Revisiting the thermodynamics of the BTZ black hole with a variable gravitational constant*

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Abstract: The thermodynamics of BTZ black holes are revisited with a variable gravitational constant. A new pair of conjugated thermodynamic variables are introduced, including the central charge C and chemical potential μ. The first law of thermodynamics and the Euler relationship, instead of the Smarr relationship in the extended phase space formalism, are matched perfectly in the proposed formalism. Compatible with the standard extensive thermodynamics of an ordinary system, the black hole mass is verified to be a first order homogeneous function of the related extensive variables, and restores the role of internal energy. In addition, the heat capacity has also resulted in a first order homogeneous function using this formalism, and asymptotic behavior is demonstrated at the high temperature limit. The non-negativity of the heat capacity indicates that the rotating and charged BTZ black holes are thermodynamically stable.

Keywords: black hole thermodynamics, holography, gravitational constant

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I. INTRODUCTION

The black hole, which is one of the most fantastic predictions in general relativity, was regarded as a thermodynamic system in the pioneering works of Hawking and Bekenstein half a century ago [1, 2]. Bekenstein argued that the entropy of the black hole should be proportional to its area of horizon, and Hawking calculated the thermal spectrum of the black hole with a Hawking temperature using a semi-classical approach. The analogy between the thermodynamics of an ordinary system and that of the black hole was subsequently analyzed, establishing the four laws of the black hole thermodynamics [3]. The thermodynamics of the AdS black hole has recently gained attention owing to the special role of the AdS spacetime in the AdS/CFT correspondence [4].

To interpret the non-trivial contribution from the non-vanishing cosmological constant for the AdS black hole, the black hole thermodynamics of the extended phase space was developed [5−9], which introduces black hole thermodynamics to a new stage that is now also referred to as black hole chemistry [10]. The main idea of the extended phase space thermodynamics is the introduction of a new thermodynamic pair pressure-volume of the black hole, where the negative cosmological constant is treated as the thermodynamic pressure of the black hole, and its conjugate variable is the thermodynamic volume of the black hole. Significant efforts have been made regarding the study of the black hole thermodynamics in the extended phase space by using the standard thermodynamic formalism and abundant thermodynamic behaviors, such as discovering the phase transitions and critical behaviors for the AdS black hole [11−19]. In addition, note that the mass of the black hole is interpreted as the enthalpy rather than the internal energy owing to the existence of the VdP term in the first thermodynamic law in the context of the extended phase space.

Black hole thermodynamics in the extended phase space has gained significant attention, inevitably questioning the framework of holography, that is, the means of variation of the cosmological constant in the boundary field theory. According to the AdS/CFT correspondence, it is argued that a change of the cosmological constant in the theory of gravity indicates a change in the number of colors N or degree of freedom N^2 (which is related to the central charge C) in the boundary theory [20−22]. Alternatively, it is also suggested that if the boundary field theory needs to be maintained as fixed in a certain situation, the cosmological (Λ) as well as gravitational (G) constants can be varied, provided that the ratio $C \sim l^{d-2}/G$ is maintained as a constant [23], where l is the AdS radius in the d-dimensional spacetime. Visser recently proposed

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holographic thermodynamics, in which the central charge is introduced as a novel thermodynamics variable, and the Euler relationship as well as the first law of thermodynamics of the boundary field theory are obtained [24]. Furthermore, the holographic duality between the bulk thermodynamics and boundary thermodynamics has been previously considered, in which the definition of pressure indicated in the bulk thermodynamics is used for the extended phase space formalism, but with the variation of G [25−27].

