T-duality as permutation of coordinates in double space*

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Abstract: We introduce the $2D$ dimensional double space with the coordinates $Z^M=(x^\mu, y_\mu)$, whose components are the coordinates of initial space x^μ and its T-dual y_μ. We shall show that in this extended space the T-duality transformations can be realized simply by exchanging the places of some coordinates x^a, along which we want to perform T-duality, and the corresponding dual coordinates y_a. In such an approach it is evident that T-duality leads to the physically equivalent theory and that a complete set of T-duality transformations forms a subgroup of the $2D$ permutation group. So, in double space we are able to represent the backgrounds of all T-dual theories in a unified manner.

Keywords: T-duality, double space, string theory

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1 Introduction

The T-duality of closed strings has been investigated for a long time [1–4]. It transforms the theory of a string moving in a toroidal background into the theory of a string moving in a different toroidal background. Generally, one suppose that the background has some continuous isometries which leaves the action invariant. In suitable adopted coordinates, where the isometry acts as translation, it means that the background does not depend on a particular set of coordinates.

In Ref. [5] a new procedure for T-duality of a closed string, moving in D dimensional weakly curved space, was considered. The generalized approach allows one to perform T-duality along coordinates on which the Kalb-Ramond field depends. In that article, T-duality transformations were performed simultaneously along all coordinates. It corresponds to the $T^{\rm full}=T^0\circ T^1\circ\cdots\circ T^{D-1}$ -duality relation with transformation of the coordinates $y_\mu=y_\mu(x^\mu)$ connecting the beginning and the end of the T-duality chain

$$\Pi_{\pm\mu\nu}, x^\mu \underset{\mathrm{T}_1}{\overset{\mathrm{T}^1}{\rightleftharpoons}} \Pi_{1\pm\mu\nu}, x_1^\mu \underset{\mathrm{T}_2}{\overset{\mathrm{T}^2}{\rightleftharpoons}} \Pi_{2\pm\mu\nu},$$

$$x_2^\mu \underset{\mathrm{T}_3}{\overset{\mathrm{T}^3}{\rightleftharpoons}} \dots \underset{\mathrm{T}_D}{\overset{\mathrm{T}^D}{\rightleftharpoons}} \Pi_{D\pm\mu\nu}={}^\star\Pi_{\pm\mu\nu}, x_D^\mu=y_\mu. \quad (1.1)$$

Here $\Pi_{i\pm\mu\nu}$ and $x_i^\mu, (i=1,2,\cdots,D)$ are background fields and the coordinates of the corresponding configurations respectively. Applying the proposed procedure $T_{\rm full}=T_0\circ T_1\circ\cdots\circ T_{D-1}$ to the T-dual theory, one can obtain the initial theory and the inverse duality relation $x^\mu=x^\mu(y_\mu)$, connecting the end and the beginning of the T-duality chain. For simplicity, in Ref. [5] T-duality was performed along all directions.

The nontrivial extension of this approach, compared with the flat space case, is a source of closed string noncommutativity [6–8]. From the canonical point of view considered in Ref. [8], there is similarity between open and closed string non-commutativity. In both cases, the initial coordinates are given not only as a functions of some effective coordinates but as a linear combination of the effective coordinates and the effective momenta. It produces nonzero Poisson brackets between coordinates. In the open string case, such a relation is a solution of boundary conditions and only endpoints, attached to the Dp-brane, are non-commutative, even in flat space.

A closed string does not have endpoints and the boundary conditions are satisfied automatically. To understand closed string non-commutativity, we should impose T-duality transformation laws and express the closed string coordinates of T-dual theory in terms of the coordinates and momenta of the original theory. Then the standard Poisson brackets of the original theory induce nontrivial Poisson brackets between coordinates in

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the T-dual theory, which are proportional to the background fluxes multiplied by the winding and momenta numbers. In order to obtain such a feature, we need to introduce background fields which depend on the coordinates. The simplest example is a weakly curved background.

In D-dimensional space it is possible to perform T-duality along any subset of coordinates x^a : $T^a = T^0 \circ T^1 \circ \cdots \circ T^{d-1}$, and along the corresponding T-dual ones $y_a : T_a = T_0 \circ T_1 \circ \cdots \circ T_{d-1}$,$(a = 0,1,\cdots,d-1)$. In Ref. [9] this was done for a string moving in a weakly curved background. For each case the T-dual actions, T-dual background fields and T-duality transformations were obtained. Let us stress that the T-dualization $\mathcal{T}^a = T^a \circ T_a$ of the present paper in the $2D$ dimensional double space contains two T-dualizations in the terminology of Ref. [9]. In fact the D dimensional T-dualizations T^a and T_a of the present paper are denoted $\mathcal{T}^a$ and $\mathcal{T}_a$ in Ref.[9].

The introduction of the extended space of double dimensions with coordinates $Z^M = (x^\mu, y_\mu)$ wil help us to reproduce all the results of Ref. [9] and offer a simple explanation for T-duality. In the present article we will demonstrate this for a flat background, while for a weakly curved background it will be presented elsewhere [10]. For example, T-duality T^{μ_1} (along fixed coordinate x^{μ_1}) and T-duality T_{μ_1} (along corresponding dual coordinate y_{μ_1}) can be performed simply by exchanging the places of the coordinates x^{μ_1} and y_{μ_1} in double space. It can be realized just by multiplying Z^M by a constant $2D \times 2D$ matrix. Similarly, an arbitrary T-duality $\mathcal{T}^a = T^a \circ T_a$ can be realized by exchanging the places of the coordinates $x^{\mu_1}, x^{\mu_2}, \cdots, x^{\mu_{d-1}}$ with the corresponding dual coordinates $y_{\mu_1}, y_{\mu_2}, \cdots, y_{\mu_{d-1}}$. From this explanation it is clear that T-duality leads to the equivalent theory, because a permutation of the coordinates in double space cannot change the physics.

A similar approach to T-duality, as a transformation in double space, appeared a long time ago [11]–[15]. Interest in this topic emerged again with Refs. [16, 17]. In Ref. [11], the beginning and the nd of the chain (1.1) was established. The relation of our approach and Ref. [16] will be discussed in Section 4.

The basic tools in our approach are T-duality transformations connecting the beginning and end of the chain. Rewriting these transformations in double space we obtain the fundamental expression, where the generalized metric relates derivatives of the extended coordinates. We will show that this expression is enough to find background fields from every node of the chain and T-duality transformations between arbitrary nodes. In such a way we unify the beginning and all corresponding T-dual theories of the chain (1.1).

