Tsallis' quantum q-fields

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Abstract: We generalize several well known quantum equations to a Tsallis' q-scenario, and provide a quantum version of some classical fields associated with them in the recent literature. We refer to the q-Schrödinger, q-Klein-Gordon, q-Dirac, and q-Proca equations advanced in, respectively, Phys. Rev. Lett. **106**, 140601 (2011), EPL **118**, 61004 (2017) and references therein. We also introduce here equations corresponding to q-Yang-Mills fields, both in the Abelian and non-Abelian instances. We show how to define the q-quantum field theories corresponding to the above equations, introduce the pertinent actions, and obtain equations of motion via the minimum action principle. These q-fields are meaningful at very high energies (TeV scale) for $q=1.15$, high energies (GeV scale) for $q=1.001$, and low energies (MeV scale) for $q=1.000001$ [Nucl. Phys. A **955** (2016) 16 and references therein]. (See the ALICE experiment at the LHC). Surprisingly enough, these q-fields are simultaneously q-exponential functions of the usual linear fields' logarithms.

Keywords: non-linear Klein-Gordon, non-linear Schrödinger and non-linear q-Dirac fields, non-linear q-Yang-Mills and non-linear q-Proca fields, classical field theory, quantum field theory

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1 Introduction

Classical field theories (CFT) associated with Tsallis' q-scenarios have received much attention recently [1–6]. Associated q-quantum field theories (q-QFTs) have also been discussed [2, 3].

These CFTs cannot be directly quantified because of non-linearity, which means the superposition principle is not applicable, and it is then impossible to introduce creation/annihilation operators for the q-fields. We will here remedy such a formidable quantification obstacle by recourse to an indirect approach.

In this paper we both extend to the quantum realm and generalize several aspects of the above mentioned works. We construct the CFTs corresponding to the q-Schrödinger, q-Klein-Gordon, and q-Dirac equations introduced in Refs. [1–4]. We do the same for the q-Proca and q-Yang-Mills (Abelian) defined in Ref. [5]. Also, and for the first time ever, we deal with the equation and q-QFT corresponding to a non-Abelian q-Yang-Mills field. It has been shown in Refs. [7, 8] that q-fields emerge at 1) very high energies (TeV) for $q=1.15$, 2) high energies (GeV) for $q=1.001$, and 3) low energies (MeV) for $q=1.000001$. The ALICE experiment at the LHC shows that Tsallis q-effects manifest themselves [9] at TeV energies.

We will see that all q-QFTs employed here transform into the well known associated QFTs for $q\rightarrow 1$, entailing going down from extremely high energies to lower ones.

Our new quantum field theory corresponds to non-linear equations. Thus, gauge and Lorentz invariance are broken. These invariances reappear in the limit $q\rightarrow 1$. A nice property of our new equation $\partial_\mu A^\mu=0$, is that as well as being valid for Abelian Yang-Mills and Proca fields, it is also valid for q-Abelian Yang-Mills and q-Proca fields.

M.A. Rego-Monteiro et al. [6] have tackled in recent years the possible need for two coupled fields, instead of only one, to properly handle classical non-linear equations. The quantification of these two coupled fields is discussed in Refs. [2] and [3].

Motivations for non-linear quantum evolution equations can be divided into two types: (A) basic equations governing phenomena at the frontiers of quantum mechanics, mainly at the boundary between quantum and gravitational physics (see Refs. [10, 11] and references therein); and (B) regarding non-linear Schrödinger-like equations (NLSE) as effective, single particle mean field descriptions of involved quantum many-body systems. A paradigmatic illustration is that of Ref. [12].

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2 Non-linear q-Schrödinger field

We proceed now to effect a transformation that requires some previous considerations. Consider two different formalisms A and B that can be connected by an appropriate mathematical transformation. Assume that we know how to solve the relevant equations for A. A legitimate question is why to bother at all with formalism B, that could be mathematically more involved than A. The answer is as follows. Even though A and B are mathematically connected, it is possible that, in some scenarios, the variables in B provide a more appropriate description of some natural phenomenon. There is some experimental evidence that such is the case with Tsallis-inspired non-linear wave equations (the ALICE experiment at CERN). Empirically, they find q-exponentials, that are solutions to the q-equations of motion, suggesting that Nature uses the non-standard scenario. Another example refers to the Schrödinger equation (SE) with variable mass, that has multiple applications. Here there exists a transformation connecting the SE with constant mass with the SE with variable mass. Why bother with such a transformation? Answer: in many problems in solid state physics, nuclear physics, etc., the relevant physics is described by the SE with variable mass. The transformation that we are advancing here reads:

$$\psi_q(\vec{x},t)=[1+(1-q)\ln\psi(\vec{x},t)]_+^{\frac{1}{1-q}}, \tag{1}$$

where ψ is the usual quantum Schrödinger field operator, and the subscript + indicates the so-called Tsallis cut-off.