Despite the rapid development of the black hole thermodynamics since the proposal of the extended phase space, certain unaddressed issues requiring further investigation remain. For instance, the thermodynamic volume conjugating to the thermodynamic pressure of the black hole lacks a physical interpretation, although occasionally its value is coincidentally equal to the geometric volume inside the horizon of the black hole. Moreover, the well-known Smarr relation of the rotating, charged AdS black holes for the d-dimensional extended phase space thermodynamics is [28]

$$(d-3)M = (d-2)TS - 2PV + (d-3)\Phi Q + (d-2)\Omega J, \quad (1)$$

which demonstrates that the thermodynamic potential M cannot be expressed as a homogeneous function for its arguments S, P, Q, J with a universal order. However, the presence of the Euler homogeneity in standard thermodynamics, which plays a critical role in understanding the equilibrium thermodynamic states, including those of the black holes. To address the questions mentioned above, a restricted version of Visser's holographic thermodynamics is proposed in Gao and Zhao's work [29], in which the cosmological constant is considered to be fixed and the gravitational constant remains to vary. Consequently, the first law of thermodynamics and the Euler homogeneity are perfectly matched in this formalism of the restricted phase space thermodynamics, which has been further verified in more general cases and higher dimensional spacetimes [30−32].

It was once thought that there was no black hole solution in the three-dimensional spacetime until the well-known BTZ black hole was discovered in the Einstein-gravity theory with a negative cosmological constant [33, 34]. This class of three-dimensional black holes gained significant attention owing to their special applications in the AdS_3/ CFT_2. The thermodynamics of the BTZ black holes have been extensively studied in the extended phase space formalism [35−39]. Nevertheless, considering the Smarr relationship indicated in Eq.(1), the lack of the Euler homogeneity becomes more prominent, that is, the left hand side of Eq.(1) is identically equal to zero and the term ΦQ vanishes regardless of the existence of the electric-magnetic field. This "reduced" Smarr relationship of the three-dimensional black hole is confusing, and largely motivates us to consider the restricted thermodynamics formalism for the BTZ black holes. Demonstrating that the three-dimensional black holes also satisfy the requirements of the standard thermodynamics would be significant, where the first law of thermodynamics and Euler relation are completely consistent.

In the next section, we study the thermodynamics of rotating and charged BTZ black holes explicitly in the restricted phase space. The thermodynamic quantities are calculated and proven to be divided into the following two groups in the formalism: extensive and intensive. Moreover, the thermodynamic stability of the BTZ black hole is analyzed considering the behaviors of the heat capacity. Finally, a conclusion is presented in Sec. III. In this study, the following units are adopted to match the Gauss's law in the three-dimensional spacetime: $\hbar = c = k_B = 1$, and $\varepsilon = 1/\mu_0 = 2\pi$.

II. EXTENSIVE THERMODYNAMICS OF THE BTZ BLACK HOLE

The action of the three-dimensional Einstein-gravity with the Maxwell electromagnetic field is as follows:

$$I = \int \mathrm{d}^3x\sqrt{-g}\left(\frac{R-2\Lambda}{2\kappa} - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}\right),$$

in which $\kappa = 8\pi G$ is the gravity coupling constant, Λ is the cosmological constant, and $F_{\mu\nu}$ is the electromagnetic field tensor defined as $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The coupled field equations are derived by variation of the action, which provides the following:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad \nabla_\nu F^{\mu\nu} = 0,$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the electromagnetic field:

$$T_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\rho}F_{\nu\sigma}g^{\rho\sigma} - \frac{1}{4}g_{\mu\nu}F^2\right).$$

The ansatz for the line element of a rotating spacetime is [40]:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta - \frac{4GJ}{r^2}\mathrm{d}t\right)^2,$$

and the Maxwell field A_μ is expressed as follows:

$$A_\mu = (-\Phi(r), 0, 0).$$

The metric function and electric potential can be directly obtained from the following field equation:

$$f(r) = -8GM + \frac{r^2}{l^2} + \frac{2GQ^2\mu_0}{\pi}\ln\left(\frac{r}{l}\right) + \frac{16G^2J^2}{r^2},$$
$$\Phi(r) = -\frac{\mu_0 Q}{2\pi}\ln\left(\frac{r}{l}\right), \tag{2}$$

where M is the black hole mass, Q is the charge, and J is the angular momentum. To avoid confusion by comparing the metric function Eq. (2) with other references, the coupling constant μ_0 remains to indicate the moment. In the following sections, the permeability μ_0 is valued as 2π for simplification.