2 T-duality in double space

Let us consider a closed bosonic string which propagates in D-dimensional space-time described by the action [18]

$$S[x] = \kappa \int_\Sigma \mathrm{d}^2\xi \sqrt{-g} \Big[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}[x] + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}[x] \Big] \partial_\alpha x^\mu \partial_\beta x^\nu, \quad (\varepsilon^{01} = -1). \tag{2.1}$$

The string, with coordinates $x^\mu(\xi)$, $\mu = 0,1,\cdots,D-1$ is moving in a non-trivial background, defined by the space-time metric $G_{\mu\nu}$ and the Kalb-Ramond field $B_{\mu\nu}$. Here $g_{\alpha\beta}$ is the intrinsic world-sheet metric and the integration goes over the two-dimensional world-sheet Σ with coordinates ξ^α ($\xi^0 = \tau$, $\xi^1 = \sigma$).

The requirement of world-sheet conformal invariance on the quantum level leads to the space-time equations of motion, which in the lowest order in slope parameter α', for the constant dilaton field $\Phi = \text{const}$ are

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0, \qquad D_\rho B^\rho{}_{\mu\nu} = 0. \tag{2.2}$$

Here $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are the Ricci tensor and covariant derivative with respect to the space-time metric respectively.

We will consider the simplest solutions of (2.2)

$$G_{\mu\nu} = \text{const}, \quad B_{\mu\nu} = \text{const}, \tag{2.3}$$

which satisfies the space-time equations of motion.

Choosing the conformal gauge $g_{\alpha\beta} = \mathrm{e}^{2F}\eta_{\alpha\beta}$, and introducing light-cone coordinates $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\partial_\pm = \partial_\tau \pm \partial_\sigma$, the action (2.1) can be rewritten in the form

$$S = \kappa \int_\Sigma \mathrm{d}^2\xi\ \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu, \tag{2.4}$$

where

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}. \tag{2.5}$$

2.1 Standard sigma-model T-duality

Applying the T-dualization procedure to all the coordinates, we obtain the T-dual action [5]

$$S[y] = \kappa \int \mathrm{d}^2\xi\ \partial_+ y_\mu {}^\star\Pi_+^{\mu\nu} \partial_- y_\nu = \frac{\kappa^2}{2} \int \mathrm{d}^2\xi\ \partial_+ y_\mu \theta_-^{\mu\nu} \partial_- y_\nu, \tag{2.6}$$

where

$$\theta_\pm^{\mu\nu} \equiv -\frac{2}{\kappa} (G_E^{-1} \Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa} (G_E^{-1})^{\mu\nu}. \tag{2.7}$$

Here we consider a flat background and omit the argument dependence of Ref. [5]. The symmetric and antisymmetric parts of $\theta^{\mu\nu}_{\pm}$ are the inverse of the effective metric $G^E_{\mu\nu}$ and the non-commutativity parameter $\theta^{\mu\nu}$

$$G^E_{\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \qquad \theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}. \tag{2.8}$$

Consequently, the T-dual background fields are

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad ^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}. \tag{2.9}$$

Note that the dual effective metric is just the inverse of the initial one

$$^\star G_E^{\mu\nu} \equiv {}^\star G^{\mu\nu} - 4({}^\star B {}^\star G^{-1} {}^\star B)^{\mu\nu} = (G^{-1})^{\mu\nu}, \tag{2.10}$$

and the following relations are valid:

$$({}^\star B {}^\star G^{-1})^\mu{}_\nu = -(G^{-1}B)^\mu{}_\nu, \qquad ({}^\star G^{-1} {}^\star B)_\mu{}^\nu = -(BG^{-1})_\mu{}^\nu. \tag{2.11}$$

2.2 T-duality transformations

The T-duality transformations between all initial coordinates x^μ and all dual coordinates y_μ of the closed string theory have been derived in Ref. [5]

$$\partial_\pm x^\mu \cong -\kappa\theta^{\mu\nu}_\pm \partial_\pm y_\nu, \qquad \partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu. \tag{2.12}$$

They are inverse to one another. We omit argument dependence and $\beta^\pm_\mu$ functions because they appear only in the weakly curved background.

We can put the above T-duality transformations in a useful form, where on the left-hand side we put the terms with world-sheet antisymmetric tensor $\varepsilon_\alpha{}^\beta$ (note that $\varepsilon_\pm{}^\pm = \pm 1$):

$$\pm\partial_\pm y_\mu \cong G^E_{\mu\nu}\partial_\pm x^\nu - 2[BG^{-1}]_\mu{}^\nu \partial_\pm y_\nu, \qquad \pm\partial_\pm x^\mu \cong 2[G^{-1}B]^\mu{}_\nu \partial_\pm x^\nu + (G^{-1})^{\mu\nu}\partial_\pm y_\nu. \tag{2.13}$$

Let us introduce the $2D$ dimensional double target space, which will play an important role in the present article. It contains both initial and T-dual coordinates

$$Z^M = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix}. \tag{2.14}$$

Here, as well as in double field theory (for recent reviews see Refs. [19]–[22]), all coordinates are doubled. It differs from the approach of Ref. [16] where only coordinates on the torus along which we perform T-dualization are doubled. The relation of this work and that of Ref. [16] will be discussed in Section 4.

In terms of double space coordinates we can rewrite the T-duality relations (2.13) in the simple form

$$\partial_\pm Z^M \cong \pm\Omega^{MN}\mathcal{H}_{NK}\partial_\pm Z^K, \tag{2.15}$$

where

$$\Omega^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.16}$$

is a constant symmetric matrix and we introduced the so called generalized metric as

$$\mathcal{H}_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}. \tag{2.17}$$

It is easy to check that

$$\mathcal{H}^T\Omega\mathcal{H} = \Omega. \tag{2.18}$$

As noticed in Ref. [11], the relation (2.18) shows that there exists manifest $O(D,D)$ symmetry. In double field theory it is usual to call Ω^{MN} the $O(D,D)$ invariant metric and denote it by η^{MN}.

2.3 Equations of motions as consistency condition of T-duality relations

It is well known that the equation of motion and the Bianchi identity of the original theory are equal to the Bianchi identity and the equation of motion of the T-dual theory [5, 7, 11, 23]. The consistency conditions of the relations (2.15)

$$\partial_+[\mathcal{H}_{MN}\partial_- Z^N] + \partial_-[\mathcal{H}_{MN}\partial_+ Z^N] \cong 0, \tag{2.19}$$

in components take the forms

$$\partial_+\partial_- x^\mu \cong 0, \qquad \partial_+\partial_- y_\nu \cong 0. \tag{2.20}$$

They are the equations of motion for both initial and T-dual theories.

The expression (2.19) originated from conservation of the topological currents $i^{\alpha M} = \varepsilon^{\alpha\beta}\partial_\beta Z^M$. It is often called the Bianchi identity. In this sense T-duality in the double space unites the equations of motion and Bianchi identities in a single relation (2.19) as shown in Ref. [11].

