Horizon thermodynamics in $f(R, R^{\mu\nu}R_{\mu\nu})$ theory*

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Abstract: We investigate whether the new horizon first law still holds in $f(R, R^{\mu\nu}R_{\mu\nu})$ theory. For this complicated theory, we first determine the entropy of a black hole by using the Wald method, and then derive the energy of the black hole by using the new horizon first law, the degenerate Legendre transformation, and the gravitational field equations. For application, we consider the quadratic-curvature gravity, and first calculate the entropy and energy of a static spherically symmetric black hole, which are in agreement with the results obtained in the literature for a Schwarzschild-(A)dS black hole.

Keywords: black hole, energy, entropy, new horizon first law

DOI: 10.1088/1674-1137/abadef

1 Introduction

The black hole, predicted by general relativity, is an object of long-standing interest to physicists. It was Bekenstein who first proposed that black holes actually possess entropy [1]. In sharp contrast to standard thermodynamic notions wherein entropy is supposed to be a function of volume, he suggested that the entropy of a black hole is proportional to the horizon area. Since then a variety of different theoretical methods have been used to calculate the Bekenstein-Hawking entropy, such as using the quantum fields near the horizon [2], quantum field theory in a fixed background [3], the entanglement entropy [4], string theory [5-10], loop quantum gravity [11, 12], Noether charge [13, 14], induced gravity [15], the causal set theory [16], the symmetry near the horizon [17, 18], and using the inherently global characteristics of a black hole spacetime [19]. It has been shown that the classical Bekenstein-Hawking entropy depends not only on the black hole parameter, but also on the coupling, which reduces the Lorentz violation [20]. Considering the complexity involved in these methods, finding a simpler way to calculate the entropy of a black hole is an important task.

Energy is another important issue besides entropy in black hole physics. In higher-order gravitational theories, the energy of a black hole is still an open problem. Several efforts to find a satisfactory answer to this issue have been carried out [21-25]. It was shown that the entropy and energy of a black hole can be simultaneously obtained in Einstein's gravity using the horizon first law [26], but it cannot work in higher-order gravitational theories. Recently a new horizon first law, in which both the entropy and the free energy are derived concepts, was suggested in Einstein's gravity and Lovelock's gravity; the standard horizon first law can be recovered by a Legendre projection [27]. In [28], it was found that the new horizon first law still works in $f(R)$ theories by introducing the effective curvature fluid: it can give not only the energy but also the entropy of black holes, which are in agreement with the known results in the literature. Here we will consider the new horizon first law and the entropy and the energy issues in $f(R, R^{\mu\nu}R_{\mu\nu})$ gravity.

The paper is organized as follows. In section 2, we briefly review the new horizon first law. In section 3, we discuss the entropy and the energy of black holes in $f(R, R^{\mu\nu}R_{\mu\nu})$ theory. In section 4, applications are considered. Finally, in section 5 we briefly summarize our results.

2 The new horizon first law

According to the suggestion proposed in [29], that the

Received 29 April 2020, Published online 18 August 2020
* Supported in part by Hebei Provincial Natural Science Foundation of China (A2014201068)
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source of thermodynamic system is also that of gravity, the radial component of the stress-energy tensor can act as the thermodynamic pressure, $P = T^r_r|_{r_+}$, then at the horizon of Schwarzschild black hole the radial Einstein equation can be written as

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2}, \tag{1}$$

which can be rewritten as a horizon first law after a imaginary displacement of the horizon, $\delta E = T\delta S - P\delta V$, with E as the quasilocal energy and S as the horizon entropy of the black hole [26]. As the temperature T in the Eq. (1) is identified from the thermal quantum field theory, independent of any gravitational field equations [27], while the pressure P in (1), according to the conjecture proposed in [29], is identified as the radial component of the matter stress-energy, it is reasonable to assume that the radial field equation of a gravitational theory under consideration takes the form [27]

$$P = D(r_+) + C(r_+)T, \tag{2}$$

where C and D are analytic functions of the radius of the black hole, r_+; in general, they depend on the gravitational theory under consideration. Varying the Eq. (1) and multiplying the geometric volume $V(r_+)$, it is straightforward to have a new horizon first law [27]