The electrostatic potential of the charged BTZ black holes asymptotically diverges with a logarithmic term. Usually, the divergence can be managed using the two following approaches: (1) a new thermodynamic parameter associated with the renormalization length scale is introduced [28]; (2) a renormalized black hole mass is introduced [40], which is the approach used in this study.

A. The rotating BTZ black hole

For a rotating BTZ black hole where $J \neq 0$, $Q = 0$, the metric function reduces to the following form:

$$f(r) = -8GM + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2},$$

which is significantly similar to that of the four and higher dimensional rotating AdS black holes. This implies the possibility of simultaneously achieving the Euler homogeneity and the first law of thermodynamics of the rotating BTZ black hole.

The mass M is determined by $f(r_0) = 0$ with the radius of the event horizon r_0, which yields the following:

$$M = \frac{r_0^2}{8Gl^2} + \frac{2GJ^2}{r_0^2}. \tag{3}$$

The entropy, angular velocity, and temperature of the rotating BTZ black holes in the three-dimensional spacetime are obtained as follows:

$$S = \frac{\mathcal{A}}{4G} = \frac{\pi r_0}{2G},$$
$$\Omega = \left(\frac{\partial M}{\partial J}\right)_{S,l} = \frac{4GJ}{r_0^2},$$
$$T = \frac{f'(r_0)}{4\pi} = \frac{r_0}{2\pi l^2} - \frac{8G^2J^2}{\pi r_0^3},$$

where $\mathcal{A} = 2\pi r_0$ is the horizon area.

In addition, there are other essential thermodynamics quantities for achieving the Euler homogeneity of the black hole, which are denoted as the chemical potential μ and its conjugate central charge C. The definition of the conjugate central charge arises from the concept of the AdS/CFT correspondence, and is $C = l/8G$ for the three-dimensional Einstein gravity. There are several candidates for the generalized central charge in the arbitrary odd $(d-1)$-dimensional conformal field theory, for which the scaling for both is l^{d-2}/G with an ambiguous coefficient [41]. The value of the central charge from the bulk and boundary can be normalized to match the holographic dictionary [42]. However, the coefficient does not matter since the pair $\{\mu, C\}$ appears in the first law, and thus only the l/G scaling is important.

The central charge C is treated as a novel thermal quantity in black hole thermodynamics, and it indicates the amount of substance in the standard thermodynamic system. Furthermore, there is a correspondence between the partition function of CFT and the gravity theory in the asymptotic AdS spacetime $Z_{\rm CFT} = Z_{\rm AdS}$ [43, 44]; thus, the following is obtained:

$$\mu C = F = -T\ln Z_{\rm CFT} = -T\ln Z_{\rm AdS} = TI_E,$$

where the thermal partition function of the CFT at a finite temperature is associated with the free energy, and the gravity partition function is calculated by the on-shell Euclidean action I_E. Then, the chemical potential μ can be directly obtained from the on-shell Euclidean action I_E of gravity, which provides the following: [45]

$$I_{\rm EH} = -\frac{1}{2\kappa}\int \mathrm{d}^3x\sqrt{g}(R - 2\Lambda),$$
$$I_{\rm GHY} = -\frac{1}{\kappa}\int_{\partial M}\mathrm{d}^2x\sqrt{h}K,$$
$$I_{\rm ct} = \frac{1}{\kappa}\int_{\partial M}\mathrm{d}^2x\sqrt{h}\left(\frac{1}{l}\right),$$

where $I_{\rm EH}$ is the Euclidean version of the Einstein-Hilbert action, $I_{\rm GHY}$ is the Gibbons-Hawking-York action, h is the reduced metric of the hypersurface, K is the trace of the extrinsic curvature, and $I_{\rm ct}$ is the counterterm to cancel the divergence. The result is as follows:

$$I_E = I_{\rm EH} + I_{\rm GHY} + I_{\rm ct} = \beta\left(-\frac{r_0^2}{8Gl^2} + \frac{2GJ^2}{r_0^2}\right),$$

