Finite-size behavior near the critical point of QCD phase-transition *

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Abstract: It is pointed out that the finite-size effect is not negligible in locating the critical point of quantum colordynamics (QCD) phase transitions at current relativistic heavy ion collisions. The finite-size scaling form of the critical related observable is suggested. Its fixed point behavior at critical incident energy can be served as a reliable identification of a critical point and nearby boundary of QCD phase transition. How to experimentally find the fixed point behavior is demonstrated by using 3D-Ising model as an example. The validity of the method at finite detector acceptances at RHIC is also discussed.

Key words: finite size scaling, QCD phase transition, fixed point, detector effects

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1 Introduction

Quantum colordynamics (QCD) has predicted quark deconfinement and chiral symmetry restoration at finite temperature and density [1]. Lattice-QCD has shown that the transition is at the crossover at the vanishing baryon chemical potential $\mu_{\rm B}$ [2]. The QCD based model indicates that the crossover turns out to be a first-order phase transition at larger values of $\mu_{\rm B}$ [3]. The endpoint of the first order phase transition to the crossover is referred to as the critical endpoint, or the critical point. All these show that the transition from the hadron phase to quark-gluon-plasma (QGP) phase can happen in 3 possible ways.

The data from current relativistic heavy ion experiments show that the QGP has been formed at RHIC energies [4]. But the position of the critical point in the QCD phase diagram is not clear from the theoretical side. The well defined character of the critical point, the divergence of correlation length, warrants the possibility of finding it experimentally. The goal of the beam energy scan at RHIC, and future heavy ion experiments at FAIR is aimed to pass through the critical incident energy.

In relativistic heavy collision, both the size and duration of formed system are finite. From event to event, the overlapping area, i.e., the formed system size, varies with impact parameter. For the finite-size system, critical behavior changes with system size (L). If the system size is too small, the correlation length can not be fully developed to cause a phase transition. If the system size is large enough and the correlation length (ξ) is much smaller in comparison to system size, the system can still be considered as infinitely large. The critical behavior under thermal limit is available. This is why the non-monotonic behavior is suggested as an indicator of the critical point [5–7] in a long period.

Nevertheless, non-monotonic behavior is not unique to the critical point. In the case of first order phase transition, or crossover, some observables also show non-monotonic behavior [2]. The absence of non-monotonic behavior does not exclude the existence of the critical point, such as the maximum cluster size in 3D-Ising model shown in Fig. 1(a).

Moreover, if the correlation length is comparable to system size, the finite-size effect is not negligible. When the correlation length is larger than $\frac{1}{6}$ of the system size, it has been shown that the finite-size effect has to be taken into account [8, 9].

Although, it is still difficult to estimate the size of the formed system and correlation length at the critical point in relativistic heavy ion collisions. A rough estimation shows that the system size at freeze-out is less than 12 fm [10, 11]. The correlation length is around 6 fm for typical nuclear collisions [11, 12]. After considering the finite evolution time, or finite-size, it is argued that the

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maximum of the correlation length may not be beyond 2–3 fm at the critical point [13–15]. Based on those estimations, the ratio of the correlation length to system size is around $\frac{1}{6}$–$\frac{1}{2}$, which is in the region larger than $\frac{1}{6}$. So in current relativistic heavy ion collisions at RHIC, the finite-size effect most probably has to be taken into account, rather than negligible [11].

In accounting for the finite-size effects, the critical behavior of all suggested observables, such as the fluctuations and correlations of transverse momentum, multiplicity, conserved charges [16], and in particular, the higher order moments of conserved quantities [17], should be re-examined under the frame of finite-size behavior of the critical point and nearby boundary.

In this, we firstly discuss the finite-size behavior of the critical point, the first order phase transition, and the crossover in general. Secondly, we suggest the finite-size scaling form of the critical related observable in relativistic heavy ion collisions. Its fixed-point behavior at critical incident energy can be served as a reliable identification of the critical point and nearby boundary of QCD phase transition. We demonstrate how to locate the fixed point from an experimental observable by using the 3D-Ising model as an example. Finally, the validity of the method at finite detector acceptance at RHIC is discussed.