At a quantum level, which is the case that we are interested in here, the cut-off has no relevance since ψ is an operator and the information is contained in the pertinent operators of creation and annihilation. At this level we have:

$$\begin{aligned}\psi_q &= [I+(1-q)\ln\psi]_+^{\frac{1}{1-q}}=\mathrm{e}^{\frac{1}{1-q}\ln\{[I+(1-q)\ln\psi]\}}\\ &=\sum_{n=0}^{\infty}a_n\frac{\phi^n}{n!}=I+\phi+\frac{(q-1)}{2}\phi^2+\ldots ,\end{aligned} \tag{2}$$

where $\psi=I+\phi$. There are no cuts or branch points then. No information is lost if one considers the whole series.

Consider now the classical instance in which ψ is just a plane wave

$$\psi(\vec{x},t)=\mathrm{e}^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-Et)}. \tag{3}$$

Replacing this into Eq. (1), we find

$$\psi_q(\vec{x},t)=[1+(1-q)\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-Et)]_+^{\frac{1}{1-q}}. \tag{4}$$

This is just the q-wave that Nobre et al. [1] used to obtain the q-Schrödinger, q-Klein-Gordon, and q-Dirac equations. Thus, the q-wave is a particular case of the quantum field defined by Eq. (1). This allows for immediate generalization to the quantum realm of the classical treatment of fields given in Ref. [1]. Accordingly, we can obtain quantum q-fields starting from the usual $q=1$ quantum fields. We can also express ψ in terms of ψ_q as

$$\psi=\mathrm{e}^{\frac{\psi_q^{(1-q)}-1}{1-q}} \tag{5}$$

The Schrödinger field action $\mathcal{S}$ is well known as

$$\mathcal{S}=\int\left(i\hbar\psi^\dagger\partial_t\psi-\frac{\hbar^2}{2m}\nabla\psi^\dagger\nabla\psi\right)\mathrm{d}t\mathrm{d}^3x. \tag{6}$$

From it one deduces the equation of motion

$$i\hbar\partial_t\psi+\frac{\hbar^2}{2m}\triangle\psi=0, \tag{7}$$

whose solution is

$$\psi(\vec{x},t)=\frac{1}{(2\pi\hbar)^{\frac{3}{2}}}\int a(\vec{p})\mathrm{e}^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-Et)}d^3p. \tag{8}$$

The action corresponding to the field ψ_q is

$$\begin{aligned}\mathcal{S}_q &= \int \mathrm{e}^{\frac{\psi_q^{(1-q)}-1}{1-q}}\mathrm{e}^{\frac{\psi_q^{\dagger(1-q)}-1}{1-q}}\psi_q^{-q}\\ &\quad\times\left(i\hbar\partial_t\psi-\frac{\hbar^2}{2m}\psi_q^{\dagger-q}\nabla\psi_q^\dagger\nabla\psi_q\right)\mathrm{d}t\mathrm{d}^3x,\end{aligned} \tag{9}$$

constructed keeping in mind that the field ψ_q satisfies

$$i\hbar\partial_t\psi_q+\frac{\hbar^2}{2m}[\triangle\psi_q+(\nabla\psi_q)^2(\psi_q^{-q}-q\psi_q^{-1})]=0. \tag{10}$$

Note that the q-exponential wave (6) is, by construction, a solution to Eq. (10). For $q\to1$ this last equation becomes the usual Schrödinger equation. The same is true for the action given by Eq. (9). One is then in a position to assert that such an action is the q-generalization of the usual one and that Eq. (10) is the q-generalization of the ordinary Schrödinger equation.

