# Power law of shear viscosity in Einstein-Maxwell-Dilaton-Axion model [*]

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**Abstract:** We construct charged black hole solutions with hyperscaling violation in the infrared (IR) region in Einstein-Maxwell-Dilaton-Axion theory and investigate the temperature behavior of the ratio of holographic shear viscosity to the entropy density. When translational symmetry breaking is relevant in the IR, the power law of the ratio is verified numerically at low temperature $T$, namely, $\eta/s \sim T^{\kappa}$, where the values of exponent $\kappa$ coincide with the analytical results. We also find that the exponent $\kappa$ is not affected by irrelevant current, but is reduced by the relevant current.

**Keywords:** shear viscosity, gauge/gravity duality, hyperscaling violation, Einstein-Maxwell-Dilaton-Axion model, breaking of translational invariance

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## 1   Introduction

In the last decade the development of holographic techniques has given new insights into hydrodynamics with strong couplings. One celebrated achievement is the Kovtun-Son-Starinets (KSS) bound for the ratio of shear viscosity to entropy density [1]

$$\frac{\eta}{s} \geqslant \frac{1}{4\pi}, \tag{1}$$

which is considered as a fundamental bound for near-perfect fluid with strong interactions. However, in recent years counter-examples which violate the KSS bound (1) have been found in holographic literature, including higher derivative gravity [2, 3], anisotropic systems [4–7], and isotropic systems without translational invariance [8–15]. In the latter case, shear viscosity loses its hydrodynamical interpretation because of the non-conservation of momentum and is usually defined by the Kubo Formula

$$\eta = \lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im} G^R_{\hat{T}^{xy}\hat{T}^{xy}}(\omega, k=0), \tag{2}$$

where $G^R_{\hat{T}^{xy}\hat{T}^{xy}}$ is the retarded Green function of the energy-momentum tensor operator $\hat{T}^{xy}$ in the dual boundary theory. Nevertheless, it is still instructive to investigate the temperature behavior of the ratio of shear viscosity to entropy density in general holographic models without translational invariance.

Historically, the breaking of translational invariance is introduced in holography to study the transportation of the dual system with momentum dissipation [8, 16–30]. In this setup, new geometries in the IR may emerge and often be accompanied by new scaling relations, leading to new scaling behavior of thermodynamic quantities or Green functions with temperature $T$ or frequency $\omega$ [31–39].

In particular, by virtue of recent progress in Ref. [9], it has been learned that when the translation symmetry breaking is relevant in the far IR, the ratio exhibits a power law behavior with temperature

$$\frac{\eta}{s} \sim T^{\kappa}, \tag{3}$$

which reflects the scaling symmetry emerging in the IR region. Moreover, a new bound for the exponent $\kappa$ was

proposed as $\kappa \leqslant 2$ there, which might be supported by a heuristic argument based on the bound for the rate of entropy production,

$$\frac{\hbar}{k_{\mathrm{B}}T} \frac{d\log(s)}{dt} \gtrsim 1. \tag{4}$$

Other holographic models have been investigated in [8, 10–12, 14, 15], which also satisfy the bound $\kappa \leqslant 2$. In Ref. [13], however, we find $\kappa > 2$ is possible when the Lifshitz scaling [40–42] or hyperscaling violation [22, 43–57] emerges in the IR. The bound for the rate of entropy production (4) is not violated, since the origin of the power law (3) should be understood as the nontrivial anomalous dimensions of (2) under the rescaling of the IR solution [9, 13].

Furthermore, in Ref. [13] we have analytically derived a formula for $\kappa$ in a general Einstein-Maxwell-Dilaton-Axion (EMD-Axion) model with spatial dimension $d$, dynamical critical exponent $z$ as well as hyperscaling violating exponent $\theta$. Specifically, we find

$$\frac{\eta}{s} \sim T^{\frac{d+z-\theta}{z}\left(-1+\sqrt{\frac{8(z-1)}{(d+z-\theta)(1+e^2)}+1}\right)}, \tag{5}$$

where $e^2$ is defined as the ratio of the Maxwell term and one of the lattice terms in the Lagrangian. However, in [13] only for the case of $e^2 = 0$ has this formula (5) been justified by numerical calculation on neutral background. In this paper we intend to extend this to verify the validity of (5) on a charged background within EMD-Axion models. Schematically, the case of $e^2 \neq 0$ can be realized by relevant currents. We will numerically construct specific charged backgrounds with ultraviolet (UV) completion and then compute the power law of the ratio of shear viscosity to entropy density. As a result, we will show that the formula for the ratio $\eta/s$ in (5) which was previously proposed in Ref. [13] based on the scaling analysis can indeed be justified even for $e^2 \neq 0$.

In this paper, we adopt the statement about '(marginally) relevant' and 'irrelevant' in Ref. [32]: the current or the axion being (marginally) relevant means that the Maxwell or the axion terms are the same order as the curvature term and the dilaton potential in the Lagrangian in the power of the radial coordinate; the current or the axion being irrelevant means that the Maxwell or the axion terms are subleading comparing to the curvature term and the dilaton potential. Roughly speaking, a field being (marginally) relevant or irrelevant depends on whether or not it would strongly deform the IR geometry.

## 2  EMD-Axion model and hyperscaling violating metric

In this section we will present our holographic setup and then outline the logic leading to the formula for the

exponent $\kappa$ in (5) based on the scaling analysis. The action of a general EMD-Axion model in $d+2$ space time reads as

$$\mathcal{S} = \int dt d^d x dr \sqrt{-g}(R + \mathcal{L}_m),$$

$$\mathcal{L}_m = -\frac{1}{2}(\partial\phi)^2 - \frac{J(\phi)}{2}\sum_{i=1}^{d}(\partial\chi_i)^2 + V(\phi) - \frac{Z(\phi)}{4}F^2, \tag{6}$$

where $\chi_i(i=1,2,\cdots,d)$ are axions and $J(\phi), Z(\phi), V(\phi)$ are the coupling functions or potential of the dilaton field $\phi$. Given the above action, the equations of motion can be derived as follows

$$R_{\mu\nu} + \frac{1}{d}g_{\mu\nu}T - T_{\mu\nu} = 0,$$

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} = \frac{1}{2}g_{\mu\nu}\mathcal{L}_m - \frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}}, \tag{7a}$$

$$\nabla^2\phi - \frac{J'(\phi)}{2}\sum_{i=1}^{d}(\partial\chi_i)^2 + V'(\phi) - \frac{Z'(\phi)}{4}F^2 = 0, \tag{7b}$$

$$\nabla^\nu(Z(\phi)F_{\mu\nu}) = 0, \tag{7c}$$

$$\nabla^\mu(J(\phi)\partial_\mu\chi_i) = 0, \quad i = 1,2,\cdots,d. \tag{7d}$$

For simplicity, we only consider isotropic solutions with the following ansatz

$$ds^2 = -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r)\sum_{i=1}^{d}dx_i^2,$$

$$\phi = \phi(r), \quad \chi_i = kx_i, \quad A = A_t(r)dt, \tag{8}$$

where translational invariance is broken by the axions $\chi_i$ but the metric and energy-momentum tensor remain homogeneous.

