 0.05
$D_0^*(2300)^0$	0^+	2334.64	2399	2302		2343 ± 10
$D_1(2430)^0$	1^+	2458.56	2456	2415		2412 ± 9
$D_1(2420)^0$	1^+	2391.87	2467	2429		2422.1 ± 0.6
$D_2^*(2460)^0$	2^+	2464.39	2502	2468		$2461.1_{-0.8}^{+0.7}$
$D_3^*(2750)$	3^-	2789.59	2833	2754		2763.1 ± 3.2

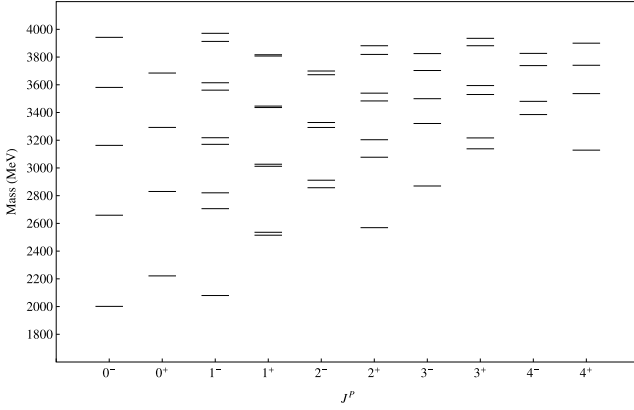


Fig. 3. Conventional charmed strange mesons spectrum.

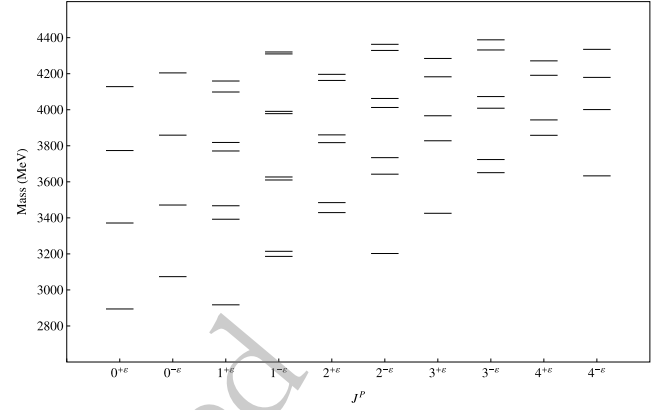


Fig. 5. Hybrid charmed strange mesons spectrum.

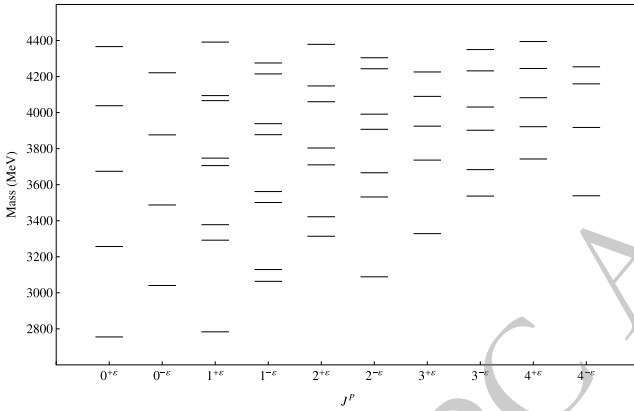


Fig. 4. Hybrid charmed mesons spectrum.

D meson, $q = u/d$ whereas for D_s meson $q = s$. $e_{\bar{c}} = -2/3$ is electric charge of anti-charm quark in units of $|e|$. $e_u = +2/3$, $e_d = -1/3$ and $e_s = -1/3$ are electric charge of up, down and strange quark respectively in units of $|e|$.

Matrix elements $\langle n'^{2S'+1}L_J' | r | n^{2S+1}L_J \rangle$ are calculated using the relativistic quark model wave functions obtained in Sec. IV. In $E1$ transitions, relativistic corrections are implicitly included due to the contribution of spin dependent interactions in the Hamiltonian.

Table 3. Comparison of our calculated spectrum (in MeV) of conventional D and D_s mesons with others works and experimental data.