We can write the action

$$S = \frac{\kappa}{4}\int d^2\xi\, \partial_+ Z^M \mathcal{H}_{MN}\partial_- Z^N, \tag{2.21}$$

which variation produces Eq. (2.19).

3 T-duality as permutation of coordinates in double space

Let us mark the T-dualization along some direction x^{μ_1} by T^{μ_1}, and its inverse along the corresponding direction y_{μ_1} by T_{μ_1}. Up to now we collected the results from T-dualizations along all directions x^μ ($\mu=0,1,\cdots,D-1$), $T^{full}=T^0\circ T^1\circ\cdots\circ T^{D-1}$ and from its inverse along all directions y_μ $T_{full}=T_0\circ T_1\circ\cdots\circ T_{D-1}$. So, the relation (2.15) in fact contains T-dualizations along all directions x^μ and y_μ $\mathcal{T}=T^{\rm full}\circ T_{\rm full}$.

In this section we will show that relation (2.15) contains information about any individual T-dualizations along some direction x^{μ_1} and the corresponding one y_{μ_1} for fixed μ_1 ($\mathcal{T}^{\mu_1}=T^{\mu_1}\circ T_{\mu_1}$). Applying the same procedure to an arbitrary subset of directions we will be able to obtain all possible T-dualizations. This means that we are able to connect any two backgrounds in the chain (1.1) and treat all theories connected by T-dualities in a unified manner.

Let us split the coordinate index μ into a and i ($a=0,\cdots,d-1$, $i=d,\cdots,D-1$), and perform T-dualization along directions x^a and y_a

$$\mathcal{T}^a=T^a\circ T_a,\quad T^a\equiv T^0\circ T^1\circ\cdots\circ T^{d-1},\quad T_a\equiv T_0\circ T_1\circ\cdots\circ T_{d-1}. \tag{3.1}$$

We will show that such T-dualization can be obtained just by exchanging the places of coordinates x^a and y_a. Note that double space contains coordinates of two spaces which are totally dual relative to one another. In the beginning these two theories are the initial one $S(x^\mu)$ and its T-dual along all coordinates $S(y_\mu)$. Arbitrary T-dualization in the double space along d coordinate with index a, $\mathcal{T}^a$, transforms at the same time $S(x^\mu)$ to $S[y_a,x^i]$ and $S(y_\mu)$ to $S[x^a,y_i]$. The obtained theories are also totally T-dual relative to one another.

3.1 Coordinate permutations in double space

We can realize permutation of the initial coordinates x^a with its T-dual y_a by multiplying the double space coordinate (2.14), now written as

$$Z^M=\begin{pmatrix} x^a\\ x^i\\ y_a\\ y_i\end{pmatrix}, \tag{3.2}$$

by the constant symmetric matrix $(\mathcal{T}^a)^T=\mathcal{T}^a$

$$\mathcal{T}^{a\,M}{}_N=\begin{pmatrix} 1-P_a & P_a\\ P_a & 1-P_a\end{pmatrix}=\begin{pmatrix} 0&0&1_a&0\\ 0&1_i&0&0\\ 1_a&0&0&0\\ 0&0&0&1_i\end{pmatrix}. \tag{3.3}$$

Here P_a is $D\times D$ projector with d units on the main diagonal

$$P_a=\begin{pmatrix} 1_a & 0\\ 0 & 0\end{pmatrix}, \tag{3.4}$$

where 1_a and 1_i are d and $D-d$ dimensional identity matrices. In Ref. [3] this transformation is called factorized duality.

Note also that

$$(\mathcal{T}^a\mathcal{T}^a)^M{}_N=\delta^M{}_N,\quad (\Omega\mathcal{T}^a\Omega)^M{}_N=(\mathcal{T}^a)^M{}_N,\quad \mathcal{T}^a\Omega\mathcal{T}^a=\Omega. \tag{3.5}$$

The last relation means that $\mathcal{T}^a\in SO(D,D)$. More precisely, we will see that $\mathcal{T}^a$ is in fact an element of the permutation group, which is a subgroup of $SO(D,D)$.

We will require that the dual extended space coordinate,

$$Z_a^M=\mathcal{T}^{a\,M}{}_N Z^N=\begin{pmatrix} y_a\\ x^i\\ x^a\\ y_i\end{pmatrix}, \tag{3.6}$$

satisfies the same form of the T-duality transformations (2.15) as the initial one

$$\partial_\pm Z_a^M\cong\pm\,\Omega^{MN}{}_a\mathcal{H}_{NK}\,\partial_\pm Z_a^K. \tag{3.7}$$

Consequently, with the help of the second equation (3.5) we find the dual generalized metric

$${}_a\mathcal{H}=\mathcal{T}^a\mathcal{H}\mathcal{T}^a, \tag{3.8}$$

or explicitly

$${}_a\mathcal{H}_{MN}=\begin{pmatrix} (G^{-1})^{ab} & 2(G^{-1}b)^a{}_j & 2(G^{-1}b)^a{}_b & (G^{-1})^{aj}\\ -2(bG^{-1})_i{}^b & g_{ij} & g_{ib} & -2(bG^{-1})_i{}^j\\ -2(bG^{-1})_a{}^b & g_{aj} & g_{ab} & -2(bG^{-1})_a{}^j\\ (G^{-1})^{ib} & 2(G^{-1}b)^i{}_j & 2(G^{-1}b)^i{}_b & (G^{-1})^{ij}\end{pmatrix}. \tag{3.9}$$

3.2 Explicit form of T-duality transformations

Rewriting Eq. (3.7) in components we get

$$\begin{aligned}
\pm\partial_\pm y_a &\cong -2(bG^{-1})_a{}^b\partial_\pm y_b + g_{aj}\partial_\pm x^j + g_{ab}\partial_\pm x^b - 2(bG^{-1})_a{}^j\partial_\pm y_j \\
\pm\partial_\pm x^i &\cong (G^{-1})^{ib}\partial_\pm y_b + 2(G^{-1}b)^i{}_j\partial_\pm x^j + 2(G^{-1}b)^i{}_b\partial_\pm x^b + (G^{-1})^{ij}\partial_\pm y_j \\
\pm\partial_\pm x^a &\cong (G^{-1})^{ab}\partial_\pm y_b + 2(G^{-1}b)^a{}_j\partial_\pm x^j + 2(G^{-1}b)^a{}_b\partial_\pm x^b + (G^{-1})^{aj}\partial_\pm y_j \\
\pm\partial_\pm y_i &\cong -2(bG^{-1})_i{}^b\partial_\pm y_b + g_{ij}\partial_\pm x^j + g_{ib}\partial_\pm x^b - 2(bG^{-1})_i{}^j\partial_\pm y_j\,.
\end{aligned} \tag{3.10}$$