As stated above, we are interested in solutions interpolating between AdS in the UV and the hyperscaling violating solution in the IR. It can be realized by the process of UV completion [46]. Firstly, we construct a hyperscaling violation solution with a running dilaton. Secondly, we modify the local behaviors of potentials to graft the solution onto AdS in the UV. Generally, solutions with AdS exist when the dilaton reaches the extremal point of its potential and the Lorentz symmetry is maintained [56].

Let us first find hyperscaling violating solutions. When the potentials behave as

$$V(\phi) \sim V_0 e^{\alpha\phi}, \quad J(\phi) \sim e^{\beta\phi}, \quad Z(\phi) \sim e^{\gamma\phi}, \quad \text{when } \phi \to \pm\infty, \tag{9}$$

it is found in Refs. [31, 32] that scaling solutions with hyperscaling violation exist, whose metric and matter fields

read as

$$ds^2 = r^{\frac{2\theta}{d}}\left(-\frac{dt^2}{r^{2z}} + \frac{L^2 dr^2}{r^2} + \frac{\sum_{i=1}^{d} dx_i^2}{r^2}\right),$$

$$A = Qr^{\zeta-z}dt, \quad e^\phi = r^\epsilon, \quad \chi_i = kx_i, \qquad (10)$$

where $k$ characterizes the scale of breaking translational invariance and $\zeta$ is called the conduction exponent [57]. Parameters $\{z, \theta, \zeta, \epsilon, k, Q\}$ in ansatz (10) are determined by the equations of motion (7). The solutions are found to be classified into four classes [32], depending on the parameters $\{\alpha, \beta, \gamma, V_0\}$ in potentials (9). Among them, the explicit solutions with (marginally) relevant axions are shown in Appendix A. The metric in (10) can be deformed into a black hole solution

$$ds^2 = r^{\frac{2\theta}{d}}\left(-\frac{f(r)dt^2}{r^{2z}} + \frac{L^2 dr^2}{r^2 f(r)} + \frac{\sum_{i=1}^{d} dx_i^2}{r^2}\right), \quad (11)$$

where the blackness function is

$$f(r) = 1 - \left(\frac{r}{r_+}\right)^{\delta_0}, \quad \delta_0 = d + z - \theta. \qquad (12)$$

Then, the Hawking temperature and entropy density are separately given by

$$T = \frac{|\delta_0|}{4\pi} r_+^{-z}, \quad s = 4\pi r_+^{\theta-d} = 4\pi\left(\frac{4\pi T}{|\delta_0|}\right)^{\frac{d-\theta}{z}}. \qquad (13)$$

From Maxwell equation (7c), we have conserved charge density

$$\rho = \sqrt{-g}Z(\phi)F^{rt}. \qquad (14)$$

It can be shown that there exists a scaling relation

$$x \to cx, r \to cr, t \to c^z t, ds \to c^{\theta/d}ds, T \to c^{-z}T, s \to c^{d-\theta}s. \qquad (15)$$

We will come back to the UV completion and modify the potentials in next section.

Now we turn to the study of shear viscosity $\eta$. From the Kubo formula (2), $\eta$ can be derived by perturbing $(\delta g)^x{}_y = g^{xx}\delta g_{xy} = h(r)e^{-i\omega t}$, which is the dual field of operator $\hat{T}^{xy}$. The Einstein equations (7a) give rise to the shear perturbation equation

$$\frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}\partial_r h(r)) + (g^{tt}\omega^2 - m(r)^2)h(r) = 0, \quad (16)$$

with varying mass

$$m(r)^2 = 2\left(g^{xx}T_{xx} - \frac{\delta T_{xy}}{\delta g_{xy}}\right). \qquad (17)$$

$h(r)$ is required to be regular at the horizon and equal to 1 at the conformal boundary $r_\partial$. The varying mass is

supposed to satisfy the condition $m(r)^2 \geqslant 0$ in the models considered so far. Here, we have $m(r)^2 = J(\phi)k^2 g^{xx}$.

$\eta/s$ can be obtained by the weaker horizon formula [9, 58]

$$\frac{\eta}{s} = \frac{1}{4\pi}h_0(r_+)^2, \qquad (18)$$

where $h_0(r)$ is the solution at $\omega = 0$ and $r_+$ is the location of the horizon.

Following the analysis presented in Ref. [13], one can calculate the exponent $\kappa$ of $\frac{\eta}{s} \sim T^\kappa$. Here we present a simpler derivation with the use of formula (18). If the axion is (marginally) relevant, by using the Einstein equations (7a) and the black hole metric (11), we have

$$m(r)^2 = M^2 r^{-\frac{2\theta}{d}}, \quad M^2 = \frac{2\delta_0(z-1)}{(1+e^2)L^2}, \qquad (19)$$

where

$$e(r)^2 = -\frac{Z(\phi)}{4}F^2 \left/ \left(\frac{1}{2}J(\phi)(\partial\chi_x)^2\right)\right. \geqslant 0, \quad \chi_x = kx. \qquad (20)$$

$e(r)^2$ is just the ratio of the Maxwell term to one of the axion terms in the Lagrangian (6). It goes to a nonzero constant in the far IR if the current is also (marginally) relevant, otherwise it goes to zero. Thus at leading order, we just set $e(r)^2 = e^2$. If the axion is irrelevant, $m(r)^2$ goes to zero in the far IR, then we set $M^2 = 0$, which is valid at leading order.