Meson	J^P	Our Results	GI Model[20]	[22]	[39]	Exp. Data
Charmed strange mesons:						
$D_{s0}^*(2590)^+$	0^-	2659	2673	2582	2557	2591 ± 9 [47]
$D_{s1}^*(3040)^\pm$	$1^+, 2^+$	3026.94, 3077.36				3044^{+31}_{-9} [49]
Charmed mesons:						
$D_0(2550)^0$	0^-	2553.34	2581	2483		2549 ± 19 [50, 51]
$D_1^*(2600)^0$	1^-	2618.66	2643	2613		2627 ± 10 [51–53]
$D_2(2740)^0$	2^-	2752.03	2816	2737		2747 ± 6 [53, 54]
$D_1^*(2760)^0$	1^-	2770.32	2817	2785		2781 ± 22 [55]
$D(3000)^0$	$3^+, 3^-, 4^-$	3184.76, 3233.29, 3275.27				3214 ± 60 [52, 54]

$M1$ radiative partial width for S -wave meson bound state is [20]

$$\begin{aligned} \Gamma(n^{2S+1}L_J \rightarrow n'^{2S'+1}L_{J'} + \gamma) \\ = \frac{\alpha}{3} \omega^3 (2J' + 1) \delta_{S,S' \pm 1} |\langle f | \frac{e_q}{m_q} j_0 \left(\frac{m_c}{m_q + m_c} \omega r \right) \\ - \frac{e_{\bar{c}}}{m_c} j_0 \left(\frac{m_q}{m_q + m_c} \omega r \right) | i \rangle|^2, \end{aligned} \quad (35)$$

where j_0 is the spherical Bessel function. Electromagnetic transitions are sensitive to the internal structure of mesons, particularly in mixed states. A mixed state is a combination of two states, only that state(s) contributes to $E1$ and $M1$ transitions, which fulfill the selection rules of these transitions. For calculating decay width of $M1$ transitions of conventional S -wave meson bound state, selection rules of $M1$ transitions and Eq. (28) implies that in this case only $1^- \leftrightarrow 0^-$ transition allowed further only 3S_1 component of 1^- state contributes to decay width. To calculate decay width of $E1$ transitions of conventional meson bound states ($L < 3$), selection rules of $E1$ transitions and Eqs. (22, 28) indicate that only four transitions contribute, i.e., $1^+ \leftrightarrow 0^-$, $1^- \leftrightarrow 0^+$, $1^- \leftrightarrow 1^+$ and $2^- \leftrightarrow 1^+$.

Table 4. Our calculated conventional D and D_s meson masses (in MeV) for $J^P = 0^-, 0^+, 1^-, 1^+, 2^-, 2^+, 3^-, 3^+, 4^-$ and 4^+ .

J^P	n	D Meson	D_s Meson
0^-	1	1890.66	2001.36
	2	2553.34	2659.00
	3	3070.94	3162.90
	4	3491.35	3581.01
0^+	1	2334.64	2221.26
	2	2895.60	2830.25
	3	3339.03	3292.43
	4	3723.79	3684.70
1^-	1	1980.63	2079.63
	2	2618.66	2706.22
	3	2770.32	2820.59
	4	3111.91	3170.35
	5	3210.21	3218.06
	6	3524.85	3561.34
	7	3599.51	3614.62
	8	3888.75	3912.91
1^+	1	2391.87	2514.83
	2	2458.56	2535.92
	3	2880.36	3012.05
	4	2955.80	3026.94
	5	3327.55	3435.81
	6	3400.51	3446.24
	7	3710.89	3807.50
	8	3784.40	3816.56
2^-	1	2752.03	2857.35
	2	2862.13	2911.54
	3	3188.32	3292.05
	4	3287.53	3327.83
	5	3571.28	3672.60
	6	3666.60	3699.38
	7	3916.99	4014.94
	8	4008.79	4037.37
2^+	1	2464.39	2569.18
	2	2959.26	3077.36
	3	3131.47	3203.58
	4	3379.56	3483.48
	5	3494.62	3539.61
	6	3750.46	3819.04
	7	3840.78	3881.70
	8	4086.20	4130.11