Additionally, since in Eq. (1) the field ψ is a quantum field, this implies that ψ_q is of such a nature too. Of course, for $q\to1$, ψ_q becomes ψ. Physically, if the energy goes down, the q-field transforms itself into the usual one (remember our assertions above on the connection between q-fields and the energy scale based on the work at ALICE at the LHC [7, 8]). Given that we speak here of a non-linear QFT, direct field quantification by appeal to creation-destruction operators is not feasible, since the superposition principle is no longer valid. The reasoning applies to the propagator notion as well. Thus, as we did here, an indirect route is necessary to quantify a classical field.

3 Non-linear q-Klein-Gordon (KG) field

In the same vein as above, we define a quantum q-Klein Gordon (KG) field $\phi_q(x_\mu)$ in terms of the ordinary KG field $\phi(x_\mu)$ as

$$\phi_q(x_\mu)=[1+(1-q)\ln\phi(x_\mu)]_+^{\frac{1}{1-q}}. \tag{11}$$

In the classical instance, if we have

$$\phi(x_\mu)=\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)}, \tag{12}$$

we re-obtain the q-wave used by Nobre et al. in Ref. [1]:

$$\phi_q(x_\mu)=[1+(1-q)\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)]_+^{\frac{1}{1-q}}. \tag{13}$$

ϕ can be given in terms of ϕ_q as

$$\phi=\mathrm{e}^{\frac{\phi_q^{(1-q)}-1}{1-q}}. \tag{14}$$

From Ref. (11) we see that ϕ_q is not Lorentz invariant (LI). We saw above that it manifests itself at very high energy. If the energy becomes smaller, and this happens for $q\to1$, ϕ_q becomes ϕ and LI is restored. The usual KG action is

$$\mathcal{S}=\int\left[\partial_\mu\phi(x_\mu)\partial^\mu\phi^\dagger(x_\mu)-m^2\phi(x_\mu)\phi^\dagger(x_\mu)\right]\mathrm{d}^4x_\mu, \tag{15}$$

from which one deduces

$$(\Box+m^2)\phi=0, \tag{16}$$

whose solution is

$$\phi(x_\mu)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{a(\vec{k})}{\sqrt{2\omega}}\mathrm{e}^{\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)}+\frac{a^\dagger(\vec{k})}{\sqrt{2\omega}}\mathrm{e}^{-\mathrm{i}(\vec{k}\cdot\vec{x}-\omega t)}\mathrm{d}^3k, \tag{17}$$

this being the field ϕ in Eq. (11). For ϕ_q one has

$$\mathcal{S}_q=\int\mathrm{e}^{\frac{\phi_q^{(1-q)}-1}{1-q}}\mathrm{e}^{\frac{\phi_q^{\dagger(1-q)}-1}{1-q}}\left(\phi_q^{-q}\phi_q^{\dagger-q}\partial_\mu\phi_q\partial_\mu\phi_q^\dagger-m^2\right)\mathrm{d}^4x_\mu, \tag{18}$$

leading to an equation of motion whose solution is ϕ_q, that is

$$\Box\phi_q+\partial_\mu\phi_q\partial^\mu\phi_q(\phi_q^{-q}-q\phi_q^{-1})+m^2\phi_q^q=0. \tag{19}$$

For $q\to1$, Eq. (18) becomes Eq. (15) while Eq. (19) goes to Eq. (16).

4 Non-linear q-Dirac field

Dirac's action is known to be:

$$\mathcal{S}=\int i\overline{\psi}\partial\!\!\!/\psi-m\overline{\psi}\psi\mathrm{d}^4x, \tag{20}$$

or

$$\mathcal{S}=\int\mathrm{i}\psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi-m\psi^\dagger\gamma^0\psi\mathrm{d}^4x. \tag{21}$$

In terms of Dirac's spinor ψ this action is

$$\mathcal{S}=\int\mathrm{i}\psi_a^\dagger(\gamma^0\gamma^\mu)_{ab}\partial_\mu\psi_b-m\psi_a^\dagger\gamma_{ab}^0\psi_b\mathrm{d}^4x. \tag{22}$$

