

The role of triangle singularity in the $B^0 \rightarrow D^- \pi^+ a_0(980)(\pi^0 \eta)$ decay*

Wei Wang (王伟)¹ Dazhuang He (何大壮)² Xuan Luo (罗旋)^{3†}

¹School of Physics, Dalian University of Technology, Dalian 116024, China

²College of Physics and Electronic Engineering, Heze University, Heze 274015, China

³School of Physics and Optoelectronics Engineering, Anhui University, Hefei 230601, China

Abstract: The triangle singularity interpretation of the $a_1(1420)$ observed by the COMPASS Collaboration has been widely accepted. In this work, we investigate the triangle mechanism in the decay $B^0 \rightarrow D^- \pi^+ a_0(980)(\pi^0 \eta)$, where the $a_0(980)$ is treated as a dynamically generated state. The $\bar{K}^{*0}-K^+-K^-$ triangle loop originates from the decay $B^0 \rightarrow D^- K^+ \bar{K}^{*0}$, which has been observed by the Belle Collaboration, followed by the subsequent decay $\bar{K}^{*0} \rightarrow K^- \pi^+$. The triangle amplitude develops a pronounced peak around 1420 MeV, which is reflected in the invariant mass spectrum of the $\pi a_0(980)$ system. The differential decay width is calculated and exhibits a narrow peak around 980 MeV in the $\pi^0 \eta$ invariant mass distribution. Furthermore, the invariant mass distribution of the $\pi^+ \pi^0 \eta$ system shows a clear peak around 1420 MeV, which further confirms the triangle singularity explanation of the $a_1(1420)$. We expect that the proposed B^0 decay mode could provide a potential platform for further exploring the triangle-singularity nature of the $a_1(1420)$ and could be tested in future experiments such as LHCb, BESIII, and Belle II.

Keywords: hadron states, triangle singularity, hadron decay

DOI: 10.1088/1674-1137/ae8005

CSTR: 32044.14.ChinesePhysicsC.50103106

I. INTRODUCTION

Over the past few decades, significant progress has been made in studies of the hadron spectrum, driven by the observation of numerous exotic states [1]. Although many of these exotic states can be interpreted as multi-quark states or hadronic molecules, some observed structures may not correspond to poles of the S -matrix but may instead arise from kinematic singularities [2–4]. Among such kinematic effects, the triangle singularity (TS), first proposed by Landau in 1959 [5, 6], has attracted increasing interest. A triangle mechanism originates from the following sequential process: an initial particle A decays into two internal particles, labeled 1 and 2, which move back-to-back in the rest frame of A . Particle 2 subsequently decays into an internal particle 3 and an external particle B , with particle 3 moving in the same direction as particle 1. The two internal particles 1 and 3 then undergo rescattering and form an external particle C . According to the Coleman-Norton theorem [7], the emergence of a TS depends on whether the above processes can be interpreted as a classical scattering process and

whether all three internal particles can simultaneously go on shell and become collinear in the rest frame of the decaying particle [5]. In reality, the internal particles have finite widths, which smear the singular behavior and transform the TS into a finite peak that can be observed experimentally.

Phenomenological studies have successfully applied the TS mechanism to explain a variety of long-standing puzzles in hadron physics. The anomalously large isospin violation observed in the decay $J/\psi \rightarrow \gamma \eta(1405/1475) \rightarrow \gamma \pi^0 f_0(980) \rightarrow \gamma 3\pi$, reported by the BESIII Collaboration [8], was interpreted in terms of the TS in Ref. [9]. This mechanism has contributed to a better understanding of the nature of the two nearby states $\eta(1405)$ and $\eta(1475)$, as well as the mixing between the $a_0(980)$ and $f_0(980)$ resonances, which has been further investigated in a series of subsequent works [10–14]. Meanwhile, the triangle mechanism has also been employed to interpret the properties of heavy exotic hadronic states, such as the $Y(4260)$, $Z_c(3900)$, $Z_b(10610)$, and $Z_b(10650)$, in various processes [15–23]. A milestone in the development of the triangle mechanism was achieved in Ref.

Received 29 January 2026; Accepted 22 June 2026; Accepted manuscript online 23 June 2026

* Dazhuang He is supported by the Heze University doctoral fund project (010008002039037). Xuan Luo is supported by the National Natural Science Foundation of China (12205002)

† E-mail: xuanluo@ahu.edu.cn



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[24], where a detailed analysis of the singularities in a triangle loop integral was presented and a compact formula was derived for evaluating the TS on the physical boundary. This formalism was successfully applied to the decay $\Lambda_b \rightarrow J/\psi K^- p$ through a Λ^* -charmonium-proton triangle loop. Thereafter, the triangle mechanism involving an $f_0(980)$ or $a_0(980)$ final state was further explored in a variety of processes [25–31]. In Ref. [26], the authors identified two nonresonant peaks associated with the TS at approximately 2850 and 3000 MeV in the invariant mass distributions of the πD_{s0} and πD_{s1} systems, respectively. That work also demonstrated the relation between the structure of the triangle amplitude and the finite widths of the internal particles in the loop. A comprehensive review of threshold cusps and various TS structures in hadronic reactions can be found in Ref. [32], where their roles in phenomena related to exotic hadron candidates are systematically summarized.

The triangle mechanism has been extensively discussed in a wide range of hadronic decays involving charmonium [33–39], the Λ baryon family [40–46], and the D -meson sector [47–50]. In addition, decays of the B -meson family provide a promising platform for probing this mechanism, such as $B^- \rightarrow D^{*0} \pi^- \pi^+ \pi^- (\pi^0 \eta)$ [26], $B^- \rightarrow K^- \pi^0 X(3872)$ [51], $B^- \rightarrow K^- X(3872)$ with $X(3872) \rightarrow \pi^0 \pi^+ \pi^-$ [52], $B^- \rightarrow K^- \pi^- D_{s0/s1}^+$ [53], $B^+ \rightarrow J/\psi \phi K^+$ [54], $B^+ \rightarrow J/\psi \pi^+ \pi^0 K^0$ and $B^+ \rightarrow J/\psi \pi^+ \pi^- K^+$ [55], $B^+ \rightarrow D^- D_s^+ \pi^+$ and $B^0 \rightarrow \bar{D}^0 D_s^+ \pi^-$ [56], $B^0 \rightarrow J/\psi K^0 f_0(980) (a_0(980))$ [31], $\bar{B}^0 \rightarrow \chi_{c1} K^- \pi^+$ [57], $B \rightarrow (J/\psi \pi^+ \pi^-) K \pi$ [58], $\bar{B}_s^0 \rightarrow J/\psi \pi^0 f_0(980)$ [29], and $B_c \rightarrow B_s \pi \pi$ [59]. Beyond hadronic decays, the triangle mechanism has also been explored in a variety of other processes, including semileptonic τ -lepton decays [60–62], photon-proton collisions [63, 64], electron-positron annihilation [65, 66], and proton-proton collisions [67, 68]. Furthermore, an interesting study [69] discussed the structure of TS involving new-physics particles at high-energy colliders. In addition, numerous other investigations have been devoted to the triangle mechanism and its phenomenological implications [70–81].