Continued on next page

Table 4-continued from previous page

J^P	n	D Meson	D_s Meson
3^-	1	2789.59	2869.84
	2	3233.29	3320.75
	3	3448.43	3499.65
	4	3615.56	3702.96
	5	3765.23	3825.09
	6	3959.01	4053.21
	7	4075.86	4114.68
	8	4274.05	4363.43
3^+	1	3031.60	3138.04
	2	3184.76	3216.67
	3	3427.33	3529.81
	4	3560.11	3594.36
	5	3781.48	3881.76
	6	3904.52	3935.16
	7	4105.19	4203.53
	8	4222.14	4248.79
4^-	1	3275.27	3385.16
	2	3466.85	3480.50
	3	3634.65	3738.40
	4	3804.95	3826.51
	5	3971.98	4073.56
	6	4125.85	4149.44
	7	4278.55	4376.37
	8	4424.86	4445.97
4^+	1	3050.12	3128.49
	2	3460.14	3536.36
	3	3727.59	3740.77
	4	3819.37	3900.19
	5	4021.00	4050.96
	6	4144.89	4230.53
	7	4303.97	4357.16
	8	4445.31	4534.66

In $1^+ \leftrightarrow 0^-$ transition, only 1P_1 component of 1^+ state fulfil selection rule of $E1$ transitions. In $1^- \leftrightarrow 0^+$ transition both components of 1^- state participate. In $1^- \leftrightarrow 1^+$, both components of the initial and final states take part except the 1P_1 component of 1^+ state. Finally, in $2^- \leftrightarrow 1^+$ transition both components of the initial and final states contribute.

VI. RESULTS AND CONCLUSIONS

In this work we present a rich spectrum of conven-

Table 5. Our calculated hybrid D and D_s meson masses (in MeV) for $J^P = 0^{-\varepsilon}, 0^{+\varepsilon}, 1^{-\varepsilon}, 1^{+\varepsilon}, 2^{-\varepsilon}, 2^{+\varepsilon}, 3^{-\varepsilon}, 3^{+\varepsilon}, 4^{-\varepsilon}$ and $4^{+\varepsilon}$.

J^P	n	Hybrid D Meson	Hybrid D_s Meson
$0^{+\varepsilon}$	1	2755.01	2894.34
	2	3256.86	3371.39
	3	3674.27	3773.52
	4	4038.16	4128.21
$0^{-\varepsilon}$	1	3040.45	3073.69
	2	3487.44	3471.19
	3	3876.36	3858.93
	4	4220.98	4204.41
$1^{+\varepsilon}$	1	2783.31	2917.32
	2	3291.98	3392.35
	3	3377.88	3467.12
	4	3705.86	3771.09
	5	3747.28	3818.40
	6	4066.32	4098.38
	7	4094.33	4159.38
	8	4391.27	4406.70
$1^{-\varepsilon}$	1	3063.57	3186.10
	2	3129.03	3214.35
	3	3501.03	3610.14
	4	3561.83	3626.80
	5	3877.47	3978.27
	6	3938.51	3991.02
	7	4214.71	4309.78
	8	4275.42	4320.49
$2^{+\varepsilon}$	1	3313.71	3429.11
	2	3422.03	3484.54
	3	3710.06	3817.38
	4	3803.52	3860.47
	5	4060.25	4162.55
	6	4148.00	4196.43
	7	4378.76	4476.95
	8	4462.95	4504.55
$2^{-\varepsilon}$	1	3088.68	3202.11
	2	3531.70	3642.38
	3	3666.11	3734.10
	4	3907.27	4012.51
	5	3991.32	4062.51
	6	4243.29	4329.07
	7	4304.42	4363.03
	8	4550.91	4612.00

Continued on next page

Table 5-continued from previous page

J^P	n	Hybrid D Meson	Hybrid D_s Meson
$3^{+\varepsilon}$	1	3328.26	3425.32
	2	3736.37	3827.75
	3	3924.76	3966.53
	4	4089.99	4182.79
	5	4225.45	4284.57
	6	4409.37	4503.65
	7	4512.24	4572.02
	8	4703.77	4797.78
$3^{-\varepsilon}$	1	3536.73	3650.73
	2	3683.47	3723.53
	3	3902.08	4008.45
	4	4030.85	4073.06
	5	4231.61	4331.70
	6	4349.85	4387.91
	7	4534.57	4619.30
	8	4645.99	4671.07
$4^{+\varepsilon}$	1	3742.56	3858.08
	2	3921.79	3943.51
	3	4082.66	4191.13
	4	4244.97	4271.20
	5	4394.41	4497.79
	6	4544.24	4572.21
	7	4683.88	4783.46
	8	4824.56	4852.37
$4^{-\varepsilon}$	1	3538.36	3633.01
	2	3917.60	4000.87
	3	4159.43	4179.46
	4	4254.05	4335.09
	5	4446.62	4483.56
	6	4560.83	4641.67
	7	4715.93	4763.97
	8	4845.22	4927.05