where $\beta = 1/T$ is the inverse temperature. Combining all the results of the thermodynamic quantities, it is easy to verify the following:

$$TI_E = \mu C = M - TS - \Omega J.$$

Instead of obtaining the Smarr relationship of the black hole thermodynamics in the extended phase space, we derive an inspiring relationship which is similar to the Euler relationship of the standard thermodynamic system:

$$M = TS + \Omega J + \mu C. \tag{4}$$

Furthermore, the first law of thermodynamics for rotating BTZ black holes can be verified straightforwardly as follows:

$$\mathrm{d}M = T\mathrm{d}S + \Omega\mathrm{d}J + \mu\mathrm{d}C. \tag{5}$$

Note, in the proof of the first law Eq. (5), the AdS radius l is restricted as a constant and the Newton' s constant G is allowed to vary; therefore, this scheme is also referred to as the black hole thermodynamics in the restricted phase space.

Subsequently, to reveal the features of the restricted formalism of the black hole thermodynamics, all the thermal quantities of the BTZ black hole are classified into two groups: one is extensive and the another is intensive, as those in a standard extensive thermodynamic system. A group of independent thermodynamic agreements S, J, C, exist, and all the thermal potentials can be expressed as a homogeneous function of these independent agreements. First, we rewrite G, r_0 as follows:

$$G = \frac{l}{8C}, \quad r_0 = \frac{Sl}{4\pi C}. \tag{6}$$

By inserting Eq. (6) into Eq. (3), the mass M is rewritten as follows:

$$M = \frac{S^2}{16\pi^2 lC} + \frac{4\pi^2 J^2 C}{lS^2}, \tag{7}$$

and the conjugated thermodynamic variables T, Ω, μ are expressed as follows:

$$T = \left(\frac{\partial M}{\partial S}\right)_{J,C} = \frac{S}{8\pi^2 lC} - \frac{8\pi^2 J^2 C}{lS^3}, \tag{8}$$

$$\Omega = \left(\frac{\partial M}{\partial J}\right)_{S,C} = \frac{8\pi^2 JC}{lS^2}, \tag{9}$$

$$\mu = \left(\frac{\partial M}{\partial C}\right)_{S,J} = -\frac{S^2}{16\pi^2 lC^2} + \frac{4\pi^2 J^2}{lS^2}. \tag{10}$$

In mathematics, an m-th order homogeneous function with respect to its agreements $(x_1,\ldots,x_n)$ satisfies:

$$f(\lambda x_1,\ldots,\lambda x_n) = \lambda^m f(x_1,\ldots,x_n), \qquad \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = mf,$$

which requires $m=1$ for the extensive quantities and $m=0$ for the intensive quantities in a standard thermodynamics system. From Eq. (7) and Eqs. (8)−(10), it is apparent that M is a first order homogeneous function and T, Ω, μ are the zeroth order homogeneous functions of S, J, C, which are the intriguing results anticipated.

Precisely, S and J are extensive variables; the following can be obtained based on Eq. (9):

$$J = \frac{lS^2\Omega}{8\pi^2 C}. \tag{11}$$

Substituting Eq. (11) into Eq. (8) obtains the following:

$$T = \frac{S - l^2 S\Omega^2}{8l\pi^2 C}. \tag{12}$$

Because $S \geq 0$ and $T \geq 0$, the following boundary is obtained:

$$\Omega \leq \frac{1}{l}, \tag{13}$$

which helps solve the entropy directly from Eq. (12):

$$S = \mathcal{S}C, \quad \mathcal{S} = \frac{8\pi^2 lT}{1 - l^2\Omega^2}. \tag{14}$$

Apparently, S is proportional to C and the coefficient $\mathcal{S}$ only depends on the intensive variables T and Ω, proving that S is an extensive variable.

Inserting Eq. (14) into Eq. (11), J is also shown to be an extensive variable:

$$J = \mathcal{J}C, \quad \mathcal{J} = \frac{8\pi^2 l^3 T^2\Omega}{(1 - l^2\Omega^2)^2},$$

where the coefficient $\mathcal{J}$ only depends on the intensive variables T and Ω.