$$\delta G = -S\,\delta T + V\delta P, \tag{3}$$

with the Gibbs free energy as

$$G = \int^{r_+} V(r)D'(r)\,\mathrm{d}r + T\int^{r_+} V(r)C'(r)\mathrm{d}r$$
$$= PV - ST - \int^{r_+} V'(r)D(r)\mathrm{d}r, \tag{4}$$

and the entropy as [27]

$$S = \int^{r_+} V'(r)C(r)\mathrm{d}r. \tag{5}$$

Under the degenerate Legendre transformation $E = G + TS - PV$, yields the energy as [28]

$$E = -\int^{r_+} V'(r)D(r)\mathrm{d}r. \tag{6}$$

This procedure was first discussed in Einstein's gravity and Lovelock's gravity, which only give rise to a second-order field equation [27]. It was generalized to $f(R)$ gravity with a static spherically symmetric black hole [28] or with a general spherically symmetric black hole [30] and was also applied to the D-dimensional $f(R)$ theory [31].

In the next section, we will investigate whether this procedure can be applied to more complicated cases, such as $f(R, R^{\mu\nu}R_{\mu\nu})$ theory, and whether Eqs. (5) and (6) can still be used to obtain the entropy and the energy in the theory we consider.

3 The entropy and the energy of black holes in $f(R, R^{\mu\nu}R_{\mu\nu})$ theory

As shown in section 2, the new horizon first law works well in Einstein's theory, Lovelock gravity [27], and $f(R)$ theory [28, 30, 31]. Does it still works in other gravitational theories such as $f(R, R^{\mu\nu}R_{\mu\nu})$ theory? We consider this question in this section. In four-dimensional spacetime, the general action of $f(R, R^{\mu\nu}R_{\mu\nu})$ theory with source is given by

$$I = \int \mathrm{d}^4x\,\sqrt{-g}\left[\frac{f(R, R^{\mu\nu}R_{\mu\nu})}{16\pi} + L_m\right], \tag{7}$$

where $L_{\rm m}$ is the matter Lagrangian and $f(R, R^{\mu\nu}R_{\mu\nu})$ is a general function of the Ricci scalar R and the square of the Ricci tensor $R_{\mu\nu}$. We take the units $G = c = \hbar = 1$. Varying the action (7) with respect to metric $g_{\mu\nu}$ yields the gravitational field equations as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\left[\frac{T_{\mu\nu}}{f_R} + \frac{1}{8\pi}\Omega_{\mu\nu}\right], \tag{8}$$

where $f_R \equiv \dfrac{\partial f}{\partial R}$ and $T_{\mu\nu} = \dfrac{2}{\sqrt{-g}}\dfrac{\delta L_m}{\delta g^{\mu\nu}}$ is the energy-momentum tensor of matter. $\Omega_{\mu\nu}$ is the tress-energy tensor of the effective curvature fluid and is given by

$$\Omega_{\mu\nu} = \frac{1}{f_R}\left[\frac{1}{2}g_{\mu\nu}(f - Rf_R) + \nabla_\mu\nabla_\nu f_R - g_{\mu\nu}\Box f_R - 2f_X R^\alpha_\mu R_{\alpha\nu}\right.$$
$$\left. - \Box(f_X R_{\mu\nu}) - g_{\mu\nu}\nabla_\alpha\nabla_\beta(f_X R^{\alpha\beta}) + 2\nabla_\alpha\nabla_{(\mu}(R^\alpha_{\nu)}f_X)\right], \tag{9}$$

where $A_{(\mu\nu)} = \dfrac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$, $X \equiv R^{\mu\nu}R_{\mu\nu}$, and $f_X \equiv \dfrac{\partial f}{\partial X}$. Inserting the following two derivational relations

$$\nabla_\alpha\nabla_\beta(f_X R^{\alpha\beta}) = R^{\alpha\beta}\nabla_\alpha\nabla_\beta f_X + (\nabla^\beta R)(\nabla_\beta f_X) + \frac{1}{2}f_X\Box R, \tag{10}$$