Eliminating y_i from the second and third equations we find

$$\Pi_{\mp ab}\partial_\pm x^b + \Pi_{\mp ai}\partial_\pm x^i + \frac{1}{2}\partial_\pm y_a \cong 0. \tag{3.11}$$

Multiplication by $2\kappa\hat\theta^{ab}_\pm$, which according to (A10) is the inverse of $\Pi_{\mp ab}$, gives

$$\partial_\pm x^a \cong -2\kappa\hat\theta^{ab}_\pm\Pi_{\mp bi}\partial_\pm x^i - \kappa\hat\theta^{ab}_\pm\partial_\pm y_b. \tag{3.12}$$

Similarly, eliminating y_a from the second and third equations we get

$$\Pi_{\mp ij}\partial_\pm x^j + \Pi_{\mp ia}\partial_\pm x^a + \frac{1}{2}\partial_\pm y_i \cong 0, \tag{3.13}$$

which after multiplication with $2\kappa\hat\theta^{ij}_\pm$, the inverse of $\Pi_{\mp ij}$, produces

$$\partial_\pm x^i \cong -2\kappa\hat\theta^{ij}_\pm\Pi_{\mp ja}\partial_\pm x^a - \kappa\hat\theta^{ij}_\pm\partial_\pm y_j. \tag{3.14}$$

Equation (3.12) is the T-duality transformations for x^a (Eq. (44) of Ref. [9]) and (3.14) is its analogue for x^i.

3.3 T-dual background fields

Requiring that the dual generalized metric (3.9) has the form (2.17) but with T-dual background fields, (denoted by lower index a on the left of background fields)

$${}_a\mathcal{H}_{MN} = \begin{pmatrix} {}_ag^{\mu\nu} & -2({}_ab\,{}_aG^{-1})^\mu{}_\nu \\ 2({}_aG^{-1}\,{}_ab)_\mu{}^\nu & ({}_aG^{-1})_{\mu\nu} \end{pmatrix}, \tag{3.15}$$

we can find expressions for the T-dual background fields in terms of the initial ones.

It is useful to consider the combination of the dual background fields in the form

$${}_a\Pi^{\mu\nu}_\pm \equiv \left({}_ab \pm \frac{1}{2}{}_aG\right)^{\mu\nu} = {}_aG^{\mu\rho}\left[({}_aG^{-1}\,{}_ab)_\rho{}^\nu \pm \frac{1}{2}\delta^\nu_\rho\right]. \tag{3.16}$$

Comparing the lower D rows of expressions (3.9) and (3.15) we find

$$({}_aG^{-1}\,{}_ab)_\mu{}^\nu = \begin{pmatrix} -(bG^{-1})_a{}^b & \frac{1}{2}g_{aj} \\ \frac{1}{2}(G^{-1})^{ib} & (G^{-1}b)^i{}_j \end{pmatrix} \equiv \begin{pmatrix} -\tilde\beta & \frac{1}{2}g^T \\ \frac{1}{2}\gamma & -\bar\beta^T \end{pmatrix}, \tag{3.17}$$

and

$$({}_aG^{-1})_{\mu\nu} = \begin{pmatrix} g_{ab} & -2(bG^{-1})_a{}^j \\ 2(G^{-1}b)^i{}_b & (G^{-1})^{ij} \end{pmatrix} \equiv \begin{pmatrix} \tilde g & -2\beta_1 \\ -2\beta_1^T & \bar\gamma \end{pmatrix}. \tag{3.18}$$

The notation in the second equality, which has been obtained using (A3), (A4), (A6) and (A7), will simplify calculations.

To obtain the background field (3.16) we need the inverse of the last expression. We will use the general expression for block-wise inversion matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}. \tag{3.19}$$

It produces

$$({}_aG)^{\mu\nu} = \begin{pmatrix} (A^{-1})^{ab} & 2(\tilde g^{-1}\beta_1 D^{-1})^a{}_j \\ 2(\bar\gamma^{-1}\beta_1^TA^{-1})_i{}^b & (D^{-1})_{ij} \end{pmatrix}, \tag{3.20}$$

where

$$A_{ab} = (\tilde g - 4\beta_1\bar\gamma^{-1}\beta_1^T)_{ab}, \qquad D^{ij} = (\bar\gamma - 4\beta_1^T\tilde g^{-1}\beta_1)^{ij}. \tag{3.21}$$

After some direct calculations it can be shown that

$$A_{ab} = (\tilde G - 4\tilde b\tilde G^{-1}\tilde b)_{ab} \equiv \hat g_{ab}, \tag{3.22}$$

where $\hat g_{ab}$ has been defined in (A8). Note that unlike $\tilde g_{ab}$, which is just the ab component of $g_{\mu\nu}$, the $\hat g_{ab}$ has the same form as the effective metric $g_{\mu\nu}$ but with all components $(\tilde{G},\tilde{b})$ defined in d dimensional subspace with indices a,b.

Using the result (3.22) we can rewrite the first equation (3.21) in the form $\hat{g}_{ab}=\tilde{g}_{ab}-4(\beta_1\bar{\gamma}^{-1}\beta_1^T)_{ab}$. Multiplying it on the left with $(\tilde{g}^{-1})^{ab}$ and on the right with $(\hat{g}^{-1})^{ab}$ we get

$$(\tilde{g}^{-1})^{ab}=(\hat{g}^{-1})^{ab}-4(\tilde{g}^{-1}\beta_1\bar{\gamma}^{-1}\beta_1^T\hat{g}^{-1})^{ab}. \qquad (3.23)$$

With the help of this relation we can verify that

$$(D^{-1})_{ij}=(\bar{\gamma}^{-1}+4\bar{\gamma}^{-1}\beta_1^T\hat{g}^{-1}\beta_1\bar{\gamma}^{-1})_{ij}, \qquad (3.24)$$

is the inverse of the second equation (3.21).