In particular, the TS explanation for the observation of the $a_1(1420)$ reported by the COMPASS Collaboration [82, 83] has been widely accepted. In this mechanism, the $a_1(1260)$ first decays into $K^* \bar{K}$, with $K^* \rightarrow \pi K$, after which the $K \bar{K}$ pair fuses to form the $f_0(980)$, giving the observed $\pi f_0(980)$ decay mode. In addition, it has also been studied in various processes [28, 29, 37, 40, 60, 84]. Based on previous studies, in this work we explore the physical effects of the TS and identify its contribution to the $B^0 \rightarrow D^- \pi^+ a_0(980)$ decay. The observation of the weak decay $B^0 \rightarrow D^- K^+ \bar{K}^{*0}$ by the Belle Collaboration [85], together with the cascade decay $\bar{K}^{*0} \rightarrow K^- \pi^+$, allows the construction of a $\bar{K}^{*0}$ - K - K triangle loop. As we will show, the invariant mass of the $\bar{K}^{*0} K^+$ system in the B^0 decay allows the production of the $a_0(980)$ in the re-

gion where the TS condition is satisfied. This leads to a pronounced peak structure in the $\pi^+ a_0(980)$ invariant mass spectrum around 1420 MeV. In the above calculation, the triangle amplitude is evaluated following the analytical formulations developed in Refs. [24, 32]. Furthermore, within the chiral unitary approach, the $a_0(980)$ can be regarded as a dynamically generated state arising from meson-meson interactions [86–88]. An enhancement around 980 MeV is expected to be reproduced in the $\pi^0 \eta$ invariant mass distribution. Consequently, the $B^0 \rightarrow D^- \pi^+ a_0(980) (\pi^0 \eta)$ decay provides a potential platform to investigate both the TS structure associated with the $\bar{K}^{*0}$ - K - K triangle loop and the nature of the $a_0(980)$ resonance.

The paper is organized as follows. In Sec. II, we present the theoretical formalism and detail the kinematic conditions for the $\bar{K}^{*0}$ - K - K triangle loop considered here. In Sec. III, the weak decay mechanism for $B^0 \rightarrow D^- K^+ \bar{K}^{*0}$ is described, and the effective coupling strength is extracted from current experimental measurements. In Sec. IV, we formulate the $B^0 \rightarrow D^- \pi^+ a_0(980)$ decay by incorporating the triangle mechanism and provide the details of the derivation. The $a_0(980)$ is further treated as a dynamically generated state, and the sequential process $B^0 \rightarrow D^- \pi^+ a_0(980)$ with $a_0(980) \rightarrow \pi^0 \eta$ is analyzed. In Sec. V, we present the numerical results for these processes. The peaks around 1420 and 980 MeV are reproduced in the invariant mass distributions of the πa_0 and $\pi^0 \eta$ systems, respectively. Finally, a brief summary is provided in Sec. VI.

II. FORMALISM

In this section, we show that a peak structure around 1420 MeV, induced by the TS, can be generated in the $B^0 \rightarrow D^- \pi^+ a_0(980)$ decay. The Feynman diagram is shown in the left panel of Fig. 1. In this process, the B^0 first decays into $D^- \bar{K}^{*0} K^+$, after which the $\bar{K}^{*0}$ decays into K^- and π^+ . The K^+ and K^- move in the same direction, with the K^- moving faster than the K^+ , allowing the $K^+ K^-$ pair to fuse into the $a_0(980)$. The channel $B^0 \rightarrow D^- \bar{K}^0 K^{*+}$ has not yet been observed experimentally. Therefore, the contribution of the K^{*+} - $\bar{K}^0$ - K^0 triangle loop from the $B^0 \rightarrow D^- \bar{K}^0 K^{*+}$ decay is not considered in the present work. Furthermore, the $a_0(980)$ can be interpreted as a dynamically generated state with isospin $I = 1$, arising from the coupled channels $\pi^0 \eta$, $K^+ K^-$, and $K^0 \bar{K}^0$ within the chiral unitary approach [87, 88]. The decay of $a_0(980)$ is shown in the right panel of Fig. 1.

The TS occurs when the kinematic variables of the internal particles satisfy the following condition

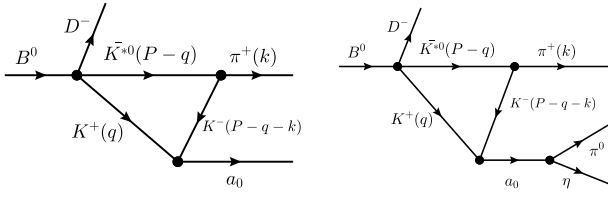


Fig. 1. The Feynman diagrams for the $B^0 \rightarrow D^- \pi^+ a_0(980)$ and $B^0 \rightarrow D^- \pi^+ a_0(980), a_0(980) \rightarrow \pi^0 \eta$ processes involving a $\bar{K}^*-K-\bar{K}$ triangle loop are shown in the left and right panels, respectively.

$$\lim_{\epsilon \rightarrow 0} (q_+^{\text{on}} - q_-^{\text{a}}) = 0, \quad (1)$$

$$q_+^{\text{on}} = \frac{\lambda^{\frac{1}{2}}(m_{\text{inv}}^2(\pi a_0), m_K^2, m_{\bar{K}^{*0}}^2)}{2m_{\text{inv}}(\pi a_0)} + i\epsilon,$$

where q_+^{on} denotes the on-shell three-momentum of K^+ in the center-of-mass frame (COM) of the $\bar{K}^{*0} K^+$ system. The quantity $m_{\text{inv}}(\pi a_0)$ is the invariant mass of the $\bar{K}^{*0} K^+$ system, and $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ is the Källén function. Meanwhile, q_-^{a} is given by

$$q_-^{\text{a}} = \gamma(vE_{K^-}^* - p_{K^-}^*) - i\epsilon, \quad (2)$$

with the definitions

$$v = \frac{k}{E_{a_0}}, \quad \gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{E_{a_0}}{m_{a_0}}, \quad (3)$$

$$E_{K^-}^* = \frac{m_{a_0}}{2}, \quad p_{K^-}^* = \frac{\lambda^{\frac{1}{2}}(m_{a_0}^2, m_K^2, m_{\bar{K}^{*0}}^2)}{2m_{a_0}},$$

where $E_{K^-}^*$ and $p_{K^-}^*$ are the energy and momentum of the K^- meson in the rest frame of a_0 . v and γ denote the velocity of the $K^+ K^-$ system and the Lorentz boost factor, respectively. Eq. (1) implies that all three particles in the loop are on shell and that K^+ in the rest frame of $a_0(980)$ and $a_0(980)$ in the COM frame of $\pi a_0(980)$ move in the same direction. The formation of $a_0(980)$ through $K^+ K^-$ fusion further requires that the momentum of K^+ in the COM frame of πa_0 be smaller than that of K^- in the rest frame of $\bar{K}^{*0}$. Equivalently, Eq. (1) can be understood as the condition that q_-^{a} and q_+^{on} represent the singularities of the triangle loop function in the upper and lower halves of the complex- q plane, respectively. The integration contour of the loop function in Eq. (21) is then pinched between q_-^{a} and q_+^{on} at the same point on the real axis. Consequently, solving Eq. (1) yields a TS around 1420 MeV in the $m_{\text{inv}}(\pi a_0)$ invariant mass distribution. When the finite width of the $\bar{K}^{*0}$ is taken into account by replacing $m_{\bar{K}^{*0}} \rightarrow m_{\bar{K}^{*0}} - i\Gamma_{\bar{K}^{*0}}/2$ with $\Gamma_{\bar{K}^{*0}} = 47$ MeV, the TS moves into the complex plane, yielding $1420.7 - i28.0$ MeV.