and

$$\nabla_\alpha\nabla_\mu(f_X R^\alpha_\nu) + \nabla_\alpha\nabla_\nu(f_X R^\alpha_\mu) = R^\alpha_\nu\nabla_\alpha\nabla_\mu f_X + R^\alpha_\mu\nabla_\alpha\nabla_\nu f_X + \frac{1}{2}(\nabla_\mu f_X)(\nabla_\nu R) + \frac{1}{2}(\nabla_\nu f_X)(\nabla_\mu R)$$
$$+ (\nabla_\alpha f_X)(\nabla_\mu R^\alpha_\nu) + (\nabla_\alpha f_X)(\nabla_\nu R^\alpha_\mu) + f_X\nabla_\mu\nabla_\nu R + 2f_X R_{\alpha\mu\nu\lambda}R^{\alpha\lambda} + 2f_X R_{\mu\lambda}R^\lambda_\nu, \tag{11}$$

into Eq. (9), the tress-energy tensor of the effective curvature fluid $\Omega_{\mu\nu}$ is simplified as

$$\Omega_{\mu\nu} = \frac{1}{f_R}\Big[\frac{1}{2}g_{\mu\nu}(f - Rf_R) + \nabla_\mu\nabla_\nu f_R - g_{\mu\nu}\Box f_R - f_X\Box R_{\mu\nu} - R_{\mu\nu}\Box f_X - g_{\mu\nu}R^{\alpha\beta}\nabla_\alpha\nabla_\beta f_X - \frac{1}{2}f_X g_{\mu\nu}\Box R$$
$$+ R^\alpha_\nu\nabla_\alpha\nabla_\mu f_X + R^\alpha_\mu\nabla_\alpha\nabla_\nu f_X - f_X\nabla_\mu\nabla_\nu R - 2f_X R_{\alpha\mu\nu\lambda}R^{\alpha\lambda}\Big]. \tag{12}$$

For a static spherically symmetric black hole whose geometry is given by

$$\mathrm{d}s^2 = -B(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + r^2\mathrm{d}\Omega^2, \tag{13}$$

where the event horizon is located at $r = r_+$ the largest positive root of $B(r_+) = 0$ with $B'(r_+) \neq 0$, the $\binom{1}{1}$ components of the Einstein tensor is

$$G^1_1 = \frac{1}{r^2}(-1 + rB' + B), \tag{14}$$

with the primes denoting the derivative with respect to r. At the horizon, since $B(r_+) = 0$, it reduces to

$$G^1_1 = \frac{1}{r^2}(-1 + rB'). \tag{15}$$

While the radial components of the tress-energy tensor of the effective curvature fluid $\Omega_{\mu\nu}$ at the horizon takes the following form

$$\Omega^1_1 = \frac{1}{f_R}\left[\frac{1}{2}(f - Rf_R) - \frac{1}{2}B' f'_R - B'(f_X R^1_1)' - \frac{2 f_X B' R^1_1}{r_+} + \frac{2 f_X B' R^3_3}{r_+} - 2 f_X (R^1_1)^2\right]. \tag{16}$$

Substituting Eqs. (15) and (16) into Eq. (8), and considering $P = T^r_r|_{r_+}$, we derive

$$8\pi P = -\frac{f_R}{r_+^2} + \frac{f_R B'}{r_+} - \frac{1}{2}(f - Rf_R) + \frac{1}{2}B' f'_R + B'(f_X R^1_1)' + \frac{2 f_X B' R^1_1}{r_+} - \frac{2 f_X B' R^3_3}{r_+} + 2 f_X (R^1_1)^2. \tag{17}$$

This equation is very complicated; therefore, how can we determine the function $C(r_+)$ in Eq. (2)? Even worse, this equation depends on higher derivatives of B, and hence we can no longer follow the same approach as the one we followed for Einstein's theory in which we obtained the entropy and energy from Eqs. (5) and (6) directly. In higher-derivative gravity, to use the horizon first law, $\delta E = T\delta S - P\delta V$, usually one should reduce the higher-derivative field equations to lower-derivative field equations via a Legendre transformation [32, 33]. Here we try a new method. If we obtain the entropy by using other methods, then using the new horizon first law (3) and the degenerate Legendre transformation $E = G + TS - PV$, we can derive the energy. As $f(R, R^{\mu\nu}R_{\mu\nu})$ is a diffeomorphism invariance of the gravitational theory, the entropy can be obtained using the Wald method, which is presented in the Appendix. Taking into account the volume of the black hole $V(r_+) = 4\pi r_+^3/3$, the pressure in Eq. (17), the Hawking temperature $T = B'(r_+)/4\pi$, and the entropy given in the Appendix, the new horizon first law (3) can be rewritten as