Now, we are able to calculate the background field (3.16)

$$_a\Pi_\pm^{\mu\nu}=\begin{pmatrix} \tilde{g}^{-1}\beta_1D^{-1}\gamma-A^{-1}\left(\tilde{\beta}\mp\frac{1}{2}\right) & \frac{1}{2}A^{-1}g^T-2\tilde{g}^{-1}\beta_1D^{-1}\left(\bar{\beta}^T\mp\frac{1}{2}\right) \\ \frac{1}{2}D^{-1}\gamma-2\bar{\gamma}^{-1}\beta_1^TA^{-1}\left(\tilde{\beta}\mp\frac{1}{2}\right) & \bar{\gamma}^{-1}\beta_1^TA^{-1}g^T-D^{-1}\left(\bar{\beta}^T\mp\frac{1}{2}\right)\end{pmatrix}. \qquad (3.25)$$

After tedious calculations using (A5)–(A7) and (A11) we can obtain

$$_a\Pi_\pm^{\mu\nu}=\begin{pmatrix} \frac{\kappa}{2}\hat{\theta}_\mp^{ab} & \kappa\hat{\theta}_\mp^{ab}\Pi_{\pm bi} \\ -\kappa\Pi_{\pm ib}\hat{\theta}_\mp^{ba} & \Pi_{\pm ij}-2\kappa\Pi_{\pm ia}\hat{\theta}_\mp^{ab}\Pi_{\pm bj}\end{pmatrix}, \qquad (3.26)$$

where $\hat{\theta}_\pm^{ab}$ has been defined in (A9).

It still remains to check that the upper D rows of (3.9) and (3.15) produce the same expressions for T-dual background fields. The field $({}_ab\,{}_aG^{-1})^\mu{}_\nu$ is just the transpose of $({}_aG^{-1}\,{}_ab)_\mu{}^\nu$. It is useful to express ${}_ag^{\mu\nu}$ in the form

$${}_ag^{\mu\nu}=({}_aG)^{\mu\rho}[\delta_\rho^\nu-4({}_aG^{-1}\,{}_ab)_\rho{}^\sigma({}_aG^{-1}\,{}_ab)_\sigma{}^\nu]. \qquad (3.27)$$

Then using (3.20), (3.17), (3.22), and (A11) we can show that

$${}_ag^{\mu\nu}=\begin{pmatrix}(G^{-1})^{ab} & 2(G^{-1}b)^a{}_j \\ -2(bG^{-1})_i{}^b & g_{ij}\end{pmatrix}, \qquad (3.28)$$

which is in agreement with (3.9).

Consequently, we obtain the T-dual background fields in the flat background after dualization along directions x^a, $(a=0,1,\cdots,d-1)$

$${}_a\Pi_\pm^{ab}=\frac{\kappa}{2}\hat{\theta}_\mp^{ab}, \qquad {}_a\Pi^a_{\pm i}=\kappa\hat{\theta}_\mp^{ab}\Pi_{\pm bi},$$
$${}_a\Pi_{\pm i}{}^a=-\kappa\Pi_{\pm ib}\hat{\theta}_\mp^{ba}, {}_a\Pi_{\pm ij}=\Pi_{\pm ij}-2\kappa\Pi_{\pm ia}\hat{\theta}_\mp^{ab}\Pi_{\pm bj}. \qquad (3.29)$$

The symmetric and antisymmetric parts of these expressions produce the T-dual metric and T-dual Kalb-Ramond field. This is in complete agreement with Refs. [9, 24]. A similar way to perform T-duality in flat space-time for $D=3$ has been described in Appendix B of Ref. [7].

This proves that exchange of the places of some coordinates x^a with their T-dual y_a in the flat double space represents T-dualities along these coordinates.

In Section 4.1. of Ref. [3], Buscher's T-dualities were derived in Eq. (4.9) in the case when there is only one isometry direction. For such a case it was concluded that "the dual background is related to the original one by the action of factorized duality". There is an essential difference between their Eq. (4.9) and relation (3.29) of the present article, where the general case of T-dualities along arbitrary sets of coordinates has been derived and proof has been given of its equivalence with the action of factorized duality.

For proof of expression (3.29) with mathematical induction, Eq. (4.9) is just the first step for $n=1$. The next step from n to $n+1$ is nontrivial because then we have three kinds of variables (beside the isometry variable θ there are a set of original variables and a set of variables along which we have already performed duality transformations). This leads to formulae different from Eq. (4.9). For example, when we perform T-dualization along more then one coordinate (let us say along $x^a, a=1,2$) in the expression for T-dual background fields, we do not carry out division by G_{aa} as in Eq. (4.9) but by $G_{ab}+2B_{ab}$, which was recorded in the expression $\hat{\theta}_-^{ab}$ of (3.29).

3.4 T-duality group

Successive T-dualization along disjunct sets of directions $\mathcal{T}^{a_1}$ and $\mathcal{T}^{a_2}$ will produce T-dualization along all directions $a=a_1\cup a_2$

$$\mathcal{T}^{a_1}\circ\mathcal{T}^{a_2}=\mathcal{T}^a. \qquad (3.30)$$

This can be represented by matrix multiplications $(\mathcal{T}^{a_1}\mathcal{T}^{a_2})^M{}_N=(\mathcal{T}^a)^M{}_N$, which is easy to check because the projectors satisfy the relations $P_{a_1}^2=P_{a_1}$, $P_{a_2}^2=P_{a_2}$, $P_{a_1}P_{a_2}=0$ and $P_{a_1}+P_{a_2}=P_a$.

The set of matrices $\mathcal{T}^a$, where the index a take the values in any of the subsets of index μ, form a commutative group with respect to matrix multiplication. The neutral element corresponds to the case when we do not perform T-duality, with $P_a=0$ and $\mathcal{T}^a=1$. Consequently, the set of all T-duality transformations form a commutative group with respect to the operation $\circ$.

This is a subgroup of the $2D$ permutational group because it acts as a replacement of some coordinates. In two-line notation, the T-duality $\mathcal{T}^a$, along $2d$ coordinates x^a and y_a can be written as

$$\begin{pmatrix} 1 & 2 & \cdots & d & d+1 & \cdots & D & D+1 & \cdots & D+d & D+d+1 & \cdots & 2D \\ D+1 & D+2 & \cdots & D+d & d+1 & \cdots & D & 1 & \cdots & d & D+d+1 & \cdots & 2D \end{pmatrix}. \tag{3.31}$$

It looks simpler in the cyclic notation

$$(1,D+1)(2,D+2)\cdots(d,D+d). \tag{3.32}$$

We will call this group the T-duality group. It is a global symmetry group of equations of motion (2.19).

4 Inclusion of dilaton field

As usual, in the standard formulation one should add the Fradkin-Tseytlin term

$$S_\phi = \int d^2\xi\sqrt{-g}R^{(2)}\phi, \tag{4.1}$$

to the action (2.1) in order to describe the dilaton field ϕ. Here $R^{(2)}$ is scalar curvature of the world sheet and the term S_ϕ is one order higher in α' than the terms in (2.1).

4.1 Path integral measure

It is well known that dilaton transformation has a quantum origin. For a constant background the Gaussian path integral produces the expression $(\det\Pi_{+\mu\nu})^{-1}$. We will show that this is just what we need in order that the change of space-time measure in the path integral is correct.