By using (19) and metric (11), we rewrite the perturbation equation (16) at $\omega = 0$

$$\partial_r(r^{1-\delta_0}f(r)\partial_r h_0(r)) - M^2 L^2 r^{-\delta_0-1}h_0(r) = 0. \qquad (21)$$

Solving this equation we can separately obtain the asymptotic expansion of $h_0(r)$ near the boundary and its value on the horizon as

$$h_0(r \to r_i) = c\left[\left(\frac{r}{r_+}\right)^{\delta_0-\delta_{\hat{T}}} + \cdots + G\left(\frac{r}{r_+}\right)^{\delta_{\hat{T}}} + \cdots\right],$$

$$h_0(r_+) = cH, \qquad (22)$$

where

$$\delta_{\hat{T}} = \frac{\delta_0}{2}\left(1 + \sqrt{1 + \left(\frac{2ML}{\delta_0}\right)^2}\right). \qquad (23)$$

We have abbreviated the series $\left(\frac{r}{r_+}\right)^{\delta_0-\delta_{\hat{T}}+n\delta_0}$ $(n \geqslant 1)$ following $\left(\frac{r}{r_+}\right)^{\delta_0-\delta_{\hat{T}}}$ and the series $\left(\frac{r}{r_+}\right)^{\delta_{\hat{T}}+m\delta_0}$ $(m \geqslant 1)$ following $\left(\frac{r}{r_+}\right)^{\delta_{\hat{T}}}$ to ellipsis. $G$ and $H$ are some $r_+$ independent constants, which only depend on $\delta_0$ and $\delta_{\hat{T}}$. Coefficient $c$ should be determined by the boundary condition $h_0(r_\partial) = 1$. $r_i$ is the boundary of the region where the black hole with hyperscaling violation

can be described by (11). When we consider the second-order variation of the action in (6) over a fixed background with the perturbed metric $h_0(r)$, the boundary part of the variation with the branch of $r^{\delta_0 - \delta_{\hat{T}}}$ ($r^{\delta_{\hat{T}}}$) in (22) is divergent (finite), thus the branch of $r^{\delta_0 - \delta_{\hat{T}}}$ ($r^{\delta_{\hat{T}}}$) is non-normalizable (normalizable). Then $h_0(r_i) \approx c \left( \dfrac{r_i}{r_+} \right)^{\delta_0 - \delta_{\hat{T}}}$ is a good approximation when the black hole is near-extremal.

Since the UV completion is taken into account, one should be cautious that $r_i$ is just an intermediate scale but not the conformal boundary $r_\partial$. The region between $r_i$ and $r_\partial$ is AdS deformed by matter fields. The boundary condition requires $h_0(r_\partial) = 1$. As explained in [9], when going from $r_\partial$ to $r_i$, $h_0(r)$ decreases monotonically from 1 to a value $\Gamma$ when $m(r)^2 > 0$. We have introduced a 'tunneling rate' $\Gamma$ to characterize how $h_0(r)$ tunnels from $r_\partial$ to $r_i$. The tunneling rate $\Gamma$ and intermediate scale $r_i$ should be independent of $T$ when the scale of $T$ is much less than other scales, such as the source of the dilaton and $k$ in the axion. This is because the scale $T$ is not the dominating scale driving the renormalization group (RG) flow from AdS to the hyperscaling violating solution. Scale $T$ becomes important only when we go into the far IR. Then we have

$$\Gamma = h_0(r_i) \approx c \left( \frac{r_i}{r_+} \right)^{\delta_0 - \delta_{\hat{T}}}. \qquad (24)$$

By working out $c$, we can determine the horizon value

$$h_0(r_+) = \Gamma H \left( \frac{r_+}{r_i} \right)^{\delta_0 - \delta_{\hat{T}}}. \qquad (25)$$

By virtue of Eq. (18), we finally obtain

$$\frac{\eta}{s} = \frac{\Gamma^2 H^2}{4\pi} \left( \frac{r_+}{r_i} \right)^{2(\delta_0 - \delta_{\hat{T}})}$$
$$\propto T^{\frac{d-\theta+z}{z}\left( -1 + \sqrt{1 + \frac{8(z-1)}{(d-\theta+z)(1+e^2)}} \right)}, \text{ when } T \to 0, \qquad (26)$$

where (13) and (23) have been used. One can also employ UV-IR matching [38, 59, 60] to reproduce the same result as was performed in Ref. [13]. As one can see, when the axion is absent or irrelevant, we have $m(r)^2 = 0$ at leading order, then $\eta/s \sim T^0$ [61, 62]. Next we will numerically verify this formula when $e^2 = 0$ or $e^2 \neq 0$.

## 3   UV completion and numerical results

In this section we specifically construct the background interpolating hyperscaling violating solution (10) in the IR and AdS solution in the UV with finite temperature and charge density. As explained in Appendix A or in [32], solution (10) can be constructed by choosing exponential potentials (9) with a running dilaton. AdS can be constructed by finding extremal points of a constant dilaton with Lorentz symmetry. Here we adopt a dilaton $\phi$ to interpolate the UV and the IR solutions. It requires some special settings for the potential $V(\phi)$.

### 3.1   UV completion

In this paper, we focus only on $\theta < d$ since in this region the entanglement entropy obeys the area to volume law, which is considered normal behavior for QFT [13, 53]. In this situation the location of IR is $r \to +\infty$. Following the discussion in [13, 56], the constraints about $(z, \theta)$ are reduced to

$$(\theta \leqslant 0 \wedge z > 1) \vee \left( 0 < \theta < d \wedge z > \frac{\theta}{d} + 1 \right). \qquad (27)$$

Then it leads to $\delta_0 > 0$. We have excluded the two cases of $\theta = d$ and $z = \dfrac{\theta}{d} + 1$, which cannot be reached by the running dilaton, as shown in Appendix A or [32]. We choose the branch of $\phi \geqslant 0$. From the requirement of the potentials (9), one can see that our solution can flow to the hyperscaling violating solution in the IR if $\phi \to +\infty (r \to +\infty)$. It requires $\epsilon > 0$ in (10). We conduct the UV completion by modifying the potential $V(\phi)$ but fixing the other two coupling potentials

$$J(\phi) = e^{\beta\phi}, \quad Z(\phi) = e^{\gamma\phi}. \qquad (28)$$

From the analysis in Appendix A, we find a universal behavior of $V(\phi) \sim r^{-\frac{2\theta}{d}}$ in the coordinate of ansatz (10). So, when approaching the UV ($r \to 0$), the qualitative behavior of $V(\phi)$ depends on the sign of $\theta$. On the side of the UV, AdS is allowable if axion and gauge field are turned off and $\phi = \phi_*$ is an extremal point of $V(\phi)$, where $V(\phi_*)$ stands for the cosmological constant. Without loss of generality, we choose $\phi_* = 0$. Then a realistic strategy is modifying $V(\phi)$ as

$$V(\phi) = \begin{cases} \dfrac{2d}{\alpha^2} \sinh^2 \left( \dfrac{\alpha\phi}{2} \right) + (d+1)d, & V_0 = \dfrac{d}{2\alpha^2}, & \text{for } \theta < 0 \\[2ex] (d+1)d, & V_0 = (d+1)d, & \text{for } \theta = 0 \\[2ex] \left( d(d+1) - \dfrac{V_0}{2} \right) \left( 1 - \tanh^2(\alpha\phi) \right) + \dfrac{V_0}{2\cosh(\alpha\phi)}, & V_0 = 2d \left( \dfrac{1}{\alpha^2} + 2d + 2 \right), & \text{for } d > \theta > 0 \end{cases} \qquad (29)$$

In Appendix A, we have $\alpha\epsilon = -\dfrac{2\theta}{d}$. Then $\alpha$ has the opposite sign of $\theta$. So, each $V(\phi)$ approaches $V_0 e^{\alpha\phi}$ when $\phi \to +\infty$ and approaches $d(d+1)$ when $\phi \to 0$. Without loss of generality, we have chosen the AdS radius to be 1. However, the intermediate behavior is not very important.