tional and hybrid charmed and charmed strange mesons. States having the same value of J^P are mixed. We have incorporated both spin and S - D mixing through the off-diagonal matrix elements of the Hamiltonian. In Fig. 2 to 5, state mixing phenomena are explicitly visualized by the spectrum pattern. Ref. [20] include spin mixing effects although they ignore S - D mixing in their calculations. In Ref. [22], spin dependent Hamiltonian part is added perturbatively after solving the eigenequation. In this work, we use the complete Hamiltonian (including spin dependent part) in the eigenequation. This allows us to

study both spin and S - D mixing effects without resorting to perturbation theory.

Our calculated spectrum of conventional D and D_s mesons is reported in Table 4 with increasing order of value of mass. n is number of eigenvalues associated with specific value of J^P . For unmixed states, it is visible that 0^+ (3P_0) states have larger mass as compared to 0^- (1S_0) states due to spin and orbital excitation. In the case of S - D mixed states, higher orbital excited states (e.g., 2^+ , $^3P_2 \leftrightarrow ^3F_2$) have large mass compared to lower one (e.g., 1^- , $^3S_1 \leftrightarrow ^3D_1$). Same is observed for spin singlet and spin triplet mixed states. States with higher orbital excitations (e.g., 2^- , $^1D_2 \leftrightarrow ^3D_2$) have large mass in comparison to lower orbital excited states (e.g., 1^+ , $^1P_1 \leftrightarrow ^3P_1$). We also compare our calculated spectrum of conventional D and D_s mesons with experimental results and other works in Tables 2 and 3. Our results have good agreement with other works as well as with experimental data. We used meson states in Table 2 for fitting the parameters value. Table 3 shows the efficiency of our results. In Table 3, we identify $D_{sJ}^*(3040)^\pm$ and $D(3000)^0$ as 1^+ and 3^- state. We tentatively identify $D_{sJ}^*(3040)^\pm$ and $D(3000)^0$ as 2^+ and 3^+ state. Possibility of $D(3000)^0$ could be 4^- state.

Since the nature of $D_{s0}^*(2317)^\pm$ and $D_{s1}(2460)^\pm$ states are not clear as discussed in Sec. I, we do not include these states in the fitting procedure. Moreover, we do not find any good match to these states in conventional as well as hybrid mass spectrum. This result is consistent with hitherto exotic nature of $D_{s0}^*(2317)$ and $D_{s1}(2460)$ mesons.

In Table 5, we report our computed masses of hybrid D and D_s mesons. For hybrid candidates, parity condition is different to that for conventional mesons; this is due to the excited gluonic field contribution as discussed in Sec. I. It is observed that hybrid states have greater masses comparative to that of conventional ones. This happen due to gluonic excitations in Π_u states. In Table 6 and Table 7 we have reported our calculated $E1$ transitions of conventional to conventional and hybrid to hybrid mesons respectively. We also present $M1$ transitions of conventional to conventional and hybrid to hybrid mesons in Table 8 and Table 9 respectively. We observed that the decay width of $D^0(c\bar{u})$ states is large as compared to $D^\pm(c\bar{d})$ states except few states in $M1$ transitions. In Table 10, we compare our calculated radiative decay width of conventional mesons with other works and ex-

Table 6. Partial widths for $E1$ decays (in KeV) of conventional D and D_s mesons.

Transition	initial state nJ^P	final state nJ^P	$\Gamma_{e1}(D \text{ meson}) (c\bar{u}, c\bar{d})$	$\Gamma_{e1}(D_s \text{ meson})$
$0^- \rightarrow 1^+ \gamma$	20^-	11^+	91.149, 3.457	0.4176
		21^+	75.435, 1.9573	0.2435
	30^-	11^+	210.5, 5.8748	0.3448
		21^+	226.65, 5.8811	0.2216
		31^+	432.72, 11.228	1.049
		41^+	109.09, 2.8305	0.697
$0^+ \rightarrow 1^- \gamma$	10^+	11^-	728.01, 19.4031	0.0859
	20^+	11^-	184.37, 4.8241	2.0001
		21^-	545.88, 14.164	0.2027
		31^-	35.948, 0.9328	0.0002
	30^+	11^-	47.770, 1.2456	0.3441
		21^-	290.88, 7.5475	1.6971
		31^-	407.65, 10.577	1.6964
		41^-	748.15, 19.413	0.5871
		51^-	158.37, 4.1092	0.162
$1^- \rightarrow 0^+ \gamma$	21^-	10^+	43.527, 1.1612	1.9786
	31^-	10^+	628.92, 16.779	3.8272
	41^-	10^+	776.18, 20.707	0.1186
		20^+	604.83, 16.136	2.8181
	51^-	10^+	49.96, 1.3328	0.2344
		20^+	604.83, 16.136	2.8181