Let us start with the relations

$$\det G_{\mu\nu} = \frac{\det G_{ab}}{\det\bar\gamma^{ij}}, \qquad \det{}_aG_{\mu\nu} = \frac{\det{}_aG^{ab}}{\det\bar\gamma^{ij}}, \tag{4.2}$$

which follow from (A1) and (3.18). Using the expressions for T-dual fields (3.29) we can find the relations between the determinants

$$\det(2\Pi_{\pm ab}) = \frac{1}{\det(2\,{}_a\Pi_\pm^{ab})} = \sqrt{\frac{\det G_{\mu\nu}}{\det{}_aG_{\mu\nu}}} = \sqrt{\frac{\det G_{ab}}{\det{}_aG^{ab}}}, \tag{4.3}$$

where the factor 2 is introduced for convenience, because $\Pi_{\pm ab} = B_{ab} \pm \frac{1}{2}G_{ab}$. So, we have

$$\begin{aligned}&\sqrt{\det G_{\mu\nu}}\,\mathrm{d}x^i\mathrm{d}x^a \to \sqrt{\det G_{\mu\nu}}\,\mathrm{d}x^i\frac{1}{\det(2\,\Pi_{+ab})}\mathrm{d}y_a\\ &= \sqrt{\det{}_aG_{\mu\nu}}\,\mathrm{d}x^i\mathrm{d}y_a,\end{aligned} \tag{4.4}$$

which means that T-dualization T^a along x^a directions produces the correct change of space-time measure in the path integral of the standard approach.

4.2 Dilaton in the double space

In double space, T-dualization $\mathcal{T}^a$ along both x^a and y_a produces

$$\begin{aligned}&\sqrt{\det G_{\mu\nu}}\sqrt{\det{}^\star G^{\mu\nu}}\,\mathrm{d}x^i\mathrm{d}y_i\mathrm{d}x^a\mathrm{d}y_a \to\\ &\sqrt{\det G_{\mu\nu}}\sqrt{\det{}^\star G^{\mu\nu}}\,\mathrm{d}x^i\mathrm{d}y_i\mathrm{d}y_a\mathrm{d}x^a\\ &\frac{1}{\det(2\,\Pi_{+ab})\det(2\,{}_a\Pi_+^{ab})}.\end{aligned} \tag{4.5}$$

According to (4.3) the last term is equal to 1 and the path integral measure is invariant under T-dual transformation. Consequently, in double space we need the new dilaton invariant under T-duality transformations.

The usual approach in the literature is to introduce the "doubled dilaton" $\Phi^{(a)}$ in term of the standard dilaton ϕ, with the requirement that $\Phi^{(a)}$ is invariant under T-dualization $\mathcal{T}^a$. From the transformation of the standard dilaton ϕ (see Refs. [1, 23])

$${}_a\phi = \phi - \ln\det(2\Pi_{+ab}) = \phi - \ln\sqrt{\frac{\det G_{ab}}{\det{}_aG^{ab}}}, \tag{4.6}$$

with the help of (4.3) we have ${}_a({}_a\phi) = \phi$, which means that

$$\Phi^{(a)} = \frac{1}{2}({}_a\phi + \phi) = \phi - \frac{1}{2}\ln\sqrt{\frac{\det G_{ab}}{\det{}_aG^{ab}}}, \tag{4.7}$$

is invariant under duality transformation along the x^a directions. If we chose the other set of coordinates $x^b\,(b \neq a)$, along which we perform T-duality, then we wil have a different "doubled dilaton" $\Phi^{(b)}$. We want to have one doubled dilaton invariant under all T-duality transformations.

Up to now, we described all T-dual transformations with one action (2.21). Using (2.17) and (2.11) the corresponding generalized metric can be expressed symmetrically in term of initial metric and Kalb-Ramond fields and their totally T-dual background fields (marked with a star)

$$\mathcal{H}_{MN} = \begin{pmatrix} ({}^\star G^{-1})_{\mu\nu} & 2({}^\star G^{-1})^{\mu\rho}\,{}^\star B_{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}. \tag{4.8}$$

We can do a similar thing with the dilaton field. As was shown in Ref. [10] the expression

$$\Phi=\phi-\ln\sqrt{\det G_{\mu\nu}}\,, \tag{4.9}$$

is duality invariant under all possible T-dualizations. So, the double space action (2.21) can be extended with an expression similar to (4.1), but with the doubled dilaton Φ instead of the standard one ϕ.

Using the fact that $\Phi={}^\star\Phi={}^\star\phi-\ln\sqrt{\det {}^\star G^{\mu\nu}}$, we can express the double dilaton Φ in term of the dilaton from the initial theory ϕ and the dilaton from its totally T-dual theory ${}^\star\phi$ as

$$\mathrm{e}^{-2\Phi}=\mathrm{e}^{-(\phi+{}^\star\phi)}\sqrt{\det G_{\mu\nu}\det {}^\star G^{\mu\nu}}\,. \tag{4.10}$$

Therefore, we can take $\mathrm{e}^{-2\Phi}\mathrm{d}x^\mu\mathrm{d}y_\mu$ as the double space integration measure, as well as in the double field theory.

5 Relation to Hull's formulation

In this section we are going to derive the action of Ref. [16] and compare its consequences with our results. Note that the background fields of Ref. [16] depend only on the coordinates Y^m (x^i in our notation) along which the T-duality has not been executed. In our approach all variables are doubled $x^\mu\to y_\mu$, while in Ref. [16] only variables along which the T-duality is performed are doubled $x^a\to y_a$. So, in our approach there are $2D$ variables x^a,x^i,y_a,y_i while in Ref. [16] there are $D+d$ variables x^a,y_a,x^i. It suggests that the formulation of Ref. [16] can be obtained from ours after elimination of the y_i variable. We already did this in Section 3.2. and obtained the T-duality relations (3.11) and (3.12), which are inverse to each other. In analogy with (2.13) we can rewrite them in a useful form, where on the left-hand side we put the terms with the world-sheet antisymmetric tensor $\varepsilon_\alpha{}^\beta$ ($\varepsilon_\pm{}^\pm=\pm1$) and obtain

$$\partial_\pm X^A=\pm\,\Omega^{AB}(\hat{\mathcal{H}}_{BC}\partial_\pm X^C+J_{\pm B})\,. \tag{5.11}$$

Here

$$X^A=\begin{pmatrix} x^a \\ y_a \end{pmatrix}, \tag{5.12}$$

is a $2d$ dimensional double space coordinate

$$\Omega^{AB}=\begin{pmatrix} 0 & 1_a \\ 1_a & 0 \end{pmatrix}, \tag{5.13}$$

and

$$\hat{\mathcal{H}}_{AB}=\begin{pmatrix} \hat{g}_{ab} & -2b_{ac}(\tilde{G}^{-1})^{cb} \\ 2(\tilde{G}^{-1})^{ac}b_{cb} & (\tilde{G}^{-1})^{ab} \end{pmatrix}, \tag{5.14}$$

is a $2d\times2d$ generalized metric. All components of $\hat{\mathcal{H}}_{AB}$ are from an ab subspace, like $\hat{g}_{ab}$ and $\hat{\theta}^{ab}$ in (A8). So, it satisfies $\hat{\mathcal{H}}^T\Omega\hat{\mathcal{H}}=\Omega$ and is an element of the $O(d,d)$ group.