III. THE $B^0 \rightarrow D^- K^+ \bar{K}^{*0}$ DECAY

To evaluate the amplitude shown in Fig. 1, the effective coupling strength of the $B^0 \rightarrow D^- K^+ \bar{K}^{*0}$ vertex is required. The B^0 meson can decay into the $D^- K^+ \bar{K}^{*0}$ final state through the weak decay of the b quark, as illustrated in Fig. 2. At the quark level, the b quark undergoes a weak transition into a c quark through the emission of a W boson and hadronizes into the D^- meson. The emitted W boson subsequently decays into a $u\bar{d}$ pair, which combines with an additional $q\bar{q}$ pair from the vacuum to form the $\bar{K}^{*0} K^+$ final state.

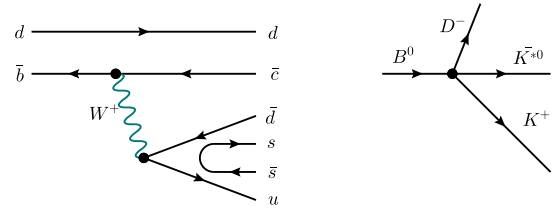


Fig. 2. (color online) The quark-level and hadron-level Feynman diagrams for the decay $B^0 \rightarrow D^- K^+ \bar{K}^{*0}$ are shown in the left and right panels, respectively.

The interaction of the effective vertex in Fig. 2 can be constructed using the P -wave interaction. Following the convention adopted in Refs. [28, 53], we write

$$-it_{B \rightarrow DK\bar{K}^{*0}} = -iC \vec{\epsilon}_{\bar{K}^{*0}} \cdot \vec{p}_D, \quad (4)$$

where $\vec{\epsilon}_{\bar{K}^{*0}}$ and $\vec{p}_D$ are the polarization vector of the $\bar{K}^{*0}$ and the momentum of the D^- , respectively. Below, we present the details of the calculation of the vertex coupling strength C . By solving Eq. (1), a TS around 1420 MeV is obtained. The three-momentum of the $\bar{K}^{*0}$ in the πa_0 rest frame is approximately 135.66 MeV, which is smaller than the $\bar{K}^{*0}$ mass of 895.81 MeV. Therefore, the time component ϵ^0 of the $\bar{K}^{*0}$ polarization vector can be safely neglected. We take the polarization sum for the $\bar{K}^{*0}$ as

$$\sum_{\mu, \nu} \epsilon_{\bar{K}^{*0} \mu} \epsilon_{\bar{K}^{*0} \nu} \sim \sum_{i, j} \epsilon_{\bar{K}^{*0} i} \epsilon_{\bar{K}^{*0} j} = \delta_{ij}; \quad \mu = i, \mu = j; \quad i, j = 1, 2, 3. \quad (5)$$

The decay width for the process $B \rightarrow DK\bar{K}^{*0}$ is given by

$$\Gamma_{B \rightarrow DK\bar{K}^{*0}} = \int dm_{\text{inv}}(\bar{K}^{*0} K^+) \times \frac{1}{(2\pi)^3} \frac{|\vec{p}_D| |\vec{p}_{\bar{K}^{*0}}|}{4m_B^2} \sum_{\text{pol}} |t_{B \rightarrow DK\bar{K}^{*0}}|^2, \quad (6)$$

where $m_{\text{inv}}(\bar{K}^{*0}K^+)$ is the invariant mass of the $\bar{K}^{*0}K^+$ system. Here, $\vec{p}_{\bar{K}^{*0}}$ and $\vec{p}_D$ denote the three-momenta of the $\bar{K}^{*0}$ in the $\bar{K}^{*0}K^+$ COM frame and of the D^- in the B^0 rest frame, respectively. They are given by

$$|\vec{p}_D| = \frac{\lambda^{\frac{1}{2}}(m_B^2, m_{\text{inv}}^2(\bar{K}^{*0}K^+), m_D^2)}{2m_B},$$

$$|\vec{p}_{\bar{K}^{*0}}| = \frac{\lambda^{\frac{1}{2}}(m_{\text{inv}}^2(\bar{K}^{*0}K^+), m_K^2, m_{\bar{K}^{*0}}^2)}{2m_{\text{inv}}(\bar{K}^{*0}K^+)}. \quad (7)$$

After squaring the amplitude $t_{B \rightarrow DK\bar{K}^{*0}}$ and applying the polarization sum in Eq. (5), we obtain

$$\sum_{\text{pol}} |t_{B \rightarrow DK\bar{K}^{*0}}|^2 = C^2 |\vec{p}_D|^2, \quad (8)$$

where $\vec{p}_D$ is the three-momentum of the D^- in the COM frame of the $\bar{K}^{*0}K^+$ system.

$$|\vec{p}_D| = \frac{\lambda^{\frac{1}{2}}(m_B^2, m_{\text{inv}}^2(\bar{K}^{*0}K^+), m_D^2)}{2m_{\text{inv}}(\bar{K}^{*0}K^+)}. \quad (9)$$

From recent measurements [1, 85], the partial decay branching fraction is given by $\text{Br}(B^0 \rightarrow D^- K^+ \bar{K}^{*0}) = (7.7 \pm 0.6) \times 10^{-4}$. Combining Eq. (6) with Eq. (8), we obtain

$$\frac{C^2}{\Gamma_{B^0}} = \frac{\text{Br}(B^0 \rightarrow D^- K^+ \bar{K}^{*0})}{\int dm_{\text{inv}}(\bar{K}^{*0}K^+) \frac{1}{(2\pi)^3} \frac{|\vec{p}_D| |\vec{p}_{\bar{K}^{*0}}|}{4m_B^2} |\vec{p}_D|^2}. \quad (10)$$