$AdS_{d+2}$ vacuum is always allowable. We choose the first type quantization, and the scaling dimensions for the dual source of the dilaton $\phi^{(0)}$ and operator $\mathcal{O}_\phi$ are determined by the small $\phi$ expansion of $V(\phi)$. When $\theta = 0$, as the dilaton is massless, we have $\Delta_{\phi^{(0)}} = 0$ and $\Delta_{\mathcal{O}_\phi} = d+1$, which is marginal deformation. We expect it to be marginally relevant to drive the solution away from AdS, as $\phi = 0$ is not a stable point when the axion and gauge field are turned on. When $\theta \neq 0$, as $V(\phi) = d(d+1) + \dfrac{d}{2}\phi^2 + \cdots$, we have $\Delta_{\phi^{(0)}} = 1$ and $\Delta_{\mathcal{O}_\phi} = d$, which is relevant deformation.

## 3.2   Numerical calculation and results

We use the following ansatz for the numerical calculation:

$$ds^2 = \frac{1}{u^2}\left(-(1-u)U(u)e^{-S(u)}dt^2 + \frac{du^2}{(1-u)U(u)} + \sum_{i=1}^{d}dx_i^2\right),$$

$$\phi = \phi(u), \quad \chi_i = kx_i, \quad A = (1-u)A(u)dt. \tag{30}$$

The conformal boundary is located at $u = 0$ while the horizon is at $u = 1$. The temperature and the entropy density are $T = \dfrac{1}{4\pi}U(1)e^{-S(1)/2}$ and $s = 4\pi$.

The $AdS_{d+2}$ vacuum corresponds to $U = 1$, $S = \phi = A = 0$. Boundary conditions at the horizon are regular conditions. Boundary conditions at the conformal boundary should satisfy the scaling dimensions, which depend on the potential $V(\phi)$ as well as the value of $\theta$.

Explicitly, the asymptotic expansions near the conformal boundary are

$$U(u) = 1 + \cdots + \varepsilon u^{d+1} + \cdots,$$
$$e^{-S(u)} = 1 + \cdots,$$
$$A(u) = \mu + \cdots + \rho u^{d-1} + \cdots,$$
$$\phi(u) = \begin{cases} \lambda + \cdots + \nu u^{d+1} + \cdots, & \theta = 0 \\ \lambda u + \cdots + \nu u^{d} + \cdots, & \theta \neq 0 \end{cases} \tag{31}$$

where $\mu$ is the chemical potential and $e^{-S(0)}$ has been set to 1 by rescaling $t$. The different boundary conditions of $\phi(u)$ come from the different choices of $V(\phi)$ in (29). We can work on either the grand canonical ensemble or canonical ensemble.

### 3.2.1   Grand canonical ensemble

In the grand canonical ensemble, we control the value of the chemical potential $\mu$.

When $\theta = 0$, the boundary conditions at conformal boundary are $U(0) = 1$, $S(0) = 0$, $A(0) = \mu$, $\phi(0) = \lambda$. We work in the unit of $k$. The dimensionless quantities parameterizing the family of black hole solutions are $\left\{\dfrac{T}{k}, \dfrac{\mu}{k}, \lambda\right\}$. When $\theta \neq 0$, the boundary conditions are $U(0) = 1$, $S(0) = 0$, $A(0) = \mu$, $\phi'(0) = \lambda$. The dimensionless quantities are $\left\{\dfrac{T}{k}, \dfrac{\mu}{k}, \dfrac{\lambda}{k}\right\}$.

We numerically construct the interpolating solutions for $\phi \geqslant 0$. When dropping down $T/k$, we should fix the values of $\left\{\dfrac{\mu}{k}, \lambda\right\}$ (for $\theta = 0$) or $\left\{\dfrac{\mu}{k}, \dfrac{\lambda}{k}\right\}$ (for $\theta \neq 0$) within an appropriate region respectively, in order to reach the hyperscaling violating solution in the IR at low $T/k$.

The dimensionless entropy density and charge density are $s/k^d$ and $\rho/k^d$. We calculate $\eta/s$ by using (18) and find $\dfrac{\eta}{s} \leqslant \dfrac{1}{4\pi}$ for all time, because of the breaking of translational invariance.

At high $T/k$, the scaling relation is controlled by AdS in the UV, which gives rise to the power laws of $s \sim T^d$ and $\dfrac{\eta}{s} \sim T^0$. At low $T/k$, however, the hyperscaling violation emerges in the IR. The power laws of $s \sim T^{\frac{d-\theta}{z}}$ and $\dfrac{\eta}{s} \sim T^\kappa$ are observed in the numerical results.

It is worth noticing that $\rho/k^d$ converges to a nonzero constant at low $T/k$. When approaching the IR, $e^2(u)$ converges to a nonzero constant for class I but goes to zero as some power of radial coordinate $u$ for class II. Similarly, at low $T/k$, the horizon value $e_h^2 = e^2(1)$ converges to a nonzero constant for class I but goes to zero as some power law for class II, whose exponent is shown in Appendix A. For the same $\{d, z, \theta\}$, it is observed that the appearance of a nonzero $e^2$ always reduces the exponent $\kappa$ of $\dfrac{\eta}{s} \sim T^\kappa$, which is consistent with the property that $\kappa$ decreases monotonously with $e^2$ in (26) when $\kappa > 0$.