We also obtained explicit expressions for the currents in terms of undualized coordinates x^i

$$J_{\pm A}=\begin{pmatrix} J_{1\pm a} \\ J^a_{2\pm} \end{pmatrix}, \tag{5.15}$$

where

$$J_{1\pm a}=-2\Pi_{\pm ab}J^b_{2\pm}\,,\qquad J^a_{2\pm}=2(\tilde{G}^{-1})^{ab}\Pi_{\mp bi}\partial_\pm x^i\,. \tag{5.16}$$

The first relation in the last expression is solution (2.44) of Ref. [16].

Therefore, instead of $2D$ component T-duality transformations (3.7) with $2D$ dimensional vector Z^M we have $2d$ component relation (5.11) with $2d$ dimensional vectors X^A and $J_{\pm A}$. The relation (5.11) is a self-duality constraint (Eq. (2.5) of Ref. [16]) imposed that halves the degrees of freedom.

Also, in Section 2.3 the consistency condition of (5.11) produces

$$\partial_+(\hat{\mathcal{H}}\partial_-X+J_-)+\partial_-(\hat{\mathcal{H}}\partial_+X+J_+)=0\,, \tag{5.17}$$

which is equation of motion (2.4) of Ref. [16]. So, we can write the action

$$S_d=\frac{\kappa}{4}\int\mathrm{d}^2\xi\Big[\partial_+X^A\hat{\mathcal{H}}_{AB}\partial_-X^B +\partial_+X^AJ_{-A}+J_{+A}\partial_-X^A+\mathcal{L}(x^i)\Big], \tag{5.18}$$

of which variation produces Eq. (5.17). This action is in complete agreement with Ref. [16], but with already constrained elements of $\hat{\mathcal{H}}_{AB}$ and an explicit expression for currents $J_{\pm A}$ in terms of undualized coordinates x^i. Because Ref. [16] starts with arbitrary $\hat{\mathcal{H}}_{AB}$ it is restricted to be a coset metric $O(d,d)/O(d)\times O(d)$ so that the T-duality would be equivalent to the standard one.

Note that the whole procedure of Ref. [16] should be performed for each node of the chain (1.1), which means for each value of d. In our approach only the case $d=D$ appears. Then the currents $J_{\pm A}$ disappear, $X^A\to Z^M$, $\hat{\mathcal{H}}_{AB}\to\mathcal{H}_{MN}$, $\Omega^{AB}\to\Omega^{MN}$ and T-duality transformations (5.11) turn to (2.15). However, the generalized metric $\mathcal{H}_{MN}$ together with basic relation (2.15) are sufficient to describe all T-dualities for each d.

6 Conclusion

Introducing the $2D$ dimensional space, which beside initial D dimensional space-time coordinates x^μ contains the corresponding T-dual coordinates y_μ, we have offered a simple formulation for T-duality transformations. The extended space with the coordinates $Z^M = (x^\mu, y_\mu)$ we call double space.

It is easy to see that after the exchanges of all initial coordinates x^μ with all T-dual coordinates y_μ we obtain the T-dual background fields of Section 2. This result is formulated in the double space in Ref. [11] in order to make global $SO(D,D)$ symmetry manifest. In the present article we show that the double space contains enough information to explain T-dualization along an arbitrary subset of coordinates x^a and corresponding T-dual y_a $(a = 0,1,\cdots,d-1)$. For this purpose we rewrite T-duality transformations for all the coordinates and their inverse in double space. We obtain the basic relation (2.15) with the generalized metric (2.17) which helps us to find all T-dual background fields for each node of the chain (1.1) and T-duality transformations between all the nodes.

We define a particular permutation of the coordinates realized by matrix $\mathcal{T}^a$, known in the literature as factorized duality (see for example Ref. [3]). It exchanges the places of some subset of the coordinates x^a and the corresponding dual coordinates y_a along which we perform T-dualization. We require that the obtained double space coordinates satisfy the same form of T-duality transformations as the initial one, or in other words, that such a permutation is a global symmetry of the T-dual transformation. We show that this permutation produces exactly the same T-dual background fields and T-duality transformations as in the standard approach of Ref. [9]. So, the double space approach clearly explains that T-duality is nonphysical, because it is equivalent to the permutation of some coordinates in double space.

In the standard formulation T-duality transforms the initial theory to the equivalent one, T-dual theory. The double space formulation contains both initial and T-dual theories and T-duality becomes the global symmetry transformation. With the help of (3.8), it is easy to see that the equations of motion (2.19) are invariant under the transformation $Z^M \to Z_a^M = (\mathcal{T}^a)^M{}_N Z^N$.

The squares of all matrices $\mathcal{T}^a$ are equal to one and therefore they are inverse themselves. The set of all $\mathcal{T}^a$ matrices forms an Abelian group with respect to matrix multiplication. Consequently, the set of all T-dualizations with respect to the successive T-dualizations also forms an Abelian group. It is a subgroup of the $2D$ permutation group, which permutes some of the first D coordinates with the corresponding last D coordinates. In the cyclic form it can be written as

$$(1,D+1)(2,D+2)\cdots(d,D+d), \qquad d=0,1,2,\cdots,D, \tag{6.19}$$

where $d=0$ formally corresponds to the neutral element (no permutations of coordinates and so no T-duality transformations) and $d=D$ corresponds to the case when T-dualization is performed along all coordinates.

The relation between our approach and the well known one of Ref. [16] has been presented in Section 4. In the approach of Ref. [16] to each node of the chain (1.1), lying d steps from the begining, it corresponds the action S_d (5.18) and self-duality constraints (5.11) with $2d$ dimensional variables X^A. Our approach unifies all nodes of the chain (1.1). The T-duality transformations (2.15), with $2D$ dimensional variable Z^M, allows us to obtain all background fields and T-duality transformations of the chain (1.1).

Let us briefly describe the significance of the obtained results. It is well known that there are five consistent superstring theories. In order to have a unique theory, the so-called M-theory, we should connect these five theories by a web of T and S dualities. If we start with any arbitrary one of these five consistent theories and find all corresponding T-dual and S-dual theories, we can achieve any of the other four consistent superstring theories. But this is not enough for the formulation of M-theory. To realize this we should construct one theory which contains the initial theory and all corresponding dual theories.