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Table 6-continued from previous page

Transition	initial state nJ^P	final state nJ^P	$\Gamma_{e1}(D \text{ meson}) (c\bar{u}, c\bar{d})$	$\Gamma_{e1}(D_s \text{ meson})$
$1^- \rightarrow 1^+ \gamma$	21^-	11^+	82.384, 2.7702	0.4364
		21^+	73.119, 1.8973	0.291
	31^-	11^+	497.36, 14.921	1.6613
		21^+	463.44, 12.025	1.263
	41^-	11^+	1028.9, 28.573	0.9155
		21^+	1016.9, 26.386	0.5979
	51^-	31^+	383.55, 9.9521	0.6226
		41^+	133.93, 3.4751	0.4148
		11^+	234.27, 6.4414	0.5734
		21^+	244.65, 6.3481	0.3518
		31^+	1126.6, 29.234	1.1176
		41^+	585.33, 15.188	0.7807
$1^+ \rightarrow 0^- \gamma$	11^+	10^-	847.18, 20.415	3.4365
	21^+	10^-	886.97, 23.498	3.7954
	31^+	10^-	0.123, 0.0032	0.1012
	41^+	20^-	443.01, 11.495	2.1211
		10^-	0.7203, 0.0189	0.0009
		20^-	764.76, 19.844	2.7341
$1^+ \rightarrow 1^- \gamma$	11^+	11^-	638.97, 14.809	2.2145
	21^+	11^-	851.13, 22.510	2.526
	31^+	11^-	80.893, 2.1169	0.4596
	41^+	21^-	313.77, 8.1415	1.8567
		31^-	4.6457, 0.1205	0.3447
		11^-	106.76, 2.7914	0.5276
		21^-	622.18, 16.144	2.2821
		31^-	20.975, 0.5442	0.4659
$1^+ \rightarrow 2^- \gamma$	31^+	12^-	65.085, 1.6888	0.5529
		22^-	0.2119, 0.0055	0.1558
	41^+	12^-	238.29, 6.1831	0.8464
		22^-	26.278, 0.6818	0.2738
	51^+	12^-	1.3736, 0.0356	0.1362
		22^-	5.0118, 0.13	0.1717
		32^-	181.45, 4.7082	0.998
		42^-	5.0814, 0.1318	0.4126
	61^+	12^-	2.6022, 0.0675	0.2015
		22^-	0.6375, 0.0165	0.2359
		32^-	554.7, 14.393	1.3456
		42^-	99.485, 2.5814	0.5961
$2^- \rightarrow 1^+ \gamma$	12^-	11^+	1324.5, 40.073	5.3991
		21^+	1235.9, 32.0705	4.27
	22^-	11^+	2821.6, 81.961	8.0433

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Table 6-continued from previous page

Transition	initial state nJ^P	final state nJ^P	$\Gamma_{e1}(D \text{ meson}) (c\bar{u}, c\bar{d})$	$\Gamma_{e1}(D_s \text{ meson})$
		21^+	2692.4, 69.861	6.5345
	32^-	11^+	135.6, 3.736	0.0116
		21^+	154.87, 4.0186	0.0318
		31^+	1541.9, 40.008	4.8947
		41^+	746.69, 19.375	3.7323
	42^-	11^+	167.93, 4.589	0.0259
		21^+	194.43, 5.045	0.0352
		31^+	3316.8, 86.062	6.8996
		41^+	2019.9, 52.411	5.3872

Table 7. Partial widths for $E1$ decays (in KeV) of hybrid D and D_s mesons.