2 The behavior of finite size scaling and fixed point

For the second order phase transition, the critical behavior is well described by finite-size scaling. It was firstly proposed from phenomenological [18] and renormalization-group [19] theories, and was approved by the Monte Carlo results of finite systems in different universal classes [20]. This scaling form not only describes the behavior of the observables at different system sizes, but also indicates the position of the critical point and the critical exponents in an infinite system. Therefore, from the finite-size scaling of critical related observables, the position and critical exponents of the critical point can be precisely extracted. This has been implemented in locating the critical point of multifragmentation nuclear liquid-gas phase transition [21].

In contrast to the critical point, the finite-size behavior of first order phase transition has not been well understood in general [22]. But the finite-size scaling behavior of first order phase transition is shown to correspond to so-called discontinuity fixed points of the renormalization group transformations, which are characterized by eigenvalue exponents equal to the spacial dimension [23]. Consequently, the finite-size scaling form pertains, and the scaling exponents are the spatial dimension, in contrary to the critical exponents of the critical point. The phenomenological theory of finite-size scaling at first-order phase transition is proposed by K. Binder and D.P. Landau, and it is found to be in good agreement with Monte Carlo simulation results [24].

Different from the critical point and the first order phase transition, at the crossover region, there is no singularity in all kinds of observables. The observables are system size independent [2, 25]. But it should be noticed that this holds only when the system size is not too small. When the system size is very small and the finite correlation length is comparable with the system size, the observables will become larger and larger when the system size becomes smaller and smaller.

Consequently, the formula of the finite-size scaling is:

$$Q(T,L)=L^{\lambda/\nu}F_{\mathrm{Q}}(tL^{1/\nu}), \tag{1}$$

where $t=(T-T_{\mathrm{c}})/T_{\mathrm{c}}$ represents the reduced temperature and λ/ν is dependent on the order of the phase transition. In the experiment, we need to re-built this formula by using the appropriate variables which are related to the scaling variables T and L shown in Eq. (1).

In heavy ion collisions, the frozen-out temperature can be parameterized by the center of mass energy $\sqrt{s}$ [26]. $\sqrt{s}$ can be taken as the scaling variable if we assume the frozen out curve is close to the phase transition. The centrality, i.e., the impact parameter, presents the overlapped area of two incident nuclei. It is directly related to the size of the formed system, and randomly fluctuates from event to event. The critical related observables are generally considered to be the fluctuations of conserved charges, like the baryon number, electric charge, and strangeness [16, 17, 27]. The incident energy and centrality dependence of some related observables are fully investigated in current heavy ion experiments [28].

Therefore, the finite-size scaling in nuclear collisions can be generalized as following. When the size of the formed matter L is much larger than the microscopic length scale (which is less than 1 fm) and the incident energy is near the critical one $\sqrt{s}_{\mathrm{c}}$, the critical related observable, e.g., $Q(\sqrt{s},L)$ in general, can be written in a finite-size scaling form [18–20],

$$Q(\sqrt{s},L)=L^{\lambda/\nu}F_{\mathrm{Q}}(\tau L^{1/\nu}). \tag{2}$$

Where $\tau=(\sqrt{s}-\sqrt{s}_{\mathrm{c}})/\sqrt{s}_{\mathrm{c}}$ is the reduced incident energy. ν and λ are the critical exponents of the correlation length $\xi=\xi_0\tau^{-\nu}$ and the observable, respectively. They characterize the universal class of the phase transition. Finite-size scaling indicates that the observable at different system sizes can be re-scaled to an identical scaling function F_{Q} with the scaled variable $\tau L^{1/\nu}$.