$$\delta G = -\frac{1}{3}\pi r_+^2 (f_R + 2 f_X R^1_1)\delta T + \frac{4\pi r_+^3 T}{3}\delta\left(\frac{f_R}{2r_+}\right) + \frac{4\pi r_+^3 T}{3}\delta\left(\frac{f_X R^1_1}{r_+}\right) + \frac{1}{3}\pi r_+^3 \delta(T f'_R) + \frac{2}{3}\pi r_+^3 \delta[T(f_X R^1_1)'] - \frac{1}{12} r_+^3 \delta(f - Rf_R) - \frac{1}{6} r_+^3 \delta\left(\frac{f_R}{r_+^2}\right) + \frac{4\pi r_+^3}{3}\delta\left(\frac{4\pi f_X T^2}{r_+^2}\right) - \frac{4\pi r_+^3}{3}\delta\left(\frac{T f_X}{r_+^3}\right) + \frac{1}{3} r_+^3 \delta[f_X (R^1_1)^2], \tag{18}$$

and $TS - PV$ is given by

$$TS - PV = \frac{1}{3}\pi r_+^2 f_R T + \frac{2}{3}\pi r_+^2 f_x R^1_1 T - \frac{2}{3}\pi r_+^3 T (f_x R^1_1)' - \frac{1}{3}\pi r_+^3 T f'_R + \frac{1}{6} f_R r_+ + \frac{1}{12} r_+^3 (f - Rf_R) - \frac{1}{3} r_+ f_X B'^2 + \frac{1}{3} f_X B' - \frac{1}{3} r_+^3 f_X (R^1_1)^2. \tag{19}$$

According to the degenerate Legendre transformation $E = G + TS - PV$, we have

$$\delta E = \delta(G + TS - PV) = \left[\frac{1}{2} f_R \delta r_+ + \frac{1}{4} r_+^2 (f - Rf_R)\delta r_+ + r_+ f_X B' R^3_3 \delta r_+ - r_+^2 f_X (R^1_1)^2\right]\delta r_+, \tag{20}$$

or equivalently

$$E = \int^{r_+}\left[\frac{1}{2} f_R + \frac{1}{4} r_+^2 (f - Rf_R) + r_+ f_X B' R^3_3 - r_+^2 f_X (R^1_1)^2\right]\mathrm{d}r_+. \tag{21}$$

When $f_X = 0$, Eq. (21) returns to the result obtained in $f(R)$ theory [28]. Using Eqs. (21) and (29), we can calculate the energy and the entropy of the black hole in $f(R, R^{\mu\nu}R_{\mu\nu})$ theory.

4 Application: quadratic-curvature gravity

For application, we consider a simple but important example: the most general quadratic-curvature gravity theory with a cosmological constant in four dimensions; its Lagrangian density is given by an arbitrary combination of scalar curvature-squared and Ricci-squared terms, namely,

$$f(R, R^{\mu\nu}R_{\mu\nu}) = R + \alpha R^{\mu\nu}R_{\mu\nu} + \lambda R^2 - 2\Lambda, \tag{22}$$

where α and λ are constants, and Λ is the cosmological constant. For this theory, we have $f_R = 1+2\lambda R$ and $f_X = \alpha$. We find from (29) that in spacetime with metric (13), the entropy is

$$S = \pi r_+^2(1+2\lambda R+2\alpha R_1^1). \tag{23}$$

Substituting Eq. (22) into Eq. (21), the energy of the black hole is given by

$$\begin{aligned} E =& \frac{1}{2}\int r_+^2\left[\frac{1}{r_+^2}+\frac{2\lambda R}{r_+^2}+\frac{1}{2}\alpha R^{\mu\nu}R_{\mu\nu}\right.\\ &\left.-\frac{1}{2}\lambda R^2+\frac{2\alpha B' R_3^3}{r_+}-2\alpha R_1^{1\,2}-\Lambda\right]dr_+\\ =& -\frac{1}{4}\int\Big[\left(6\alpha R_1^{1\,2}+2R_1^1\alpha B''+2\alpha R_2^{2\,2}+2\Lambda+\lambda R^2\right)r_+^2\\ &+4R_1^1\alpha r_+B'-2(2\alpha R_2^2+2\lambda R+1)\Big]\mathrm{d}r_+. \end{aligned} \tag{24}$$