Transition	initial state nJ^P	final state nJ^P	$\Gamma_{e1}(\text{hybrid } D \text{ meson}) (c\bar{u}, c\bar{d})$	$\Gamma_{e1}(\text{hybrid } D_s \text{ meson})$
$0^{+\varepsilon} \rightarrow 1^{-\varepsilon}\gamma$	$20^{+\varepsilon}$	$11^{-\varepsilon}$	261.12, 6.7754	1.059
		$21^{-\varepsilon}$	81.811, 2.1228	0.6407
	$30^{+\varepsilon}$	$11^{-\varepsilon}$	57.046, 1.4802	0.2516
		$21^{-\varepsilon}$	60.926, 1.5809	0.2588
		$31^{-\varepsilon}$	385.99, 10.016	1.5485
		$41^{-\varepsilon}$	117.44, 3.0472	1.0873
$0^{-\varepsilon} \rightarrow 1^{+\varepsilon}\gamma$	$10^{-\varepsilon}$	$11^{+\varepsilon}$	618.71, 16.054	0.4149
	$20^{-\varepsilon}$	$11^{+\varepsilon}$	387.02, 10.0423	1.9049
		$21^{+\varepsilon}$	264.43, 6.8614	0.0996
		$31^{+\varepsilon}$	63.071, 1.6365	0.00002
	$30^{-\varepsilon}$	$11^{+\varepsilon}$	6.5882, 0.1709	0.2337
		$21^{+\varepsilon}$	251.11, 6.5156	1.8328
		$31^{+\varepsilon}$	153.29, 3.9775	2.48
		$41^{+\varepsilon}$	301.26, 7.817	0.3068
		$51^{+\varepsilon}$	158.79, 4.1204	0.0377
$1^{+\varepsilon} \rightarrow 0^{-\varepsilon}\gamma$	$21^{+\varepsilon}$	$10^{-\varepsilon}$	199.35, 5.3184	1.3168
	$31^{+\varepsilon}$	$10^{-\varepsilon}$	511.52, 13.646	2.395
	$41^{+\varepsilon}$	$10^{-\varepsilon}$	3.2826, 0.0876	0.0766
		$20^{-\varepsilon}$	231.81, 6.1844	2.0572
	$51^{+\varepsilon}$	$10^{-\varepsilon}$	16.243, 0.4333	0.2632
		$20^{-\varepsilon}$	417.04, 11.126	2.48
$1^{+\varepsilon} \rightarrow 1^{-\varepsilon}\gamma$	$21^{+\varepsilon}$	$11^{-\varepsilon}$	222.27, 5.7673	0.7543
		$21^{-\varepsilon}$	85.868, 2.2281	0.4825
	$31^{+\varepsilon}$	$11^{-\varepsilon}$	436.29, 11.3207	1.7273
		$21^{-\varepsilon}$	229.88, 5.9648	1.2476
	$41^{+\varepsilon}$	$11^{-\varepsilon}$	71.744, 1.8616	0.4743
		$21^{-\varepsilon}$	61.15, 1.5867	0.3914
		$31^{-\varepsilon}$	303.65, 7.8791	0.8211

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Table 7-continued from previous page