At critical energy, $\sqrt{s}=\sqrt{s}_{\mathrm{c}}$, the scaled variable ($\tau L^{1/\nu}=0$) is independent of system size L, and the scaling function becomes a constant,

$$F_{\mathrm{Q}}(0)=Q(\sqrt{s}_{\mathrm{c}},L)L^{-\lambda/\nu}. \tag{3}$$

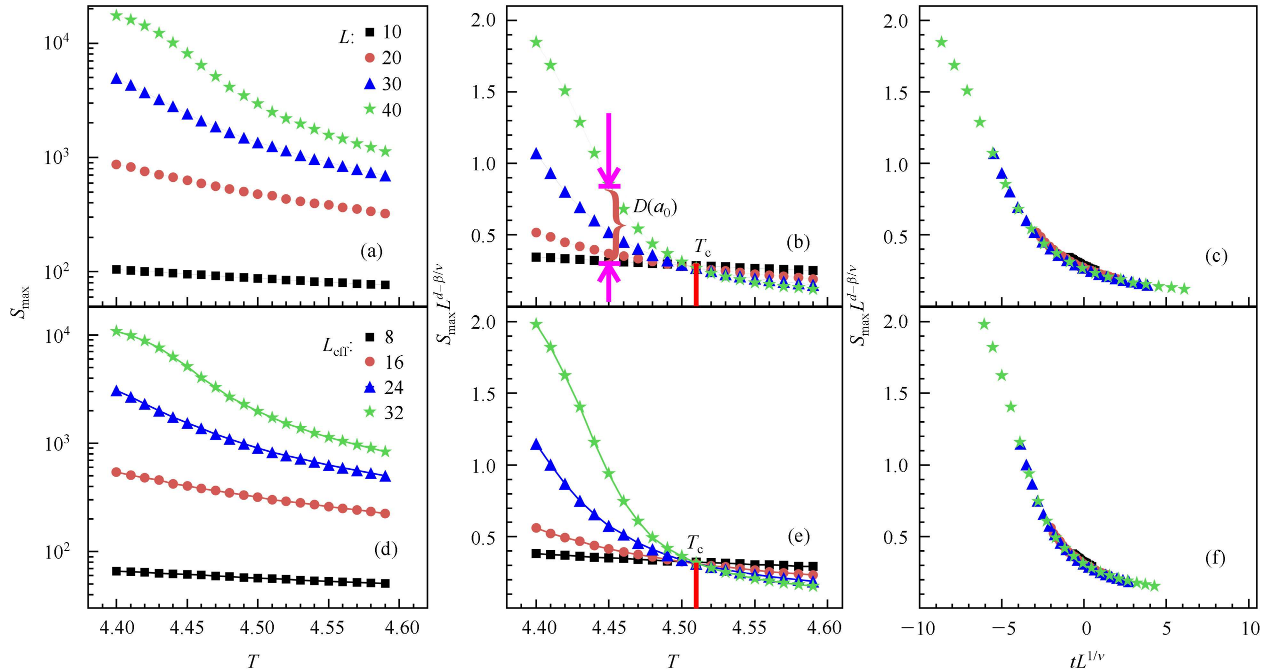

Fig. 1. (color online) Upper panel: (a) and (b) are the temperature dependence of maximum cluster size and maximum cluster size scaled by $L^{d-\beta/\nu}$, and (c) is the scaling function of maximum cluster size in 3D-Ising model at different lattice sizes L. Lower panel: the same measures as in corresponding upper panel, but in a sub-system containing 50% of the lattice sites.

It shows that the fluctuation of a critical related observable is self-similar at different size scales. In this case, the energy dependence of the observable at various of system sizes will intersect to this point, i.e., fixed point. The energy of the fixed point indicates the critical incident energy. As an example, we show in Fig. 1(b) the fixed point behavior of the maximum cluster in the 3D-Ising model, which is supposed to be the same university of the de-confinement [29]. We can see that the maximum cluster at different lattice sizes intersect exactly at the fixed point, i.e., the critical point.

Reversely, if we can find the fixed point from the incident energy dependence of a properly scaled observable in heavy ion collisions, which are measured at different centralities, i.e., system sizes, it will indicate the existence of the critical point.