We conduct the numerical calculation for $d = 2$ and $\theta = \dfrac{4}{3}$, $0$, $-4$ as representatives of three cases, namely $\theta < 0$, $\theta = 0$ and $0 < \theta < d$. Different values of $\gamma$ are chosen to represent class I or class II. The specific results are shown in Figs. 1, 2, 3. All the numerical results match the analytical ones.
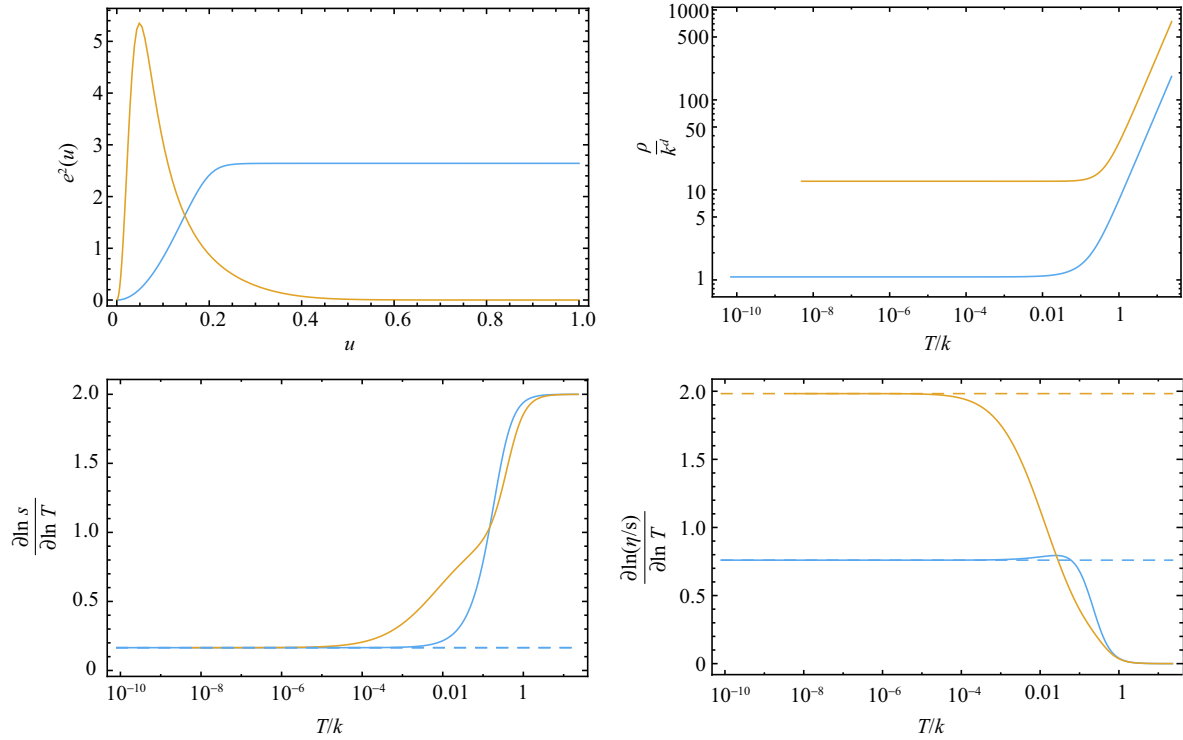
Fig. 1.    (color online) $d = 2$, $\theta = 0$.  Parameters are $\alpha = 0$, $\beta = -0.3$, $\gamma = 0.6$ (blue) or 2 (orange), then $z = 12.1$. Among them, $\gamma = 0.6$ belongs to class I; $\gamma = 2$ belongs to class II. Solid lines denote numerical results and dashed lines denote analytical results from (13) and (26). Upper-left: Plot of $e^2(u)$ at rather low temperatures, where it converges to a nonzero constant in the IR for $\gamma = 0.6$. Upper-right: Log-log plot of $\rho/k^d$ as a function of $T/k$, where $\rho/k^d$ converges to a constant at low temperature. Lower-left: Exponent $\lambda$ of $s \sim T^\lambda$ as a function of $T/k$. Lower-right: Exponent $\kappa$ of $\frac{\eta}{s} \sim T^\kappa$ as a function of $T/k$.
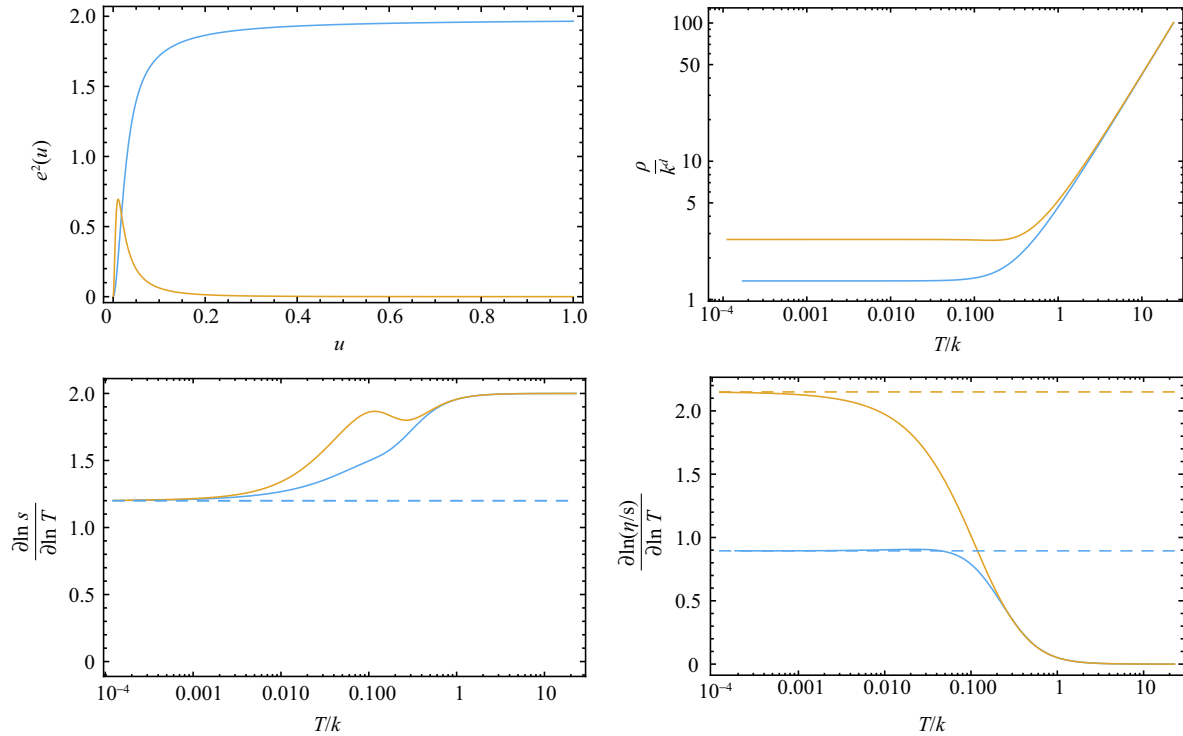