Transition	initial state nJ^P	final state nJ^P	Γ_{el} (hybrid D meson) ($c\bar{u}$, $c\bar{d}$)	Γ_{el} (hybrid D_s meson)
	$51^{+\varepsilon}$	$41^{-\varepsilon}$	115.48, 2.9963	0.5718
		$11^{-\varepsilon}$	132.82, 3.4463	0.1873
		$21^{-\varepsilon}$	113.01, 2.9322	0.156
		$31^{-\varepsilon}$	374.27, 9.7114	1.452
		$41^{-\varepsilon}$	174.23, 4.5208	1.0843
$1^{-\varepsilon} \rightarrow 0^{+\varepsilon}\gamma$	$11^{-\varepsilon}$	$10^{+\varepsilon}$	420.74, 10.917	1.4283
	$21^{-\varepsilon}$	$10^{+\varepsilon}$	714.93, 18.551	1.8474
	$31^{-\varepsilon}$	$10^{+\varepsilon}$	21.638, 0.5615	0.000008
	$41^{-\varepsilon}$	$20^{+\varepsilon}$	350.18, 9.0864	1.3169
		$10^{+\varepsilon}$	23.361, 0.6062	0.0033
		$20^{+\varepsilon}$	652.03, 16.918	1.6334
$1^{-\varepsilon} \rightarrow 1^{+\varepsilon}\gamma$	$11^{-\varepsilon}$	$11^{+\varepsilon}$	462.52, 12.001	1.4309
	$21^{-\varepsilon}$	$11^{+\varepsilon}$	827.38, 21.468	1.8929
	$31^{-\varepsilon}$	$11^{+\varepsilon}$	163.51, 4.2426	0.5119
	$41^{-\varepsilon}$	$21^{+\varepsilon}$	265.17, 6.8805	1.2207
		$31^{+\varepsilon}$	49.942, 1.2959	0.473
		$11^{+\varepsilon}$	196.41, 5.0965	0.577
		$21^{+\varepsilon}$	543.57, 14.104	1.5197
		$31^{+\varepsilon}$	158.73, 4.1187	0.6515
$1^{-\varepsilon} \rightarrow 2^{+\varepsilon}\gamma$	$31^{-\varepsilon}$	$12^{+\varepsilon}$	251.90, 6.5361	1.0387
		$22^{+\varepsilon}$	22.292, 0.5784	0.3843
	$41^{-\varepsilon}$	$12^{+\varepsilon}$	549.57, 14.26	1.3888
		$22^{+\varepsilon}$	116.31, 3.0179	0.5716
	$51^{-\varepsilon}$	$12^{+\varepsilon}$	12.435, 0.3227	0.0632
		$22^{+\varepsilon}$	30.026, 0.7791	0.1367
		$32^{+\varepsilon}$	377.98, 9.8075	1.5399
		$42^{+\varepsilon}$	38.496, 0.9989	0.6371
	$61^{-\varepsilon}$	$12^{+\varepsilon}$	1.709, 0.0443	0.0708
		$22^{+\varepsilon}$	18.596, 0.4825	0.1541
		$32^{+\varepsilon}$	860.24, 22.321	2.0341
		$42^{+\varepsilon}$	211.48, 5.4873	0.9093
$2^{+\varepsilon} \rightarrow 1^{-\varepsilon}\gamma$	$12^{+\varepsilon}$	$11^{-\varepsilon}$	872.62, 22.642	3.3811
		$21^{-\varepsilon}$	373.06, 9.6801	2.3478
	$22^{+\varepsilon}$	$11^{-\varepsilon}$	2390.7, 62.033	6.0388
		$21^{-\varepsilon}$	1386.2, 35.968	4.5079
	$32^{+\varepsilon}$	$11^{-\varepsilon}$	74.05, 1.9214	0.0924
		$21^{-\varepsilon}$	74.711, 1.9386	0.0787
		$31^{-\varepsilon}$	736, 19.097	3.0566
		$41^{-\varepsilon}$	284.58, 7.3841	2.3242
	$42^{+\varepsilon}$	$11^{-\varepsilon}$	57.592, 1.4944	0.0451
		$21^{-\varepsilon}$	66.767, 1.7324	0.0386

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Table 7-continued from previous page

Transition	initial state nJ^P	final state nJ^P	Γ_{e1} (hybrid D meson) ($c\bar{u}$, $c\bar{d}$)	Γ_{e1} (hybrid D_s meson)
		$31^{-\varepsilon}$	2063.6, 53.545	5.1521
		$41^{-\varepsilon}$	1143.8, 29.679	4.0959

Table 8. Partial widths for $M1$ decays (in KeV) of conventional D and D_s mesons.

Transition	initial state nJ^P	final state nJ^P	Γ_{m1} (D meson) ($c\bar{u}$, $c\bar{d}$)	Γ_{m1} (D_s meson)
$0^- \rightarrow 1^- \gamma$	20^-	11^-	11.097, 7.4747	2.184
		11^-	0.1487, 6.2222	2.3694
		21^-	22.198, 10.062	2.9894
		31^-	98.771, 1.2647	0.209
	40^-	11^-	0.1319, 1.566	0.5123
		21^-	25.08, 18.489	5.9304
		31^-	248.86, 10.857	0.1343
		41^-	16.733, 7.8301	3.7441
		51^-	68.052, 0.0079	2.0575
$1^- \rightarrow 0^- \gamma$	11^-	10^-	23.648, 0.6064	0.0665
		10^-	127.55, 17.5012	4.0203
		20^-	2.5479, 0.0644	0.0042
	31^-	10^-	2085, 19.985	7.9164
		20^-	4.8418, 0.19	0.0919
	41^-	10^-	112.9, 17.719	1.622
		20^-	82.281, 10.879	1.9332
		30^-	0.6405, 0.0163	0.00001
		10^-	1350, 0.9538	5.1259
	51^-	20^-	42.254, 1.3827	2.8697
		30^-	7.1297, 0.123	0.004

Table 9. Partial widths for $M1$ decays (in KeV) of hybrid D and D_s mesons.