In order to find the exponent of the scale, and the incident energy of the fixed point, we can firstly present the incident energy dependence of critical related observable at different system sizes, similar to Fig. 1(a). Then multiply a size factor to the observable $Q(\sqrt{s},L)$, i.e., $Q(\sqrt{s},L)L^{-a}$, and change the parameter a from $-\infty$ to ∞ to see if all size curves interact to a point for a certain value of a_0 at a certain incident energy, e.g., Fig. 1(b).

In the experiment, the point liked behavior can be quantified by the width of all size points. At a given incident energy, the width is usually defined as the square root of χ^2 of all size points, i.e.,

$$D(\sqrt{s},a)=\sqrt{\frac{\chi^2_{Q(\sqrt{s},L)L^{-a}}}{N_{\rm L}-1}}. \quad (4)$$

$N_{\rm L}$ is the number of points, and $\chi^2_{Q(\sqrt{s},L)L^{-a}}$ is the error weighted variance of all size points,

$$\chi^2_{Q(\sqrt{s},L)L^{-a}}=\sum_{i=1}^{N_{\rm L}}\frac{\left[Q(\sqrt{s},L_i)L_i^{-a}-\langle Q(\sqrt{s},L)L^{-a}\rangle\right]^2}{w_i^2}, \quad (5)$$

$w_i=\delta\left[Q(\sqrt{s},L_i)L_i^{-a}\right]$ are the experimental errors of $\left[Q(\sqrt{s},L_i)L_i^{-a}\right]$, where both the errors of the observable $Q(\sqrt{s},L_i)$ and system size L_i^{-a} contribute to. $\langle Q(\sqrt{s},L)L^{-a}\rangle$ is also the error weighted mean,

$$\langle Q(\sqrt{s},L)L^{-a}\rangle=\frac{\sum\limits_{i=1}^{N_{\rm L}}Q(\sqrt{s},L_i)L_i^{-a}/w_i^2}{\sum\limits_{i=1}^{N_L}1/w_i^2}. \quad (6)$$

For example, in Fig. 1(b), this width at a given temperature is the distance between two violet arrows. Therefore, if at a given incident energy, the minimum of $D(\sqrt{s},a)$ is around 1 at a_0, i.e., $D_{\min}(\sqrt{s},a_0)\sim 1$, it can be recognized as an experimental point. While, if it is larger than 1, there is no point liked behavior.

For the QGP formed system [4], the following 3 cases should be expected. (1) $D_{\min}(\sqrt{s},a_0)$ at a certain incident energy is around 1, and at nearby incident energies, it is always larger than 1, and correspondingly a_0 is not an integer, as the green curve shows in the middle of Fig. 2. This may indicate the existence of the fixed point, i.e., the critical point in Eq. (3). The critical incident energy is the energy of the fixed point and the obtained parameter a_0 is the ratio of critical exponents, i.e., $\lambda/\nu{=}a_0$, see Fig. 1(b).

In this case, the critical behavior should be further confirmed by the scaling function,

$$F_{\rm Q}(\tau L^{1/\nu}){=}L^{-a_0}Q(\sqrt{s},L). \tag{7}$$

Here the critical exponent of correlation length ν is a fitting parameter. If the data at all incident energies and system sizes can be well fitted by the scaling function, the critical point and the critical exponents are finally determined, see Fig. 1(c).

(2) $D_{\rm min}(\sqrt{s},a_0)$ at a certain incident energy is around 1, and at nearby incident energies, it is always larger than 1, and correspondingly a_0 is an integer. This also indicates the existence of the fixed point, but scaled power is a trivial integer. It implies the region of the first order phase transition. The incident energy of the fixed point is the transition energy of the first order phase transition. The scaling function of the observable should be simply formulated by the spatial dimension, instead of the critical exponents in Eq. (1).

If a_0 is zero, there are two possibilities. It could be the critical point with the critical exponent $\lambda=0$, like the Binder cumulant ratio [30], or the region of the first order phase transition. The final identification is their specified scaling functions, as discussed above.