Fig. 2.    (color online) $d = 2$, $\theta < 0$.  Parameters are $\alpha = 0.471$, $\beta = -0.236$, $\gamma = 0.943$ (blue, class I) or 2 (orange, class II), then $z = 5$ and $\theta = -4$.
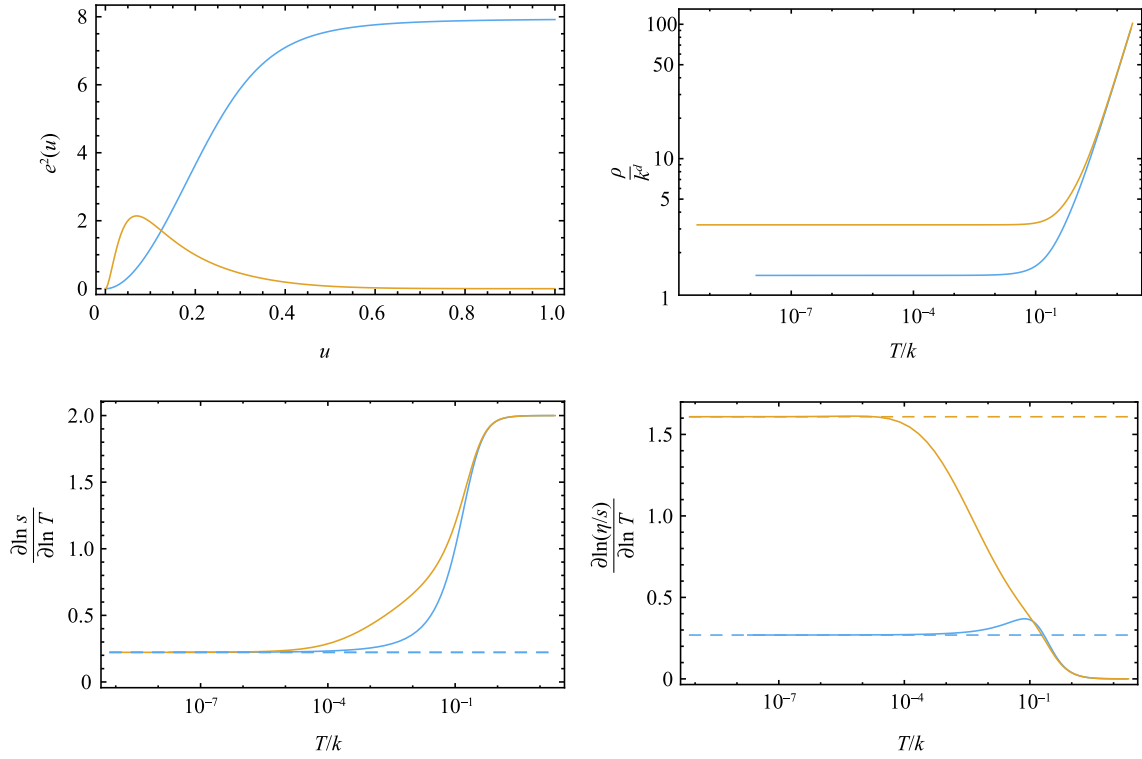
Fig. 3.   (color online) $d = 2$, $\theta > 0$. Parameters are $\alpha = -1$, $\beta = -1.5$, $\gamma = 2$ (blue, class I) or 4 (orange, class II), then $z = 3$ and $\theta = 1.33$.
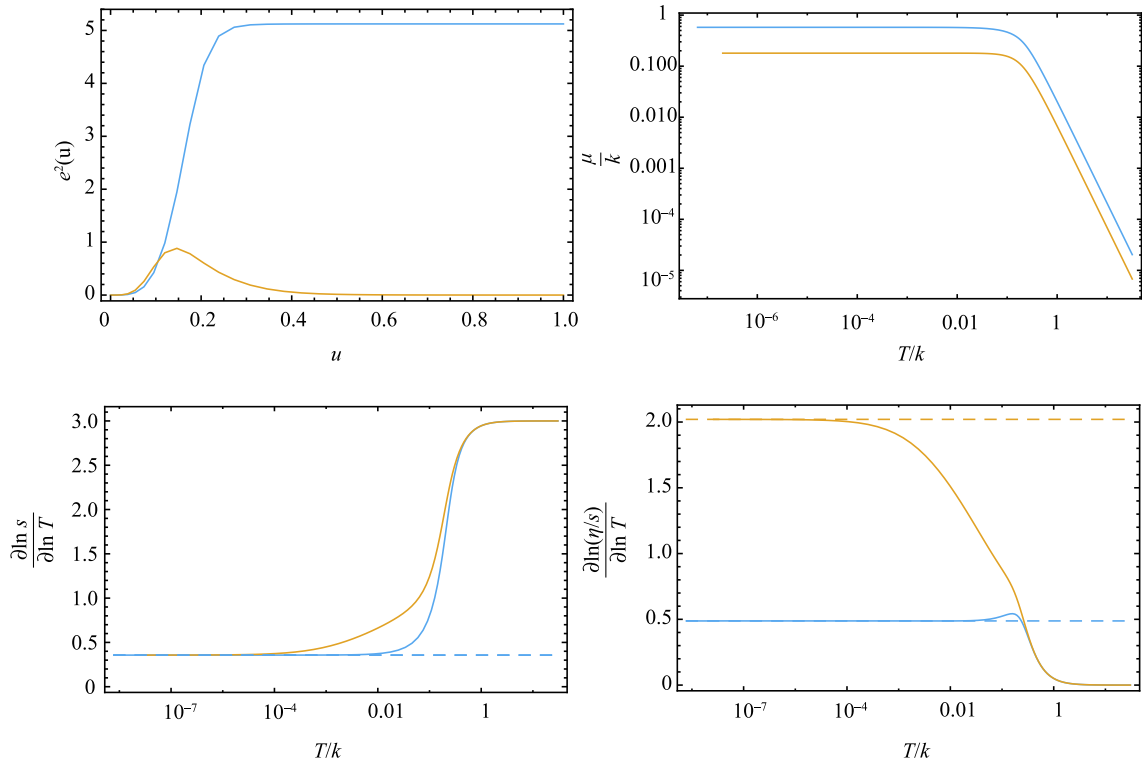