We deduce now that the spinor's components obey the equations of motion

$$\mathrm{i}\gamma_{ab}^\mu\partial_\mu\psi_b-m\psi_a=0, \tag{23}$$

$$(\Box+m^2)\psi_a=0, \tag{24}$$

that, of course, are the Dirac and Klein-Gordon equations, respectively. We define now a very high energy field ψ_{qa} as

$$\psi_{qa}=[1+(1-q)\ln\psi_a]_+^{\frac{1}{1-q}}, \tag{25}$$

not Lorentz invariant. ψ_{qa} is not a component of the Dirac-spinor. Let us now cast ψ_a in terms of ψ_{qa}:

$$\psi_a=\mathrm{e}^{\frac{\psi_{qa}^{(1-q)}-1}{1-q}}. \tag{26}$$

The ψ_{qa}-associated action is

$$\mathcal{S}_q=\sum_{ab}\int\mathrm{i}\mathrm{e}^{\frac{\psi_{qa}^{\dagger(1-q)}-1}{1-q}}(\gamma^0\gamma^\mu)_{ab}\psi_{qb}^{\dagger(-q)}\partial_\mu\psi_{qb}\mathrm{e}^{\frac{\psi_{qb}^{(1-q)}-1}{1-q}} -m\mathrm{e}^{\frac{\psi_{qa}^{\dagger(1-q)}-1}{1-q}}\gamma_{ab}^0\mathrm{e}^{\frac{\psi_{qb}^{(1-q)}-1}{1-q}}\mathrm{d}^4x_\mu. \tag{27}$$

Given the lack of Lorentz invariance, Einstein's convention on repeated indexes cannot be used. This action becomes that of Eq. (22) for $q\to1$. From Eq. (27) one deduces the equations of motion for ψ_q as

$$\mathrm{i}\gamma_{ab}^\mu\psi_{qb}^{-q}\partial_\mu\psi_{qb}\mathrm{e}^{\frac{\psi_{qb}^{(1-q)}-1}{1-q}}-m\mathrm{e}^{\frac{\psi_{qa}^{(1-q)}-1}{1-q}}=0 \tag{28}$$

$$\Box\psi_{qa}+\partial_\mu\psi_{qa}\partial^\mu\psi_{qa}(\psi_{qa}^{-q}-q\psi_{qa}^{-1})+m^2\psi_{qa}^q=0, \tag{29}$$

that become Eq. (23) – Eq. (24) when $q\to1$. The energy considerations in this limit made above in the KG case also hold here.

5 Advancing a non-linear Abelian Yang-Mills q-field

It is well known that the action for an Abelian Yang-Mills field reads

$$\mathcal{S}=-\frac{1}{4}\int\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\mathrm{d}^4x, \tag{30}$$

where

$$\mathcal{F}_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \tag{31}$$

and the associated equation of motion is

$$\partial_\mu\mathcal{F}^{\mu\nu}=0, \tag{32}$$

which can be recast as two equations

$$\Box A_\mu=0\ \partial_\mu A^\mu=0. \tag{33}$$

Our present q-extension begins by defining

$$A_{q\mu}=[1+(1-q)\ln A_\mu]_+^{\frac{1}{1-q}}, \tag{34}$$

breaking Lorentz invariance (LI) once again. Conversely, we can write

$$A_\mu=\mathrm{e}^{\frac{A_{q\mu}^{(1-q)}-1}{1-q}}, \tag{35}$$

leading to

$$\partial^\mu A_\mu=\mathrm{e}^{\frac{A_{q\mu}^{(1-q)}-1}{1-q}}A_{q\mu}^{-q}\partial^\mu A_{q\mu}=0, \tag{36}$$

and then

$$\partial^\mu A_{q\mu}=0, \tag{37}$$

so that the field $A_{q\mu}$ fulfills the Lorentz gauge, a surprising result given the above LI-breaking. Our associated q-action $A_{q\mu}$ is

$$\begin{aligned}\mathcal{S}_q = -\frac{1}{4}\sum_{\mu,\nu,\rho,\eta} g^{\mu\rho}g^{\nu\eta}\int\Big[&\mathrm{e}^{\frac{A_{q\eta}^{(1-q)}-1}{1-q}}A_{q\eta}^{-q}\partial_\rho A_{q\eta}\\ &-\mathrm{e}^{\frac{A_{q\rho}^{(1-q)}-1}{1-q}}A_{q\rho}^{-q}\partial_\eta A_{q\rho}\Big]\\ &\otimes\Big[\mathrm{e}^{\frac{A_{q\nu}^{(1-q)}-1}{1-q}}A_{q\nu}^{-q}\partial_\mu A_{q\nu}-\mathrm{e}^{\frac{A_{q\mu}^{(1-q)}-1}{1-q}}A_{q\mu}^{-q}\partial_\nu A_{q\mu}\Big]\mathrm{d}^4x,\end{aligned} \tag{38}$$

leading to the equation of motion

$$\Box A_{q\mu}+\partial_\nu A_{q\mu}\partial^\nu A_{q\mu}(A_{q\mu}^{-q}-qA_{q\mu}^{-1})=0, \tag{39}$$

obeyed by $A_{q\mu}$. It is clear that for $q\rightarrow1$ our new theory becomes the customary Abelian Yang-Mills theory.