Moreover, the Gibbs-Duhem equation can be expressed based on Eq. (5) and Eq. (4):

$$\mathrm{d}\mu = -\mathcal{S}\mathrm{d}T - \mathcal{J}\mathrm{d}\Omega,$$

where $\mathcal{S} = S/C, \mathcal{J} = J/C$ are the zeroth order homogeneous functions of S, J, C. The Gibbs-Duhem equation suggests that μ is dependent on Ω and T, indicating that μ is not an independent variable. Combining Eqs. (8)−(10), the partial derivative μ with respect to T at the constant Ω is straightforward to obtain:

$$\left(\frac{\partial\mu}{\partial T}\right)_{\Omega} = -\frac{8\pi^2 lT}{(1-l^2\Omega^2)} < 0, \tag{15}$$

which indicates that μ is monotonically decreasing with T, and is demonstrated by the following relationship:

$$\mu|_{T=0} = 0,$$

which implies that μ is identically negative. For the rotating BTZ black holes in the restricted phase space, there is neither an inflection point nor an extremal point in the $T-S$ and $\Omega-J$ sub-phase spaces. Therefore, the phase transition is inexistent in the thermodynamics of the rotating BTZ black holes.

Furthermore, we consider the heat capacity of the rotating BTZ black hole at a constant angular momentum J to study its thermodynamic stability:

$$C_J = T\left(\frac{\partial S}{\partial T}\right)_J = \frac{S^5 - 64J^2\pi^4 S C^2}{S^4 + 192J^2\pi^4 C^2}.$$

Similarly, the heat capacity C_J is also apparently a first order homogeneous function of S, J, C, which is expected in extensive thermodynamics. Moreover, the non-negative temperature T constrains $S^4 \geq 64J^2\pi^4C^2$ from Eq. (8), providing a non-negative heat capacity, *that is*, $C_J \geq 0$, which is always true, indicating that rotating BTZ black holes are thermodynamically stable. In addition, the heat capacity C_J explicitly depends on S, while implicitly depending on T via Eq. (8). Intriguingly, in a high temperature limit, C_J is asymptotically valued as follows:

$$\lim_{T\to\infty} C_J \sim 8\pi^2 lCT, \tag{16}$$

which is independent of the angular momentum J. This asymptotic behavior of the heat capacity is consistent with the conjecture $C \sim T^{d-2}$ in a d-dimensional spacetime [31], where $d-2$ is the spatial dimension of the event horizon. Considering Debye's theory for ordinary non-metallic solid matter in a low temperature limit, the heat capacity behaves as $C_V \sim T^D$ in spatial D dimensions; these similar behaviors deserve more attention for a better understanding.

B. The charged BTZ black hole

For a charged BTZ black hole with $J = 0, Q \neq 0$, the form of the metric function contains a logarithmic term. This feature makes charged BTZ black holes significantly different from the charged AdS black holes in four and higher dimensions, thus whether there are extensive thermodynamic properties for the charged BTZ black holes in the restricted formalism is considered. From Eq. (2), the metric function of the charged BTZ black hole is as follows:

$$f(r) = -8GM + \frac{r^2}{l^2} - 4GQ^2\ln\left(\frac{r}{l}\right), \quad \Phi(r) = -Q\ln\left(\frac{r}{l}\right).$$

As indicated before, the thermodynamic variables of the charged BTZ black holes are obtained as follows:

$$\begin{aligned} M &= \frac{r_0^2}{8Gl^2} - \frac{Q^2}{2}\ln\left(\frac{r_0}{l}\right), \\ T &= \frac{f'(r_0)}{4\pi} = \frac{r_0}{2\pi l^2} - \frac{GQ^2}{\pi r_0}, \\ S &= \frac{\mathcal{A}}{4G} = \frac{\pi r_0}{2G}, \end{aligned}$$