Transition	initial state nJ^P	final state nJ^P	Γ_{m1} (hybrid D meson) ($c\bar{u}$, $c\bar{d}$)	Γ_{m1} (hybrid D_s meson)
$0^{+\varepsilon} \rightarrow 1^{+\varepsilon} \gamma$	$20^{+\varepsilon}$	$11^{+\varepsilon}$	22.206, 9.4979	2.1082
		$11^{+\varepsilon}$	8.4751, 10.6	2.2568
		$21^{+\varepsilon}$	14.759, 6.9874	2.1905
	$40^{+\varepsilon}$	$31^{+\varepsilon}$	2.6918, 1.303	0.5497
		$11^{+\varepsilon}$	9.428, 5.1214	0.7265
		$21^{+\varepsilon}$	21.874, 16.759	4.8863
		$31^{+\varepsilon}$	13.313, 3.7718	0.6848
		$41^{+\varepsilon}$	12.489, 5.5476	2.5828
		$51^{+\varepsilon}$	10.924, 2.235	1.2671
$1^{+\varepsilon} \rightarrow 0^{+\varepsilon} \gamma$	$11^{+\varepsilon}$	$10^{+\varepsilon}$	0.2161, 0.0056	0.0003
	$21^{+\varepsilon}$	$10^{+\varepsilon}$	53.308, 7.458	1.1989
		$20^{+\varepsilon}$	0.4085, 0.0105	0.0002

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Table 9-continued from previous page

Transition	initial state nJ^P	final state nJ^P	Γ_{m1} (hybrid D meson) ($c\bar{u}$, $c\bar{d}$)	Γ_{m1} (hybrid D_s meson)
	$31^{+\epsilon}$	$10^{+\epsilon}$	425.28, 17.191	2.2738
		$20^{+\epsilon}$	12.745, 0.2962	0.0171
	$41^{+\epsilon}$	$10^{+\epsilon}$	60.381, 11.716	0.9809
		$20^{+\epsilon}$	46.49, 6.0773	0.9054
	$51^{+\epsilon}$	$10^{+\epsilon}$	760.03, 29.874	2.861
		$20^{+\epsilon}$	234.39, 9.8806	1.5053
		$30^{+\epsilon}$	2.5875, 0.0632	0.0022

Table 10. Comparison of our calculated radiative decay partial widths (in KeV) with other works. D_1 and D'_1 are mixed states ($J^P = 1^+$) of charmed mesons.

Mode	Our Results	GI Model [20]	[57]	[58]	[39]	PDG [18]
Charmed strange mesons						
$D_s^{*\pm} \rightarrow D_s^\pm \gamma$	0.0665	1.03	2.39	1.12		< 1776.5
$D_{s1}(2536)^\pm \rightarrow D_s^\pm \gamma$	3.7954	9.23	61.2	37.7	3.53	
$D_{s1}(2536)^\pm \rightarrow D_s^{*\pm} \gamma$	2.526	9.61	9.21	5.74	4.74	
Charmed mesons ($c\bar{u}$)						
$D^*(2007)^0 \rightarrow D^0 \gamma$	23.648	106				< 741
$D_0^*(2300)^0 \rightarrow D^*(2007)^0 \gamma$	728.01	288				
$D_1(2430)^0 \rightarrow D^0 \gamma$	847.18	640				
$D_1(2420) \rightarrow D^0 \gamma$	886.97	156				
$D_1(2430)^0 \rightarrow D^*(2007)^0 \gamma$	638.97	82.8				
$D_1(2420) \rightarrow D^*(2007)^0 \gamma$	851.13	386				
Charmed mesons ($c\bar{d}$)						
$D^*(2010)^\pm \rightarrow D^\pm \gamma$	0.6064	10.8				≈ 1.334
$D_0^* \rightarrow D^*(2010)^\pm \gamma$	19.4031	30				
$D_1 \rightarrow D^\pm \gamma$	20.415	66				
$D'_1 \rightarrow D^\pm \gamma$	23.498	16.1				
$D_1 \rightarrow D^*(2010)^\pm \gamma$	14.809	8.6				
$D'_1 \rightarrow D^*(2010)^\pm \gamma$	22.51	39.9				

perimental results. Our results are consistent with other works except for a few states of D^0 mesons. We expect

our predicted results will be useful in detection of new charmed and charmed strange mesons.

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