6 Introducing our non-linear non-Abelian Yang-Mills q-field

The Yang-Mills theory is a gauge theory, constructed from a Lie algebra, that attempts to describe the behavior of elementary particles via non-Abelian Lie groups. This lies at the core of i) the unification of the weak and electromagnetic forces, as well as ii) quantum chromodynamics. It constitutes the foundation of our understanding of the standard model. The corresponding action is

$$\mathcal{S}=-\frac{1}{2g^2}\int \mathrm{tr}(\boldsymbol{\mathcal{F}}^{\mu\nu}\boldsymbol{\mathcal{F}}_{\mu\nu})\mathrm{d}^4x, \tag{40}$$

where

$$\boldsymbol{\mathcal{F}}^{\mu\nu}=\mathcal{F}_C^{\mu\nu}T^C. \tag{41}$$

Here the matrices T^C correspond to a non-Abelian, semi-simple Lie group. One has

$$[T_A,T_B]=f_{AB}^CT^C, \tag{42}$$

$$\mathrm{tr}(T_AT_B)=\frac{\delta_{AB}}{2}, \tag{43}$$

where $\boldsymbol{\mathcal{F}}^{\mu\nu}$ is

$$\boldsymbol{\mathcal{F}}^{\mu\nu}=\partial_\mu\boldsymbol{A}_\nu-\partial_\nu\boldsymbol{A}_\mu-\mathrm{i}g[\boldsymbol{A}_\mu,\boldsymbol{A}_\nu], \tag{44}$$

with

$$\boldsymbol{A}_\mu=A_\mu^CT_C \tag{45}$$

and

$$\mathcal{F}_{\mu\nu}^C=\partial_\mu A_\nu^C-\partial_\nu A_\mu^C+gf_{AB}^CA_\mu^AA_\nu^B. \tag{46}$$

Because of the relation

$$\mathrm{tr}(\boldsymbol{\mathcal{F}}^{\mu\nu}\boldsymbol{\mathcal{F}}_{\mu\nu})=\frac{1}{2}\mathcal{F}^{\mu\nu C}\mathcal{F}_{\mu\nu C}, \tag{47}$$

the action becomes

$$\mathcal{S}=-\frac{1}{4g^2}\int\mathcal{F}^{\mu\nu C}\mathcal{F}_{\mu\nu C}\mathrm{d}^4x, \tag{48}$$

leading to the equation of motion

$$\partial_\rho\mathcal{F}^{\rho\sigma D}-g\mathcal{F}^{\rho\sigma C}f_{AD}^CA_\rho^A=0. \tag{49}$$

We now define our q-extension

$$A_{q\mu}^C=[1+(1-q)\ln A_\mu^C]_+^{\frac{1}{1-q}}, \tag{50}$$

again breaking both Lorentz and gauge invariance for $q>1$. From Eq. (50) we obtain

$$A_\mu^C=\mathrm{e}^{\frac{A_{q\mu}^C-1}{1-q}}, \tag{51}$$

and the action associated with the field (50) is

$$\mathcal{S} = -\frac{1}{4g^2}\sum_{\mu,\nu,C}g^{\mu\mu}g^{\nu\nu}\int\mathcal{F}_{q\mu\nu}^C\mathcal{F}_{q\mu\nu}^C\mathrm{d}^4x, \tag{52}$$

where

$$\begin{aligned}\mathcal{F}_{q\mu\nu}^C = &A_{q\nu}^{-qC}\mathrm{e}^{\frac{A_{q\nu}^C-1}{1-q}}\partial_\mu A_{q\nu}^C-A_{q\mu}^{-qC}\mathrm{e}^{\frac{A_{q\mu}^C-1}{1-q}}\partial_\nu A_{q\mu}^C\\ &+gf_{AB}^C\sum_{A,B}\mathrm{e}^{\frac{A_{q\mu}^A-1}{1-q}}\mathrm{e}^{\frac{A_{q\nu}^B-1}{1-q}},\end{aligned} \tag{53}$$

leading to the equation of motion

$$\sum_\rho g^{\rho\rho}\partial_\rho\mathcal{F}_{q\rho\sigma}^D-g\sum_{\rho AC}g^{\rho\rho}\mathcal{F}_{q\rho\sigma}^Cf_{AD}^C\mathrm{e}^{\frac{A_{q\rho}^A-1}{1-q}}=0. \tag{54}$$

The field $A_{q\mu}^C$ satisfies this equation, of course. Whenever the energy becomes low enough, $q\rightarrow1$, and one recovers LI and gauge invariance.