Similar to the previous subsection, the on-shell action of the charged BTZ black hole is given by: [46]

$$\begin{aligned} I_{\rm EH} &= -\frac{1}{2\kappa}\int \mathrm{d}^3x\sqrt{-g}(R-2\Lambda) + \frac{1}{4\mu_0}\int \mathrm{d}^3x F_{\mu\nu}F^{\mu\nu}, \\ I_{GHY} &= -\frac{1}{\kappa}\int_{\partial M}\mathrm{d}^2x\sqrt{h}K - \frac{1}{\mu_0}\int_{\partial M}\mathrm{d}^2x n_r F^{rt}A_t, \\ I_{ct} &= \frac{1}{\kappa}\int_{\partial M}\mathrm{d}^2x\sqrt{h}(\frac{1}{l}). \end{aligned}$$

By summing all these contributions, the result is as follows:

$$I_E = I_{\rm EH} + I_{\rm GHY} + I_{\rm ct} = \beta\left(\frac{Q^2}{2}\ln(\frac{r_0}{l}) + \frac{Q^2}{2} - \frac{r_0^2}{8Gl^2}\right),$$

from which we can prove $\mu C = TI_E = M - TS - \hat{\Phi}\hat{Q}$ with $T = 1/\beta$, where $\hat{\Phi} = \frac{\Phi\sqrt{G}}{l}, \hat{Q} = \frac{Ql}{\sqrt{G}}$ [1)]. Based on the expressions of the thermodynamic variables, the first law of the black hole thermodynamics and the Euler relationship are easily verified:

$$\mathrm{d}M = T\mathrm{d}S + \hat{\Phi}\mathrm{d}\hat{Q} + \mu\mathrm{d}C, \tag{17}$$

$$M = TS + \hat{\Phi}\hat{Q} + \mu C. \tag{18}$$

Here, the AdS radius l remains restricted and the gravita-

1) The definition of the charge and potential are different with the holographic dictionary $\tilde{\Phi} = \Phi/l, \tilde{Q} = Ql$ [23, 47], since bulk action Eq.(2) in this paper is slightly different in the coefficients with the usual bulk action $I = \frac{1}{16\pi G}\int d^dx\sqrt{-g}\left(R - 2\Lambda - F^2\right)$.

tional constant is allowed to vary.

In addition, we intend to prove that extensive and intensive thermal quantities exist in this restricted formalism of the charged BTZ black hole. According to Eq. (6), the mass M of the charged BTZ black hole is rewritten as follows:

$$M=\frac{S^2}{16l\pi^2C}-\frac{\hat{Q}^2\ln[S/(4\pi C)]}{16lC}. \tag{19}$$

Other thermodynamic variables are as follows:

$$T=\left(\frac{\partial M}{\partial S}\right)_{\hat{Q},C}=-\frac{\hat{Q}^2}{16lSC}+\frac{S}{8l\pi^2C}, \tag{20}$$

$$\hat{\Phi}=\left(\frac{\partial M}{\partial \hat{Q}}\right)_{S,C}=-\frac{\hat{Q}\ln[S/(4\pi C)]}{8lC}, \tag{21}$$

$$\mu=\left(\frac{\partial M}{\partial C}\right)_{\hat{Q},S}=\frac{\hat{Q}^2}{16lC^2}-\frac{S^2}{16\pi^2lC^2}+\frac{\hat{Q}^2\ln[S/(4\pi C)]}{16lC^2}. \tag{22}$$

When $S\to\lambda S,\hat{Q}\to\lambda\hat{Q},C\to\lambda C$, M scales as $M\to\lambda M$, while $T,\hat{\Phi},\mu$ are not rescaled. This proves the first order homogeneity of M and the zeroth order homogeneity of $T,\hat{\Phi},\mu$ in $S,\hat{Q},C$.