Fig. 4.   (color online) $d = 3$, $\theta = 0$. Parameters are $\alpha = 0$, $\beta = -0.3$, $\gamma = 0.9$ (blue, class I) or 2 (orange, class II), then $z = 8.41$. In the lower-right plot, it is worth noticing that $\kappa$ converges to a quantity which is greater than 2 at low temperature.
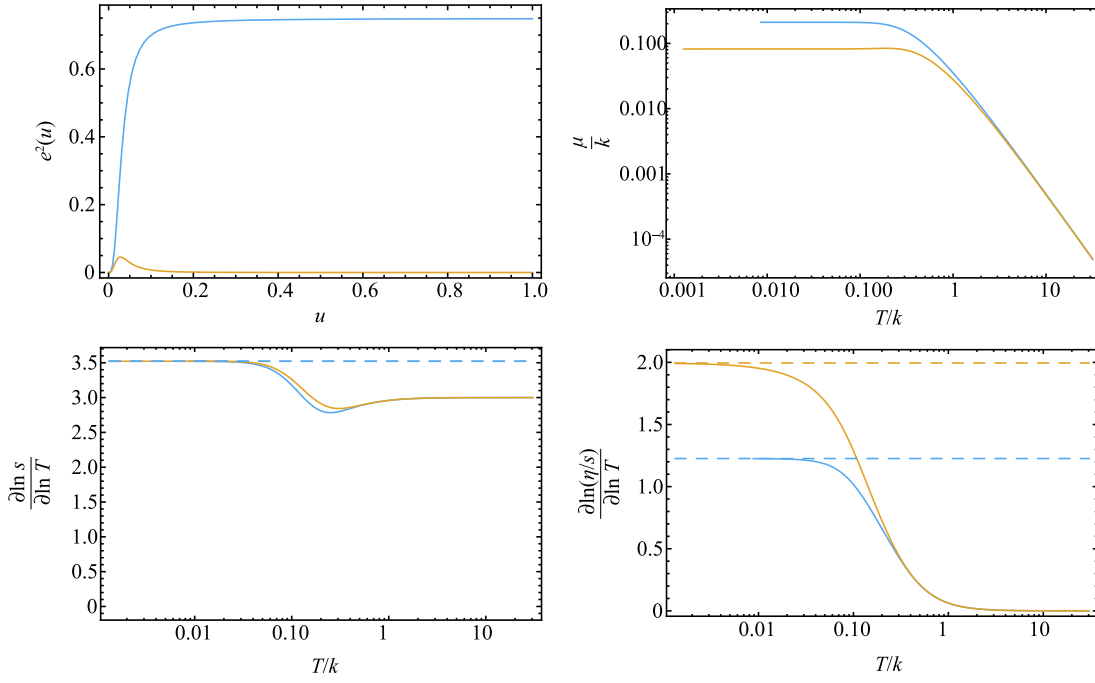
Fig. 5.   (color online) $d = 3$, $\theta < 0$. Parameters are $\alpha = 0.5$, $\beta = -0.25$, $\gamma = 1.75$ (blue, class I) or 3 (orange, class II), then $z = 2.56$ and $\theta = -6$.
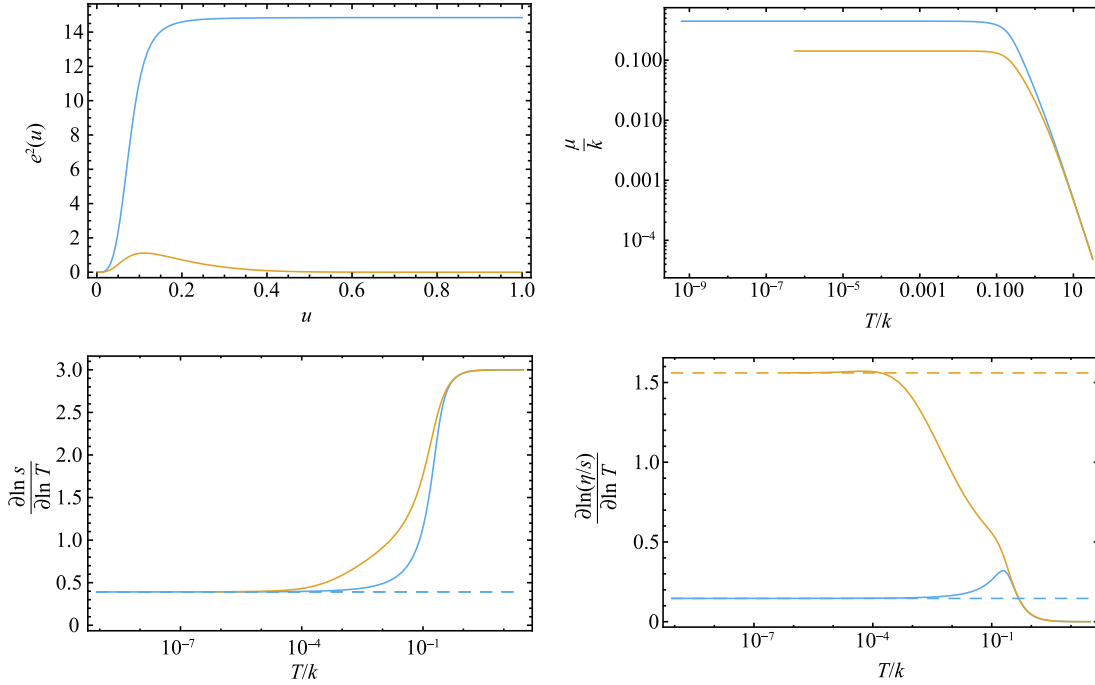


Fig. 6.   (color online) $d = 3$, $\theta > 0$. Parameters are $\alpha = -1$, $\beta = -1.5$, $\gamma = 2.5$ (blue, class I) or 5 (orange, class II), then $z = 2.56$ and $\theta = 2$.

### 3.2.2   Canonical ensemble

In the canonical ensemble, we control the value of charge density $\rho$. The $t$ component of Maxwell equations (7c) can be replaced by (14).

When $\theta = 0$, the three-parameter family of solutions is characterized by $\left\{ \dfrac{T}{k}, \ \dfrac{\rho}{k^d}, \ \lambda \right\}$. When $\theta \neq 0$, it is characterized by $\left\{ \dfrac{T}{k}, \ \dfrac{\rho}{k^d}, \ \dfrac{\lambda}{k} \right\}$. When lowering $T/k$, we fix

the other two dimensionless parameters within an appropriate region. Then $\mu/k$ converges to a nonzero constant at low $T/k$ instead of $\rho/k^d$ as in the grand canonical ensemble. The behaviors of $\eta/s$ and $e^2$ are similar to those in the grand canonical ensemble. By using the method in Appendix A, we can foresee the value to which $e^2$ converges in the IR at low temperature for class I.