IV. THE $B^0 \rightarrow D^- \pi^+ a_0(980)$ DECAY

The amplitude for the $B^0 \rightarrow D^- \pi^+ a_0(980)$ decay (left panel of Fig. 1) can be written as

$$-it_{B \rightarrow D\pi a_0} = i \sum_{\text{pol}} \int \frac{d^4 q}{(2\pi)^4} \frac{it_{B \rightarrow DK\bar{K}^{*0}}}{q^2 - m_K^2 + i\epsilon} \times \frac{it_{\bar{K}^{*0}K^- \pi^+}}{(P-q)^2 - m_{\bar{K}^{*0}}^2 + i\epsilon} \frac{it_{K^+ K^- a_0}}{(P-q-k)^2 - m_K^2 + i\epsilon}, \quad (11)$$

in the COM frame of the πa_0 system, where the π^+ originates from the $\bar{K}^{*0}$ decay. In Eq. (11), the amplitude $t_{B \rightarrow DK\bar{K}^{*0}}$ was calculated in the previous section (Sec. III). The amplitude $t_{\bar{K}^{*0}K^- \pi^+}$ can be calculated using the chiral-invariant Lagrangian with local hidden symmetry [89, 90] and is given by

$$\mathcal{L}_{\text{VPP}} = -ig \langle V^\mu [P, \partial_\mu P] \rangle, \quad (12)$$

where bracket $\langle \cdots \rangle$ denotes the $SU(3)$ trace, and the coupling constant is given by $g = m_V/2f_\pi$, with $m_V = 800$ MeV and $f_\pi = 93$ MeV, within the local hidden gauge formalism. P and V^μ represent the pseudoscalar and vector meson octets, respectively, which are given by

$$P = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix},$$

$$V = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix}. \quad (13)$$

Then, the amplitude for the decay $\bar{K}^{*0} \rightarrow K^- \pi^+$ can be written as

$$-it_{\bar{K}^{*0}K^- \pi^+} = -ig \vec{\epsilon}_{\bar{K}^{*0}} \cdot (\vec{p}'_{\pi^+} - \vec{p}'_{K^-}). \quad (14)$$

As in Eq. (8), we neglect the time component of the polarization vector in Eq. (14) and employ the polarization sum given in Eq. (5). Here, $\vec{p}'_{\pi^+}$ and $\vec{p}'_{K^-}$ denote the three-momenta of the π^+ and K^- in the COM frame of the πa_0 system, respectively. The momentum $\vec{p}'_{\pi^+}$ is given by

$$|\vec{p}'_{\pi^+}| = |\vec{k}| = \frac{\lambda^{\frac{1}{2}}(m_{\text{inv}}^2(\pi a_0), m_\pi^2, m_{a_0}^2)}{2m_{\text{inv}}(\pi a_0)}. \quad (15)$$

As discussed in Sec. II, the $a_0(980)$ is considered to be a dynamically generated state. The amplitude $t_{K^+ K^- a_0}$ can be written directly as

$$t_{K^+ K^- a_0} = g_{K^+ K^- a_0}. \quad (16)$$

Above, we set $g_{K^+ K^- a_0} = 3875$ MeV [87]. Thus, the amplitude expression in Eq. (11) can be simplified to

$$t_{B \rightarrow D\pi a_0(980)} = ig_{K^+ K^- a_0} g_C \sum_{\text{pol}} \int \frac{d^4 q}{(2\pi)^4} \times \frac{\vec{\epsilon}_{\bar{K}^{*0}} \cdot \vec{p}'_D}{q^2 - m_K^2 + i\epsilon} \frac{\vec{\epsilon}_{\bar{K}^{*0}} \cdot (\vec{p}'_{\pi^+} - \vec{p}'_{K^-})}{(P-q)^2 - m_{\bar{K}^{*0}}^2 + i\epsilon} \times \frac{1}{(P-q-k)^2 - m_K^2 + i\epsilon}, \quad (17)$$

where $\vec{p}'_{\pi^+}$ is given in Eq. (15). The quantity $\vec{p}'_D$ is the momentum of the D^- in the COM frame of πa_0 , which originates from the B^0 decay.

$$|\vec{p}'_D| = \frac{\lambda^{\frac{1}{2}}(m_B^2, m_{\text{inv}}^2(\pi a_0), m_D^2)}{2m_{\text{inv}}(\pi a_0)}. \quad (18)$$

Equation (5) is used to perform the polarization sum in Eq. (17), yielding

$$\begin{aligned} t_{B \rightarrow D\pi a_0(980)} &= i g_{K^+ K^- a_0} g_C \int \frac{d^4 q}{(2\pi)^4} \\ &\times \frac{1}{q^2 - m_K^2 + i\epsilon} \frac{\vec{p}'_D \cdot (2\vec{k} + \vec{q})}{(P - q)^2 - m_{\bar{K}^{*0}}^2 + i\epsilon} \\ &\times \frac{1}{(P - q - k)^2 - m_K^2 + i\epsilon}, \end{aligned} \quad (19)$$

where $\vec{p}'_{\pi^+} - \vec{p}'_{K^-} = \vec{k} - (-\vec{k} - \vec{q}) = 2\vec{k} + \vec{q}$. P is the momentum of the πa_0 system in its rest frame.

$$P = (m_{\text{inv}}(\pi a_0), 0, 0, 0). \quad (20)$$

The quantity t_T is defined to describe the loop integral in Eq. (19) as

$$\begin{aligned} t_T &= i \int \frac{d^4 q}{(2\pi)^4} \vec{p}'_D \cdot (2\vec{k} + \vec{q}) \frac{1}{q^2 - m_K^2 + i\epsilon} \\ &\times \frac{1}{(P - q)^2 - m_{\bar{K}^{*0}}^2 + i\epsilon} \frac{1}{(P - q - k)^2 - m_K^2 + i\epsilon}. \end{aligned} \quad (21)$$

Following Refs. [24, 32], the negative-energy part of the $\bar{K}^{*0}$ propagator in t_T can be neglected. The integration over dq^0 in t_T is performed by applying the residue theorem and using the following formula

$$\int d^3 q \vec{q}_i f(\vec{q}, \vec{k}) = \vec{k}_i \int d^3 q \frac{\vec{q} \cdot \vec{k}}{|\vec{k}|^2} f(\vec{q}, \vec{k}). \quad (22)$$