Here, we prove that S and $\hat{Q}$ are extensive variables. From Eq. (21) and Eq. (22), we obtain the following:

$$\mu=\frac{\hat{Q}^2}{16lC^2}-\frac{S^2}{16l\pi^2C^2}-\frac{\hat{Q}\hat{\Phi}}{2C}. \tag{23}$$

From Eq. (23), we obtain the physical expression for $\hat{Q}$

$$\hat{Q}=\frac{4\pi C\hat{\Phi}l+\sqrt{S^2+16\pi^2lC^2\left(\mu+l\hat{\Phi}^2\right)}}{\pi}.$$

Assuming $S=C\mathcal{S}$, the expression for $\hat{Q}$ reduces to:

$$\hat{Q}=C\hat{\mathcal{Q}},\quad \hat{\mathcal{Q}}=4\hat{\Phi}l+\sqrt{\left[\mathcal{S}^2+16\pi^2l\left(\mu+l\hat{\Phi}^2\right)\right]}\,\pi. \tag{24}$$

Aquiring Eq. (24) into Eq. (20), except for $S<0$, we obtain the following expression:

$$S=C\mathcal{S},\quad \mathcal{S}=4\pi^2lT+\pi\sqrt{(\hat{\mathcal{Q}}^2+32\pi^2l^2T^2)/2}. \tag{25}$$

Combining Eq. (24) and Eq. (25), the expressions of $\hat{\mathcal{Q}},\mathcal{S}$ can be directly solved in principle. However, the formulas are significantly tedious, and the explicit forms are unnecessary here, thus we do not present these in detail. Apparently, $\hat{\mathcal{Q}},\mathcal{S}$ can be determined from the implicit functions, which indicates that they only depend on the intensive variables T, $\hat{\Phi}$, and μ. Therefore, $S,\hat{Q}$ are proved to be extensive variables proportional to C.

The Gibbs-Duhem equation derived from Eq. (17) and Eq. (18) is as follows:

$$\mathrm{d}\mu=-\hat{\mathcal{Q}}\mathrm{d}\hat{\Phi}-\mathcal{S}\mathrm{d}T,$$

where $\hat{\mathcal{Q}}=\hat{Q}/C$ and $\mathcal{S}=S/C$ are the zeroth order homogeneous functions in S, $\hat{Q}$, C. Note, the chemical potential μ depends only on T, $\hat{\Phi}$, which is determined by Eqs. (20)−(22). The behaviors of $\mu-T$ are considered with a fixed $\hat{\Phi}$ to investigate the possible phase transition. The direct calculation of the relationship $\mu(T,\hat{\Phi})$ is difficult, thus it is derived with the help of the chain rule of the partial derivative:

$$\begin{aligned}\left(\frac{\partial\mu}{\partial T}\right)_{\hat{\Phi}}&=\left(\frac{\partial\mu}{\partial\hat{Q}}\right)_{S}\left(\frac{\partial S}{\partial T}\right)_{\hat{\Phi}}+\left(\frac{\partial\mu}{\partial S}\right)_{\hat{Q}}\left(\frac{\partial\hat{Q}}{\partial T}\right)_{\hat{\Phi}}\\&=\frac{4\pi\hat{Q}^3(1+\ln(\frac{S}{4\pi C}))\mathrm{e}^{8lC\hat{\Phi}/\hat{Q}}}{128lC^3\hat{\Phi}-\hat{Q}^2(\hat{Q}-4lC\hat{\Phi})\mathrm{e}^{16lC\hat{\Phi}/\hat{Q}}}+\frac{S\left(\pi^2\hat{Q}^2-2S^2\right)\left(\ln\left(\frac{S}{4\pi C}\right)\right)^3}{2\left(64\pi^2C^3l^2\hat{\Phi}^2+32\pi^2C^3l^2\Phi^2\ln\left(\frac{S}{4\pi C}\right)+CS^2\left(\ln\left(\frac{S}{4\pi C}\right)\right)^3\right)}.\end{aligned}$$

Although the relationship indicated above appears complicated for analysis, the fact that $\hat{\Phi}<0$ from Eq. (20) and $2S^2\geq\pi^2\hat{Q}^2$ from the requirement of the non-negative temperature, this explicitly demonstrates that $\left(\frac{\partial\mu}{\partial T}\right)_{\hat{\Phi}}<0$. This implies that the chemical potential is monotonically decreasing with T, and there are no swallow tail behaviors present as those in Van der Walls systems. For charged BTZ black holes, we also verify that neither an extremal point nor an inflection point exists in the $T-S$ or $\hat{\Phi}-\hat{Q}$ phase space, which suggests that there is no second order phase transition.