In a parallel manner, we conduct the numerical calculation for $d=3$ and $\theta=-6,\ 0,\ 2$ to represent the three regions of $\theta<0$, $\theta=0$ and $0<\theta<d$. The specific results are shown in Figs. 4, 5 and 6. All the numerical results match the analytical ones.

## 4   Conclusion and outlook

In this paper we have numerically constructed charged solutions with emerging hyperscaling violation in the EMD-Axion model and investigated the temperature behavior of the ratio of shear viscosity to entropy density. We have found that the relevant axion, which breaks the translational invariance, leads to the power law of $\eta/s \sim T^\kappa$. In particular, the relevant current reduces the exponent $\kappa$. This reduction is characterized by the quantity $e^2$, which can be derived from the dimen-

sionless conserved charge density $\rho/k^d$. Irrelevant current does not affect the exponent $\kappa$ since $e^2 \to 0$ in the far IR at low temperature. Our analytical results for the exponent $\kappa$ coincide with our numerical calculation, indicating that our proposed formula for $\kappa$ in (5) is robust at least for generic backgrounds within the EMD-Axion model.

Especially, our results for the Lifshitz case verify that $\kappa>2$ can happen for $d>2$, indicating that the hyperscaling violation is not the essential ingredient leading to the exponent $\kappa>2$. Moreover, as conjectured in [13], the upper bounds for $\kappa$ coincide with the behavior of entanglement entropy.

Analytically, it is possible that irrelevant current or axions can affect the temperature behavior of $\eta/s$ at subleading orders. One should consider their backreaction to the background, then solve the shear perturbation equation (16). It is related to the issue whether the temperature $T$ is still the unique scale in entropy production, as it is when the axion is relevant. This is worthy of investigation in the future.

## Appendix A

## The classification of IR solutions

We focus on relevant axion solutions, otherwise $\eta/s$ just converges to a non-zero constant at low temperature at leading order. We require that the solutions should have positive specific heat, and the temperature deformation is the only allowed relevant deformation. In addition, we give an extra requirement of $\theta<d$. The scaling solutions have been obtained and classified in Ref. [31, 32].

### A.1   Class I: marginally relevant current, marginally relevant axion

If both the Maxwell and the axion terms are the same order in the power of the radial coordinate as the curvature term and the dilaton potential $V(\phi)$ in the Lagrangian (6), the scaling solutions obtained form a one-parameter family

$$\beta\epsilon=-2,\quad \alpha\epsilon=-\frac{2\theta}{d},\quad \gamma=\alpha(d-1)-\beta d,$$

$$\epsilon^2=\frac{2(d-\theta)(d(z-1)-\theta)}{d},$$

$$\zeta=\theta-d,\quad L^2=\frac{2(\delta_0-1)\delta_0}{2V_0-(d-1)k^2},$$

$$Q^2=\frac{2\left(k^2(dz-\theta)+2V_0(1-z)\right)}{\delta_0\left((d-1)k^2-2V_0\right)}, \tag{A1}$$

which can be parameterized by $k$ in coordinates of (10). The charge related quantities are

$$\rho^2 = \frac{\delta_0^2 Q^2}{L^2} = \frac{k^2(\theta-dz)+2V_0(z-1)}{\delta_0-1}, \tag{A2}$$

$$e^2 = \frac{\delta_0^2 Q^2}{L^2 k^2} = \frac{k^2(\theta-dz)+2V_0(z-1)}{k^2(\delta_0-1)}. \tag{A3}$$

The mode analysis in Ref. [32] indicates that there are three pairs of conjugate modes summing to $\delta_0$. Two pairs are degenerate with $\beta_{1,-}=\beta_{2,-}=0$ and $\beta_{1,+}=\beta_{2,+}=\delta_0$. $\beta_{1,-}$ rescales the time. $\beta_{1,+}$ is temperature deformation and is responsible for creating a small black hole (11). $\beta_{2,-}$ changes $k$ and shifts the solution along the one-parameter family. $\beta_{2,+}$ changes the chemical potential and belongs to the transformation of gauge symmetry. The expression for the last pair of $\beta_{3,\pm}$ is too tedious to show here. We require that $\beta_{1,+}$ is relevant and $\beta_{3,-}$ is irrelevant, with IR located at $r\to\infty$. Then the final allowed parameter space here is found to be $\rho^2>0$ and (27).

Since the quantity $\frac{\rho}{k^d}=\frac{\sqrt{-g_{tt}g_{rr}g_{xx}^d}\,Z(\phi)F^{rt}}{(\partial_x\chi)^d}$ is conserved and invariant under coordinate transformation within (8), we can use it to connect UV with IR and determine the solution in the one-parameter family by using (A2). Finally, $e^2$ can be obtained from $\frac{\rho}{k^d}$ by using (A3), which is rather convenient in canonical ensemble.

At zero temperature, one can integrate the three modes of $\beta_{2,-}$, $\beta_{2,+}$ and $\beta_{3,-}$ to the UV and adjust them to satisfy the boundary conditions specified by $\{\lambda, \rho, \mu\}$ at the conformal boundary. A finite temperature solution can be driven by $\beta_{1,+}$.

### A.2   Class II: irrelevant current, marginally relevant axion

If only the Maxwell term turns to be subleading, a single scaling solution at leading order is obtained as

$$\beta\epsilon = -2, \quad \alpha\epsilon = -\frac{2\theta}{d}, \quad \epsilon^2 = \frac{2(d-\theta)(d(z-1)-\theta)}{d},$$
$$L^2 = \frac{\delta_0(dz-\theta)}{V_0}, \quad k^2 = \frac{2V_0(z-1)}{dz-\theta}. \tag{A4}$$

There are three pairs of modes, in which two pairs sum to $\delta_0$. The first pair is $\beta_{1,-} = 0$ and $\beta_{1,+} = \delta_0$ which are rescaling of time and temperature deformation. The second pair is relevant $\beta_{2,+}$ and irrelevant $\beta_{2,-}$. The third pair is gauge field modes with

$$A(r) = A_1 + A_2 r^{\zeta-z}, \zeta = d - \theta + \frac{2\theta}{d} - \epsilon\gamma. \tag{A5}$$

The mode of $A_2$ is irrelevant when $\delta_0(\zeta + d - \theta) < 0$. We find $e^2(r) \sim r^{\zeta-d+\theta}$ and $\rho \sim T^0, e_h^2 \sim T^{\frac{\zeta-d+\theta}{-z}}$, when $T \to 0$. Similar to class I, we can integrate $\beta_{2,-}$ and gauge field modes to the UV.

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