Now, the triangle loop amplitude t_T reduces to

$$\begin{aligned} t_T &= \vec{p}'_D \cdot \vec{k} \int \frac{d^3 q}{(2\pi)^3} \left(2 + \frac{\vec{q} \cdot \vec{k}}{|\vec{k}|^2} \right) \\ &\times \frac{1}{8\omega_{K^+}(\vec{q}) \omega_{K^-}(\vec{q} + \vec{k}) \omega_{K^*}(\vec{q})} \\ &\times \frac{1}{k^0 - \omega_{K^-}(\vec{q} + \vec{k}) - \omega_{K^*}(\vec{q}) + i\epsilon} \\ &\times \frac{1}{P^0 - \omega_{K^*}(\vec{q}) - \omega_{K^+}(\vec{q}) + i\epsilon} \\ &\times \frac{1}{P^0 - \omega_{K^+}(\vec{q}) - \omega_{K^-}(\vec{q} + \vec{k}) - k^0 + i\epsilon} \\ &\times \frac{1}{P^0 + \omega_{K^+}(\vec{q}) + \omega_{K^-}(\vec{q} + \vec{k}) - k^0 + i\epsilon} \end{aligned}$$

$$\begin{aligned} &\times \left\{ 2P^0 \omega_{K^+}(\vec{q}) + 2k^0 \omega_{K^-}(\vec{q} + \vec{k}) \right. \\ &- 2 [\omega_{K^+}(\vec{q}) + \omega_{K^-}(\vec{q} + \vec{k})] \\ &\times [\omega_{K^+}(\vec{q}) + \omega_{K^-}(\vec{q} + \vec{k}) + \omega_{K^*}(\vec{q})] \left. \right\}, \\ &= \vec{p}'_D \cdot \vec{k} \times \tilde{t}_T, \end{aligned} \quad (23)$$

where $\omega_{K^+}(\vec{q}) = \sqrt{m_K^2 + \vec{q}^2}$, $\omega_{K^{*0}}(\vec{q}) = \sqrt{m_{\bar{K}^{*0}}^2 + \vec{q}^2}$, and $\omega_{K^-}(\vec{q} + \vec{k}) = \sqrt{m_K^2 + (\vec{k} + \vec{q})^2}$. From Eq. (20), $P^0 = m_{\text{inv}}(\pi a_0)$ is the invariant mass of the πa_0 system. Although the integral in Eq. (23) is convergent, the chiral unitary approach naturally provides an upper limit q_{max} [10, 13]. In this work, we use $q_{\text{max}} = 600$ MeV for $a_0(980)$ production [86, 91]. To reproduce the peak observed in experiments, the finite width of the $\bar{K}^{*0}$ in the triangle loop must be taken into account. This is implemented by replacing ω_{K^*} with $\omega_{K^*} - i\Gamma_{\bar{K}^{*0}}/2$ in the denominator of Eq. (23).

Finally, the differential decay width for the three-body final state $B^0 \rightarrow D^- \pi^+ a_0(980)$ is given by

$$\begin{aligned} \frac{d\Gamma'}{dm_{\text{inv}}(\pi a_0)} &= \frac{2}{3} \frac{1}{(2\pi)^3} \frac{C^2 g_{K^+ K^- a_0}^2}{8m_B^2} \\ &\times |\vec{p}_D| |\vec{p}'_D|^2 |\vec{k}|^3 |\tilde{t}_T|^2. \end{aligned} \quad (24)$$

The factor $2/3$ in Eq. (24) originates from the angular phase-space integration of the integrand in $\vec{p}'_D \cdot \vec{k} = |\vec{p}'_D| |\vec{k}| \cos \theta$. Here, $\vec{p}_D$ denotes the momentum of the D^- in the B^0 rest frame.

$$|\vec{p}_D| = \frac{\lambda^{\frac{1}{2}}(m_B^2, M_{D^-}^2, m_{\text{inv}}^2(\pi a_0))}{2m_B}. \quad (25)$$

In addition, we further explore the possibility of observing the $a_0(980)$ through the $\pi^0 \eta$ final state. Following Ref. [53], the $a_0(980)$ appears as an unstable resonance in the B^0 decay and subsequently decays into $\pi^0 \eta$. Therefore, we consider the corresponding four-body decay, as illustrated in the right panel of Fig. 1. In this process, the $a_0(980)$ resonance can be treated as an intermediate particle, allowing us to include an additional differential distribution with respect to the $\pi \eta$ invariant mass in Eq. 24. The differential decay width for $B^0 \rightarrow D^- \pi^+ \pi^0 \eta$ is then given by

$$\begin{aligned} \frac{d\Gamma'}{dm_{\text{inv}}(\pi a_0)} &= \frac{1}{3} \frac{1}{(2\pi)^3} \frac{g^2 C^2}{4m_B^2} \int dm_{\text{inv}}^2(\pi \eta) \\ &\times \left(-\frac{1}{\pi} \text{Im} D \right) g_{K^- K^+ a_0}^2 |\vec{p}_D| |\vec{p}'_D|^2 |\vec{k}|^3 |\tilde{t}_T|^2, \end{aligned} \quad (26)$$

with

$$D = \frac{1}{m_{\text{inv}}^2(\pi\eta) - m_{a_0}^2 + i m_{a_0} \Gamma_{a_0}}, \quad (27)$$

where $m_{\text{inv}}(\pi\eta)$ is the invariant mass of the $\pi\eta$ system produced in the $a_0(980)$ decay, and m_{a_0} is the mass of $a_0(980)$. Eq. (26) reduces to Eq. (24) in the limit $\Gamma_{a_0} \rightarrow 0$, where $i \text{Im} D \rightarrow -i\pi \delta(m_{\text{inv}}^2(\pi\eta) - m_{a_0}^2)$. Thus, Eq. (26) simplifies to

$$\begin{aligned} \frac{d\Gamma'}{dm_{\text{inv}}(\pi a_0)} &= \frac{1}{3\pi} \frac{1}{(2\pi)^3} \frac{g^2 C^2}{4m_B^2} \int dm_{\text{inv}}^2(\pi\eta) \\ &\times \frac{g_{K^+K^-a_0}^2 |\vec{p}_D| |\vec{p}_D'|^2 |\vec{k}|^3 |\vec{t}_T|^2 m_{a_0} \Gamma_{a_0}}{[m_{\text{inv}}^2(\pi\eta) - m_{a_0}^2]^2 + (m_{a_0} \Gamma_{a_0})^2}. \end{aligned} \quad (28)$$

The decay width Γ_{a_0} is calculated by considering the $a_0 \rightarrow \pi^0 \eta$ decay

$$\Gamma_{a_0} = \frac{1}{8\pi} \frac{g_{a_0\pi\eta}^2}{m_{\text{inv}}^2(\pi\eta)} |\vec{q}_\eta|^2, \quad (29)$$

where $|\vec{q}_\eta|$ denotes the magnitude of the η momentum in the rest frame of $a_0(980)$ and is given by

$$|\vec{q}_\eta| = \frac{\lambda^{\frac{1}{2}}(m_{\text{inv}}^2(\pi\eta), m_\pi^2, m_\eta^2)}{2m_{\text{inv}}(\pi\eta)}. \quad (30)$$

By substituting Eq. (29) into Eq. (28), we obtain

$$\begin{aligned} \frac{d\Gamma'}{dm_{\text{inv}}(\pi a_0)} &= \frac{1}{(2\pi)^5} \frac{g^2 C^2}{24m_B^2} \int dm_{\text{inv}}^2(\pi\eta) \\ &\times |\vec{p}_D| |\vec{p}_D'|^2 |\vec{k}|^3 |\vec{t}_T|^2 \frac{|\vec{q}_\eta|^2}{m_{\text{inv}}^2(\pi\eta)} \\ &\times \frac{m_{a_0} g_{K^+K^-a_0}^2 g_{a_0\pi\eta}^2}{[m_{\text{inv}}^2(\pi\eta) - m_{a_0}^2]^2 + (m_{a_0} \Gamma_{a_0})^2}. \end{aligned} \quad (31)$$