The heat capacity at a constant charge of the charged BTZ black holes is given by the following:

$$C_{\hat{Q}} = T\left(\frac{\partial S}{\partial T}\right)_{\hat{Q}} = \frac{S\left(-\pi^2\hat{Q}^2+2S^2\right)}{\pi^2\hat{Q}^2+2S^2}.$$

The requirement of the Hawking temperature $T\geq 0$, *that is*, $2S^2\geq\pi^2\hat{Q}^2$, implies that the heat capacity is $C_{\hat{Q}}\geq 0$ and the charged BTZ black hole is always thermodynamically stable. In addition, the heat capacity $C_{\hat{Q}}$ is also a first order homogeneous function of S, $\hat{Q}$, C as expected. In a high temperature asymptotic region, the value of $C_{\hat{Q}}$ is given by the following:

$$\lim_{T\to\infty} C_{\hat{Q}} \sim 8\pi^2 lCT, \tag{26}$$

which is independent of the charge $\hat{Q}$ and is the same as that of the rotating BTZ black hole.

III. CONCLUSION

In this study, the thermodynamic properties of the three-dimensional BTZ black holes are restudied, motivated by the recent consideration of holographic thermodynamics. In the previous extended phase space formalism, we noticed certain confusing problems for the Smarr relationship of the three-dimensional BTZ black holes: 1) the mass term and the terms related to the electric field disappear, and 2) the thermodynamic potential does not satisfy the Euler homogeneity, which is critical for understanding the equilibrium state of thermodynamics.

To address the aforementioned problems, we attempted to restrict the AdS radius l as a constant and varied the Newtonian constant G. In this formalism, a pair of new conjugate variables (μ, C) was introduced from the AdS/CFT correspondence. The resulting thermodynamics are compatible with the standard extensive thermodynamics; all the thermodynamic quantities are classified into either extensive or intensive. The entropy S and angular momentum J (or charge $\hat{Q}$) are proven to be extensive; thus, they are simply additive. The other thermal quantities, T,Ω,μ are intensive. The black hole mass M is regarded as an internal energy and is a first order homogeneous function of the relevant extensive variables. The first law and Euler relationship, as well as the Gibbs-Duhem equation of the BTZ black holes, were found to hold simultaneously.

In addition to the aforementioned, the phase spaces $T-S$ and $\Omega-J$ of the rotating BTZ black holes (or the phase spaces $T-S$ and $\hat{\Phi}-\hat{Q}$ of the charged BTZ black holes), constructed by the independent variables, were investigated and it was found that there is neither an extremum nor an inflection point. Thus, the rotating and charged BTZ black holes have no critical phenomena in this formalism, which is similar to the phase structure of the BTZ black holes in the extended phase space. We also found that both rotating and charged BTZ black holes are thermodynamically stable, which is determined by the non-negativity of the corresponding heat capacity.

In summary, we achieved the extensibility of the thermodynamic variables in the BTZ black holes by considering Newton's constant variable; in fact, this approach holds universally for asymptotically flat or de Sitter spacetimes. In a previous study [48], the variable l represents the maximum radius for a horizon that a black hole can reach in the thermodynamic process; it was introduced by considering the scale, which is independent of the cosmological constant. Based on this, the thermodynamic degrees of freedom N of a black hole is related to l and Newton's constant G through scaling considerations, and it is regarded as the number of pieces of the size of the Planck area. The thermodynamics of non-AdS black holes was demonstrated to belong to the standard framework of extensive thermodynamics if Newton's constant is allowed to vary. Furthermore, there are usually certain coupling parameters in modified gravity theory models, and it is undetermined whether these parameters can be treated as thermodynamic variables. For the requirements regarding the extension of thermodynamics, the constraints that will be imposed on these coupling parameters is a noteworthy topic that we aim to investigate in the near future.

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Data Availability Statement

No Data associated in the manuscript.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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