(3) $D_{\rm min}(\sqrt{s},a_0)$ is around 1 at all incident energies and correspondingly a_0 is an integer. This indicates all size curves are overlapped, and there is in fact no fixed point. It corresponds to the transition of crossover.

3 Detector effects

It should be stressed that the observables mentioned here are intensive variables, like susceptibility. If the observables are extensive variables, such as the fluctuation of the particle number, $\langle(N{-}\overline{N})^2\rangle{=}TV\chi$, the trivial size dependence is included, and can be merged to the power a.

The size of the formed matter in heavy collisions is mainly determined by the overlapping area of two incident nuclei. This area is proportional to the number of participant nucleons and is quantified as centrality. The initial size of the formed matter can be approximately estimated by the square root of the number of participants, $\sqrt{N_{\rm part}}$. The maximum size is $\sqrt{2N_{\rm A}}$, $N_{\rm A}$ is the number of nucleons of incident nucleus. The ratio,

$$L{=}\sqrt{N_{\rm part}}/\sqrt{2N_{\rm A}}, \tag{8}$$

presents the relative size of the initial system.

The system size L' at transition should be larger than the initial one L and monotonically increase the function of L, i.e., $L'{=}cL^{1+\delta}$ with $\delta\geqslant 0$ in general. Whether we take L' or L in Eq. (2), the scaling exponents will be different, but the position of the critical point will be the same. So the initial size is a good approximation in locating the position of the critical point.

It should also be noticed that the detectors at current relativistic heavy ion experiments cover a part of the phase space, and only a part of final state particles is accepted. Even if the critical related information survived in the final state observables, whether the finite-size behavior of detected subsystem is preserved has to be examined further.

Due to the universality of the phase transition, the 3D-Ising model is extensively used to study the properties of the QCD deconfinement phase transition in heavy ion collisions [31]. In this section, the finite size behavior of a sub-system is demonstrated in the 3D-Ising model. The size of the sub-system is chosen to be a certain percent of the whole lattice sites. Changing the lattice of the whole system, the effective sites of the sub-system, $L_{\rm eff}$, vary with it. We find that the finite size behavior of the sub-system remains valid as long as the size of the sub-system is within the range of finite size scaling.

In the lower panel of Fig. 1, the finite size behavior of the maximum cluster size at various $L_{\rm eff}$ is presented. Where the size of the sub-system is 50% of the whole system. In comparison with the corresponding results of the whole system shown in the upper panel of Fig. 1, the susceptibilities of the sub-system is different from that of the whole one, but the position of the fixed point indicates the same critical temperature, $T_{\rm c}{=}4.51$ J. Moreover, the maximum cluster at different sub-system sizes is well scaled to an identical scaling function. Therefore, the suggested finite size behavior should be visible at a detector with a relatively large acceptance, like RHIC/STAR.

4 Summary

In summary, it is pointed out that the finite-size effects are not negligible in locating the critical point of the QCD phase transition at current relativistic heavy ion collisions. At the crossover, critical point and first order QCD phase transition, the finite-size scaling behaviors of the critical related observable are suggested.

The critical point of QCD phase transition can be found by the appearance of the fixed point with a non-integer power in the scaled size factor, and the finite-size scaling function of the observable. The region of the first order phase transition is identified by the fixed point with an integer power in the scaled size factor and the scaling function which is determined by spatial dimension.

At the region of the crossover, the behavior of the fixed point is absent, and the scaling function reduces to the incident energy dependence of the observable, which is system size independent.

At a given incident energy, the width of the observables at various centralities is suggested as a quantification of point liked behavior. The energy dependence of the width at different orders of phase transitions are shown. When incident energy scans from high to low, the deviation of minimum width from point like behavior will indicate the appearance of the critical point.

Finally, for a finite acceptance detector, we demonstrate that the finite-size behavior of critical related observables remains valid as long as the detected subsystem is large enough.

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