Meanwhile, the amplitude for the $K^+K^- \rightarrow \pi\eta$ process mediated by exchange of the $a_0(980)$ resonance can be written as

$$\frac{g_{K^+K^-a_0}^2 g_{a_0\pi\eta}^2}{[m_{\text{inv}}^2(\pi\eta) - m_{a_0}^2]^2 + (m_{a_0} \Gamma_{a_0})^2} = |t_{K^+K^- \rightarrow \pi\eta}|^2. \quad (32)$$

In contrast, when the $a_0(980)$ is treated as a dynamically generated state, the amplitude $t_{K^+K^- \rightarrow \pi\eta}$ can be calculated by solving the Bethe–Salpeter (BS) equation within the chiral unitary approach. The BS equation is given by

$$T = [1 - VG]^{-1} V. \quad (33)$$

Here, G denotes the loop function for two meson propagators [86].

$$G(s) = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(P-q)^2 - m_1^2 + i\epsilon} \frac{1}{q^2 - m_2^2 + i\epsilon}, \quad (34)$$

and is regularized by a cutoff q_{max} as

$$\begin{aligned} G(s) &= \frac{1}{16\pi^2 s} \left\{ \sigma \left(\arctan \frac{s+\Delta}{\sigma\lambda_1} + \arctan \frac{s-\Delta}{\sigma\lambda_2} \right) \right. \\ &\quad \left. - \left[(s+\Delta) \ln \frac{(1+\lambda_1)q_{\text{max}}}{m_1} + (s-\Delta) \ln \frac{(1+\lambda_2)q_{\text{max}}}{m_2} \right] \right\}, \end{aligned}$$

with

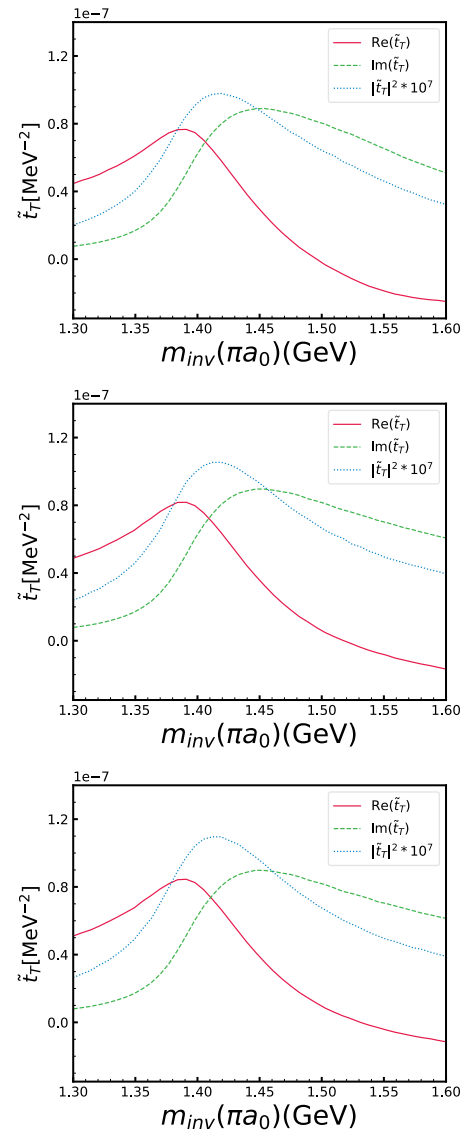


Fig. 3. (color online) The $\bar{K}^* K^+ K^-$ triangle amplitudes $\tilde{t}_T$ for $q_{\text{max}} = 600, 800, 1000$ MeV are shown from top to bottom, respectively. We set $m_{a_0} = 980$ MeV, and t_T^2 is multiplied by 10^7 .

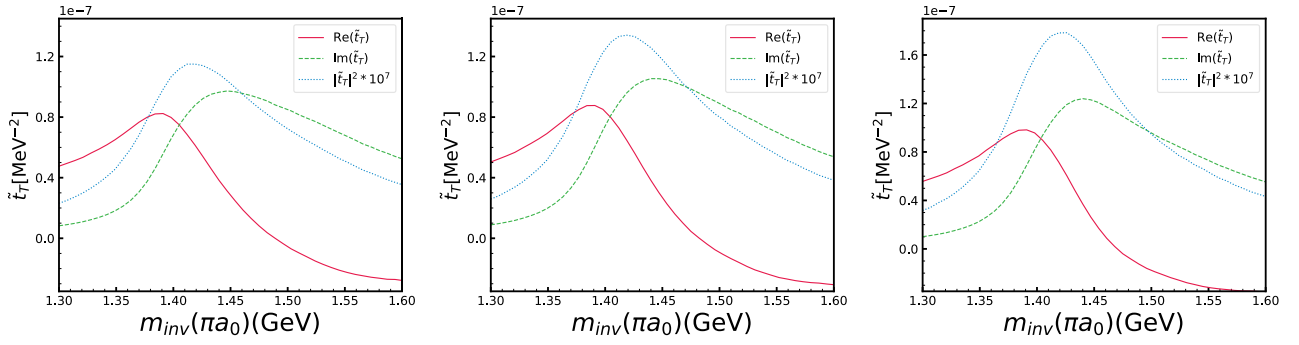


Fig. 4. (color online) As in Fig. 3, the triangle amplitudes for $m_{a_0} = 983, 985$, and 987 MeV with $q_{\max} = 600$ MeV are shown.

$$\sigma = [-s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)]^{1/2},$$

$$\Delta = m_1^2 - m_2^2, \quad \lambda_1 = \sqrt{1 + \frac{m_1^2}{q_{\max}^2}}, \quad \lambda_2 = \sqrt{1 + \frac{m_2^2}{q_{\max}^2}}.$$

A value of $q_{\max} = 600$ MeV is used to reproduce the $a_0(980)$ around 980 MeV. The meson-meson interaction potentials for the case $I = 1$ are given as follows [86]:

$$\begin{aligned} V_{K^+ K^- \rightarrow K^+ K^-} &= -\frac{1}{2f_\pi^2} s, & V_{K^+ K^- \rightarrow K^0 \bar{K}^0} &= -\frac{1}{4f_\pi^2} s, \\ V_{K^+ K^- \rightarrow \pi^0 \eta} &= \frac{-\sqrt{3}}{12f_\pi^2} \left(3s - \frac{8}{3} m_K^2 - \frac{1}{3} m_\pi^2 - m_\eta^2 \right), \\ V_{K^0 \bar{K}^0 \rightarrow K^0 \bar{K}^0} &= -\frac{1}{2f_\pi^2} s, & V_{\pi^0 \eta \rightarrow \pi^0 \eta} &= -\frac{m_\pi^2}{3f_\pi^2}, \\ V_{K^0 \bar{K}^0 \rightarrow \pi^0 \eta} &= -V_{K^+ K^- \rightarrow \pi^0 \eta}. \end{aligned} \quad (35)$$