In the case of a Schwarzschild-(A)ds black hole, for example, whose metric is given by $B(r)=1-\dfrac{2M}{r}-\dfrac{\Lambda r^2}{3}$, the entropy (23) and the energy (29), respectively, return to

$$S = \pi r_+^2(1+2\lambda R+2\alpha R_1^1) = \pi r_+^2[1+2(\alpha+4\lambda)\Lambda], \tag{25}$$

and

$$E = [1+2(\alpha+4\lambda)\Lambda]M, \tag{26}$$

where we have used $B(r_+) = 0$. Eqs. (25) and (26) are consistent with the results obtained in [34, 35]. The energy in [35] was computed using the Abbott-Deser-Tekin method and a qualitatively different way of regularizing the Iyer-Wald charges. On the other hand, in [34] the energy was calculated using the horizon first law after taking a Legendre transformation. When $\alpha = 0$, we obtain the results in $R+\lambda R^2-2\Lambda$ theory. While for $\alpha=0$ and $\lambda=0$, we get the results in Einstein's gravity with the cosmological constant.

5 Conclusion

We have discussed whether the new horizons first law is still valid in $f(R, R^{\mu\nu}R_{\mu\nu})$ theory. Unlike the approach taken in Einstein's gravity, we cannot directly derive the entropy and energy via Eqs. (5) and (6). We must first obtain the entropy using other methods, such as Wald formula, then we can use the new horizon first law, the degenerate Legendre transformation, and the gravitational field equations to derive the energy of the black hole in $f(R, R^{\mu\nu}R_{\mu\nu})$ theory. For application, we have considered quadratic-curvature gravity and have presented the entropy and the energy for a static spherically symmetric black hole, especially for a Schwarzschild-(A)ds black hole where the results are consistent with those obtained in the literature. Whether this procedure can be applied to other complicated cases, such as Einstein-Horndeski-Maxwell theory [36], is worth studying in the future. In future works, we also plan to compare our results with those obtained by using other methods such as the Misner-Sharp and Abbott-Deser-Tekin methods.

We thank J. Zhai for the helpful advice.

Appendix A

In the bulk of the present paper, we studied the entropy and energy of a black hole (13) in $f(R, R^{\mu\nu}R_{\mu\nu})$ theory. To confirm whether our results are reasonable, here we calculate the entropy by using the Wald formula, which takes the following form

$$S = -2\pi\oint\frac{\delta L}{\delta R_{abcd}}\epsilon_{ab}\epsilon_{cd}\,\mathrm{d}V_2, \tag{A1}$$

where L is Lagrangian density of the gravitational field, $\mathrm{d}V_2$ is the volume element on the bifurcation surface Σ, and ϵ_{ab} is the binormal vector to Σ normalized as $\epsilon_{ab}\epsilon^{ab}=-2$. For the metric (13) the binormal vectors can easily be found as $\epsilon_{01}=1$ and $\epsilon_{10}=-1$. For $L=\dfrac{f(R,R_{\mu\nu}R^{\mu\nu})}{16\pi}$, we can get

$$\begin{aligned} \frac{\delta L}{\delta R_{abcd}} &= \frac{1}{16\pi}\left(f_R\frac{\delta R}{\delta R_{abcd}}+f_X\frac{\delta R_{\mu\nu}R^{\mu\nu}}{\delta R_{abcd}}\right)\\ &= \frac{1}{16\pi}\left(g^{c[a}g^{b]d}f_R+2R^{\mu\nu}g^{\sigma\rho}\delta^a_{[\mu}\delta^b_{\sigma]}\delta^c_\nu\delta^d_\rho f_X\right). \end{aligned} \tag{A2}$$

For the metric (13), we have $g^{c[a}g^{b]d}\epsilon_{ab}\epsilon_{cd}=-2$ and $2R^{\mu\nu}g^{\sigma\rho}\delta^a_{[\mu}\delta^b_{\sigma]}\delta^c_\nu\delta^d_\rho\xi_{ab}\xi_{cd}=4R^{00}g^{11}=-4R_1^1$. Since the integral (27) is to be evaluated on shell, finally we have the entropy as

$$S = \frac{A(r_+)}{4}\left(f_R+2f_XR_1^1\right) \tag{A3}$$

in $f(R, R^{\mu\nu}R_{\mu\nu})$ theory for the black hole (13).

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