The present article is a realization of such a program for T-duality in the bosonic case for a flat background, which is substantially simpler that the supersymmetric one. In fact, the theory with all doubled coordinates contains the initial and all corresponding T-dual theories. We hope that S-duality, which can be understood as a transformation of the dilaton background field, can be successfully incorporated into our procedure. The same program for a bosonic string but in a weakly curved background, with linear dependence on coordinates, will be investigated in Ref. [10].

Unfortunately, the solution for the bosonic case is not enough for construction of M-theory, because the T-duality for superstrings is a non-trivial extension of the bosonic case. In Ref. [25] we have tried to extend such an approach to type II theories. In fact, doubling all bosonic coordinates we have unified types IIA, IIB as well as type II* [26] (obtained by T-dualization along time-like direction) theories.

We expect that, in our approach to the formulation of M-theory we should also include T-dualization along fermionic variables. This means that we should also double these fermionic variables. A necessary step for understanding T-dualization along all fermionic coordinates in fermionic double space has been considered in Ref. [27]. We expect that the final step in the construction of M-theory will be unification of all theories obtained after T-dualization along all bosonic and all fermionic variables. In that case we should double all coordinates in superspace, anticipating that some super-permutation will connect an arbitrary two of our five consistent supersymmetric string theories.

Appendix A

Block-wise expressions for background fields

In order to simplify notation and to write expressions without indices (as matrix multiplication) we will introduce notations for component fields.

For the metric tensor and the Kalb-Ramond background fields we define

$$G_{\mu\nu}=\begin{pmatrix}\tilde{G}_{ab} & G_{aj}\\ G_{ib} & \bar{G}_{ij}\end{pmatrix}\equiv\begin{pmatrix}\tilde{G} & G^T\\ G & \bar{G}\end{pmatrix}, \tag{A1}$$

and

$$b_{\mu\nu}=\begin{pmatrix}\tilde{b}_{ab} & b_{aj}\\ b_{ib} & \bar{b}_{ij}\end{pmatrix}\equiv\begin{pmatrix}\tilde{b} & -b^T\\ b & \bar{b}\end{pmatrix}. \tag{A2}$$

We also define notation for the inverse of the matrix

$$(G^{-1})^{\mu\nu}=\begin{pmatrix}\tilde{\gamma}^{ab} & \gamma^{aj}\\ \gamma^{ib} & \bar{\gamma}^{ij}\end{pmatrix}\equiv\begin{pmatrix}\tilde{\gamma} & \gamma^T\\ \gamma & \bar{\gamma}\end{pmatrix}, \tag{A3}$$

and for the effective matrix

$$g_{\mu\nu}=G_{\mu\nu}-4b_{\mu\rho}(G^{-1})^{\rho\sigma}b_{\sigma\nu}=\begin{pmatrix}\tilde{g}_{ab} & g_{aj}\\ g_{ib} & \bar{g}_{ij}\end{pmatrix}\equiv\begin{pmatrix}\tilde{g} & g^T\\ g & \bar{g}\end{pmatrix}. \tag{A4}$$

Note that because $G^{\mu\nu}$ is the inverse of $G_{\mu\nu}$, we have

$$\begin{aligned}\gamma&=-\bar{G}^{-1}G\tilde{\gamma}=-\bar{\gamma}G\tilde{G}^{-1}, &\quad \gamma^T&=-\tilde{G}^{-1}G^T\bar{\gamma}=-\tilde{\gamma}G^T\bar{G}^{-1},\\ \tilde{\gamma}&=(\tilde{G}-G^T\bar{G}^{-1}G)^{-1}, & \bar{\gamma}&=(\bar{G}-G\tilde{G}^{-1}G^T)^{-1},\\ \tilde{G}^{-1}&=\tilde{\gamma}-\gamma^T\bar{\gamma}^{-1}\gamma, & \bar{G}^{-1}&=\bar{\gamma}-\gamma\tilde{\gamma}^{-1}\gamma^T.\end{aligned} \tag{A5}$$

It is also useful to introduce new notation for the expressions

$$(bG^{-1})_\mu{}^\nu=\begin{pmatrix}\tilde{b}\tilde{\gamma}-b^T\gamma & \tilde{b}\gamma^T-b^T\bar{\gamma}\\ b\tilde{\gamma}+\bar{b}\gamma & b\gamma^T+\bar{b}\bar{\gamma}\end{pmatrix}\equiv\begin{pmatrix}\tilde{\beta} & \beta_1\\ \beta_2 & \bar{\beta}\end{pmatrix}, \tag{A6}$$

and

$$(G^{-1}b)^\mu{}_\nu=\begin{pmatrix}\tilde{\gamma}\tilde{b}+\gamma^T b & -\tilde{\gamma}b^T+\gamma^T\bar{b}\\ \gamma\tilde{b}+\bar{\gamma}b & -\gamma b^T+\bar{\gamma}\bar{b}\end{pmatrix}\equiv\begin{pmatrix}-\tilde{\beta}^T & -\beta_2^T\\ -\beta_1^T & -\bar{\beta}^T\end{pmatrix}. \tag{A7}$$

We denote by ˆ expressions similar to the effective metric (A4) and non-commutativity parameters but with all contributions from the ab subspace

$$\hat{g}_{ab}=(\tilde{G}-4\tilde{b}\tilde{G}^{-1}\tilde{b})_{ab}, \qquad \hat{\theta}^{ab}=-\frac{2}{\kappa}(\hat{g}^{-1}\tilde{b}\tilde{G}^{-1})^{ab}. \tag{A8}$$

Note that $\hat{g}_{ab}\neq\tilde{g}_{ab}$ because $\tilde{g}_{ab}$ is the projection of $g_{\mu\nu}$ on subspace ab. It is extremely useful to introduce background field combinations

$$\begin{aligned}&\Pi_{\pm ab}=b_{ab}\pm\frac{1}{2}G_{ab}\\ &\hat{\theta}^{ab}_\pm=-\frac{2}{\kappa}(\hat{g}^{-1}\tilde{\Pi}_\pm\tilde{G}^{-1})^{ab}=\hat{\theta}^{ab}\mp\frac{1}{\kappa}(\hat{g}^{-1})^{ab},\end{aligned} \tag{A9}$$

which are inverse to each other

$$\hat{\theta}^{ac}_\pm\Pi_{\mp cb}=\frac{1}{2\kappa}\delta^a_b. \tag{A10}$$

With the help of (3.23) one can prove the relation

$$(\tilde{g}^{-1}\beta_1 D^{-1})^a{}_i=(\hat{g}^{-1}\beta_1\bar{\gamma}^{-1})^a{}_i, \tag{A11}$$

where D^{ij} is defined in (3.21).

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