Finally, the double-differential distribution of the B^0 decay can be written as

$$\begin{aligned} \frac{1}{\Gamma_{B^0}} \frac{d^2 \Gamma'}{dm_{\text{inv}}(\pi a_0) dm_{\text{inv}}(\pi \eta)} &= \frac{C^2}{\Gamma_{B^0}} \frac{g^2}{(2\pi)^5} \\ &\times \frac{|\vec{p}_D| |\vec{p}_D'|^2 |\vec{k}|^3 |\vec{q}_\eta|}{12m_B^2} |\tilde{t}_T \times t_{K^+ K^- \rightarrow \pi \eta}|^2. \end{aligned} \quad (36)$$

V. RESULTS

In Fig. 3, we first present the triangle amplitudes $\tilde{t}_T$, $\text{Im}(\tilde{t}_T)$, $\text{Re}(\tilde{t}_T)$, and $|\tilde{t}_T|^2 \times 10^7$ as functions of the invariant mass $m_{\text{inv}}(\pi a_0)$, with the $a_0(980)$ mass fixed at $m_{a_0} = 980$ MeV. The results for $|\tilde{t}_T|^2$ exhibit a clear peak around 1420 MeV, consistent with the condition in Eq. (1). The peaks of $\text{Im}(\tilde{t}_T)$ and $\text{Re}(\tilde{t}_T)$ appear at 1440 MeV and 1390 MeV, respectively, arising from the TS and the $\bar{K}^{*0} K^+$ threshold. From top to bottom, the results correspond to the cutoff parameters $q_{\max} = 600, 800$ and 1000 MeV, respectively. As $q_{\max}$ increases, the peak positions in $\text{Im}(\tilde{t}_T)$, $\text{Re}(\tilde{t}_T)$, and $|\tilde{t}_T|^2$ remain unchanged, while

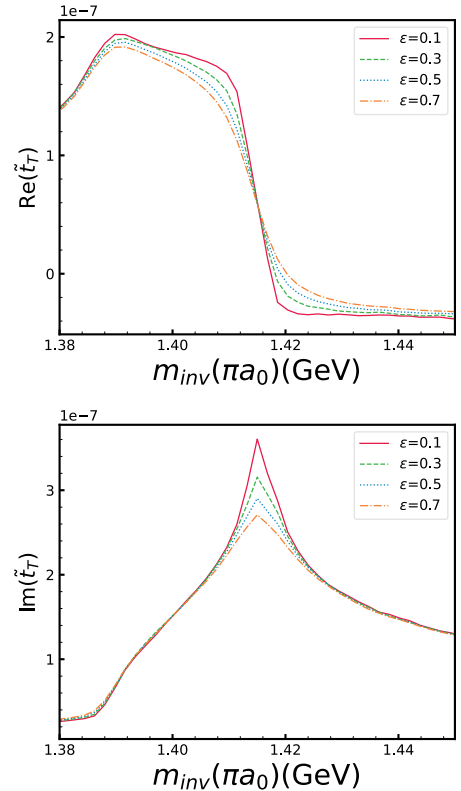


Fig. 5. (color online) The dependence of the triangle amplitude $\text{Re}(\tilde{t}_T)$ and $\text{Im}(\tilde{t}_T)$ on the widths of the internal particles is shown in the upper and lower panels, respectively. We set $m_{a_0} = 990$ MeV, which is slightly larger than the $K^- K^+$ threshold, and use $\Gamma_{\bar{K}^{*0}}/2 = \epsilon = 0.1, 0.3, 0.5$, and 0.7 MeV.

their magnitudes generally increase. The triangle amplitudes for different masses of $a_0(980)$ are shown in Fig. 4. The results indicate that the peaks remain near 1420 MeV, while the strength of the triangle amplitude increases as m_{a_0} increases. Next, we examine the behavior of the triangle amplitudes $\text{Im}(\tilde{t}_T)$ and $\text{Re}(\tilde{t}_T)$ as functions of the internal particle width.

Following Ref. [53], Fig. 5 illustrates the development of the TS by fixing $\Gamma_{K^+}/2 = \epsilon$ at different finite values close to zero. The formation of the TS in the $\bar{K}^{*0} K^- K$

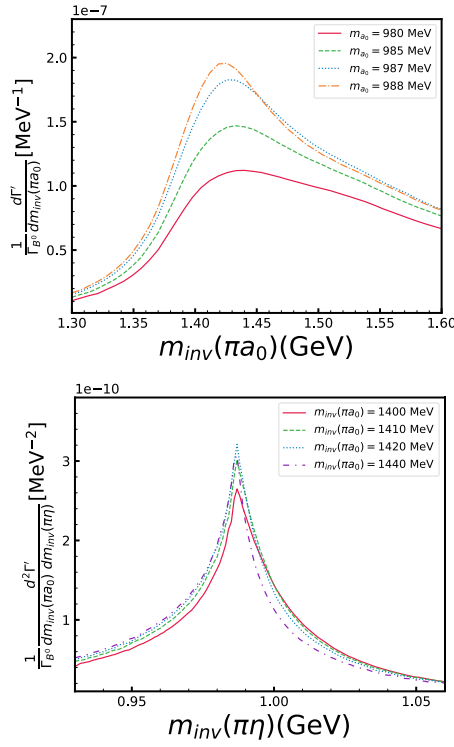


Fig. 6. (color online) The upper and lower panels show, respectively, the differential distribution $\frac{d\Gamma'}{dm_{\text{inv}}(\pi a_0)}$ (Eq. 24) as a function of the invariant mass $m_{\text{inv}}(\pi a_0)$ and the double differential distribution $\frac{1}{\Gamma_{B^0}} \frac{d^2\Gamma'}{dm_{\text{inv}}(\pi a_0)dm_{\text{inv}}(\pi\eta)}$ (Eq. 36) as a function of the invariant masses $m_{\text{inv}}(\pi a_0)$ and $m_{\text{inv}}(\pi\eta)$.

triangle loop requires the $a_0(980)$ mass to be slightly above the K^-K^+ threshold; therefore, we take $m_{a_0} = 990$ MeV. The distinct origins of the peaks in $\text{Re}(\tilde{t}_T)$ and $\text{Im}(\tilde{t}_T)$ indicate that these two components exhibit different behaviors, as shown in the upper and lower panels of Fig. 5. Specifically, $\text{Re}(\tilde{t}_T)$ exhibits a cusp at the $\bar{K}^{*0}K$ threshold around 1390 MeV, followed by a sharp decrease near the TS at 1420 MeV. In contrast, $\text{Im}(\tilde{t}_T)$ develops a narrow peak at 1420 MeV, directly reflecting the emergence of the triangle singularity. Fig. 5 also shows that, as ϵ decreases, the decrease in $\text{Re}(\tilde{t}_T)$ near 1420 MeV becomes steeper, the cusp near 1390 MeV becomes more pronounced, and the peak in $\text{Im}(\tilde{t}_T)$ becomes increasingly sharp. In the limit $\Gamma_{K^*}/2 = \epsilon = 0$, the peak of $\text{Im}(\tilde{t}_T)$ develops into a genuine singularity.