7 Our non-linear quantum Proca q-field

The Proca action gives a detailed account of a massive spin-1 field of mass m in a Minkowskian space-time. The associated equation is a relativistic-wave equation, called the Proca equation. The action is

$$\mathcal{S}=-\frac{1}{2}\int\mathcal{F}^{\dagger\mu\nu}\mathcal{F}_{\mu\nu}-2m^2A_\mu^\dagger A^\mu\mathrm{d}^4x, \tag{55}$$

where

$$\mathcal{F}_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{56}$$

the equations of motion being

$$(\Box+m^2)A_\mu=0\ \partial_\mu A^\mu=0. \tag{57}$$

At this stage we define our q-action

$$A_{q\mu}=[1+(1-q)\ln A_\mu]^{\frac{1}{1-q}}. \tag{58}$$

breaking LI. Inversion of Eq. (58) gives

$$A_\mu=\mathrm{e}^{\frac{A_{q\mu}^{(1-q)}-1}{1-q}} \tag{59}$$

From the second relation in Eq. (57) and from Eq. (59) we find

$$\partial^\mu A_{q\mu}=0, \tag{60}$$

whose associated action is

$$\mathcal{S} = -\frac{1}{2}\int \sum_{\mu,\nu} g^{\mu\mu}g^{\nu\nu}\mathcal{F}_q^{\dagger\mu\nu}\mathcal{F}_{q\mu\nu} -2m^2\sum_{\mu}\mathrm{e}^{\frac{A_{q\mu}^{\dagger(1-q)}-1}{1-q}}\mathrm{e}^{\frac{A_{q\mu}^{(1-q)}-1}{1-q}}\mathrm{d}^4x, \tag{61}$$

with

$$\mathcal{F}_{q\mu\nu}=A_{q\nu}^{-q}\mathrm{e}^{\frac{A_{q\nu}^{(1-q)}-1}{1-q}}\partial_\mu A_{q\nu}-A_{q\mu}^{-q}\mathrm{e}^{\frac{A_{q\mu}^{(1-q)}-1}{1-q}}\partial_\nu A_{q\mu}. \tag{62}$$

From both this and Eq. (60) one finds the equation of motion

$$\Box A_{q\mu}+(A_{q\mu}^{-q}-qA_{q\mu}^{-1})\sum_\nu g^{\nu\nu}(\partial_\nu A_{q\mu})^2+m^2A_{q\mu}^q=0, \tag{63}$$

satisfied by $A_{q\mu}$. LI is recovered in the limit $q\rightarrow 1$.

8 Conclusions

We have obtained some new quantum results that may be regarded as interesting.

More specifically, we have generalized to the quantum realm the classical Tsallis' q-Schrödinger, q-Klein-Gordon, q-Dirac, and q-Proca equations obtained in Refs. [1–6]. We have also added equations corresponding to q-Yang-Mills fields, both Abelian and non-Abelian.

We have obtained the q-quantum field theories corresponding to all the above equations, and showed that in the limit $q\rightarrow 1$ they become the customary ones.

These results agree with our previous results [7, 8] concerning the energies involved. One needs energies of up to 1 TeV in order to clearly distinguish between q-theories and $q=1$, ordinary ones.

All our new quantum q-fields are q-exponential functions of the logarithms of the conventional $q=1$ fields. We have seen that these cannot be directly quantified because of non-linearity, which makes the superposition principle non-applicable, and then it is impossible to introduce creation/annihilation operators for the q-fields. To remedy such a formidable quantification obstacle we have here devised an indirect approach that has been shown to work correctly.

An interesting fact is that a Tsallis' q-exponential wave is a solution of the equations of motion (10), (19), (28), (29), (39), (54), and (63), although these all look quite different!

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