In Fig. 6, the differential distributions $\frac{d\Gamma'}{dm_{\text{inv}}(\pi a_0)}$, Eq. (24), and $\frac{1}{\Gamma_{B^0}} \frac{d^2\Gamma'}{dm_{\text{inv}}(\pi a_0)dm_{\text{inv}}(\pi\eta)}$, Eq. (36), are shown as functions of the invariant masses $m_{\text{inv}}(\pi a_0)$ and $m_{\text{inv}}(\pi\eta)$ in the upper and lower panels, respectively. In the upper panel of Fig. 6, the differential mass distribution for the $B^0 \rightarrow D^- \pi^+ a_0(980)$ decay with respect to $m_{\text{inv}}(\pi a_0)$ is

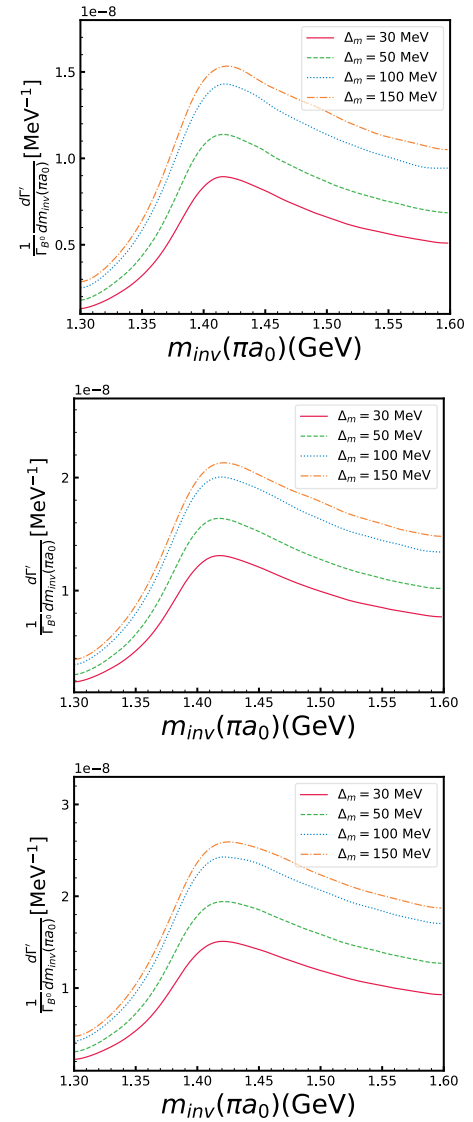


Fig. 7. (color online) The differential decay width $\frac{1}{\Gamma_{B^0}} \frac{d\Gamma'}{dm_{\text{inv}}(\pi a_0)}$ is shown as a function of the invariant mass $m_{\text{inv}}(\pi a_0)$, integrated over $m_{\text{inv}}(\pi\eta) \in [980 \text{ MeV} - \Delta_m, 980 \text{ MeV} + \Delta_m]$. From top to bottom, the curves correspond to $q_{\text{max}} = 600, 800$ and 1000 MeV, respectively.

shown for fixed values $m_{\text{inv}}(\pi\eta) = 980, 985, 987$ and 988 MeV with $q_{\text{max}} = 600$ MeV. It is evident that increasing m_{a_0} enhances the decay width of B^0 , while the peak position remains at the TS, $m_{\text{inv}}(\pi a_0) \approx 1420$ MeV. In the lower panel of Fig. 6, the dependence of $\frac{1}{\Gamma_{B^0}} \frac{d^2\Gamma'}{dm_{\text{inv}}(\pi a_0)dm_{\text{inv}}(\pi\eta)}$ on $m_{\text{inv}}(\pi\eta)$ is shown for fixed values $m_{\text{inv}}(\pi a_0) = 1400, 1410, 1420$ and 1440 MeV around the TS. A pronounced peak appears around 980 MeV, indicating that the main contribution to the decay width comes from the region $m_{\text{inv}}(\pi\eta) \approx 980$ MeV. Moreover, the peak for $m_{\text{inv}}(\pi a_0) = 1420$ MeV, close to the TS, is signi-

ificantly sharper than those at the other values. Therefore, when focusing on the $a_0(980)$ region, we can integrate the double differential distribution $\frac{1}{\Gamma_B} \frac{d^2\Gamma'}{dm_{\text{inv}}(\pi a_0) dm_{\text{inv}}(\pi \eta)}$ over $m_{\text{inv}}(\pi \eta)$ around $m_{\text{inv}}(\pi \eta) = 980$ MeV. In Fig. 7, we present the results obtained by integrating over $m_{\text{inv}}(\pi \eta) \in [980 \text{ MeV} - \Delta_m, 980 \text{ MeV} + \Delta_m]$ with $\Delta_m = 30, 50, 100$ and 150 MeV. The results show a clear peak at 1420 MeV when the $a_0(980)$ is considered a dynamically generated state, consistent with the results in Fig. 6. A comparison of the subfigures for $q_{\text{max}} = 600, 800$ and 1000 MeV shows that the decay width exhibits a slight enhancement as the cutoff parameter increases.

VI. SUMMARY

In this work, we investigate the triangle mechanism in the $B^0 \rightarrow D^- \pi^+ a_0(980)(\pi^0 \eta)$ decay and explore the nature of $a_0(980)$ as a dynamically generated state arising from meson-meson interactions. The $\bar{K}^{*0}-K^+-K^-$ triangle loop originates from the weak decay $B^0 \rightarrow D^- K^+ \bar{K}^{*0}$, followed

by $\bar{K}^{*0} \rightarrow \pi^+ K^-$ and the subsequent rescattering of $K^+ K^-$ to form $a_0(980)$. We provide analytical expressions for the differential decay widths $\frac{d\Gamma}{dm_{\text{inv}}(\pi a_0)}$ and $\frac{d^2\Gamma}{dm_{\text{inv}}(\pi a_0) dm_{\text{inv}}(\pi \eta)}$ corresponding to the three-body and four-body final states, respectively, including the $\bar{K}^{*0}-K^+-K^-$ triangle loop amplitude. The triangle amplitude produces a pronounced peak near 1420 MeV in the $\pi a_0(980)$ invariant mass distribution, which can be interpreted as a TS signal. The dependence of the TS on the internal $\bar{K}^{*0}$ width is also investigated. The $K^+ K^-$ rescattering generates a pronounced peak near 980 MeV in the $\pi^0 \eta$ invariant mass distribution, reflecting the formation of the $a_0(980)$. Meanwhile, the invariant mass spectrum of the $\pi^+ \pi^0 \eta$ system exhibits a peak around 1420 MeV, consistent with the results obtained from the three-body final state. Based on the current observations and analyses by the COMPASS Collaboration, we expect that B^0 decays can serve as a potential platform for further confirming the TS nature of the $a_1(1420)$ in future experiments.

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