Quantum phase transitions in matrix product states of one-dimensional spin-1 chains *

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Abstract: We present a new model of quantum phase transitions in matrix product systems of one-dimensional spin-1 chains and study the phases coexistence phenomenon. We find that in the thermodynamic limit the proposed system has three different quantum phases and by adjusting the control parameters we are able to realize any phase, any two phases equal coexistence and the three phases equal coexistence. At every critical point the physical quantities including the entanglement are not discontinuous and the matrix product system has long-range correlation and N-spin maximal entanglement. We believe that our work is helpful for having a comprehensive understanding of quantum phase transitions in matrix product states of one-dimensional spin chains and of certain directive significance to the preparation and control of one-dimensional spin lattice models with stable coherence and N-spin maximal entanglement.

Key words: matrix product state, quantum phase transition, long-range correlation, entanglement entropy

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1 Introduction

The study of quantum many-body systems is always a much more intensive research subject in the fields of condensed matter and quantum information due to the inherent richness and complexity of a large number of interacting particles as well as the potential application prospect in solid-state quantum computing, among which quantum phase transitions (QPTs) occupy a distinguished position. These transitions, taking place at zero temperature, are driven by fluctuations due to the Heisenberg uncertainty principle even in the ground states (GSs) [1]. The fundamental starting point of studying quantum many-body systems in the fields of condensed matter is the system Hamiltonian and then to study the properties of its GS. At one time we encountered a bottle-neck due to the seldomness of the ground-state analytic solution of the Hamiltonian. New blood was brought into the study of quantum many-body systems through the addition of some knowledge of quantum information and the quantum information approach which deals primarily with the quantum ground state, and the corresponding parent Hamiltonian may be constructed such that the state is exactly the GS bypassing the aforementioned difficulty.

However, related articles [2, 3] showed that for one-dimensional spin lattice models, every many-body state, in particular, every GS of a finite many-body system dictated and characterized by a local Hamiltonian can be represented as a matrix product state (MPS). The power of this representation stems from the fact that in many cases a low-dimensional MPS already yields a very good approximation of the state [4], such as the Greenberger-Horne-Zeilinger (GHZ) state of the form $|\Psi\rangle=|1\cdots1\rangle+|0\cdots0\rangle$ with $A_1=|0\rangle\langle0|$ and $A_2=|1\rangle\langle1|$ [5], the cluster state which is the unique ground state of the three-body interactions $\sum_i\sigma_i^z\sigma_{i+1}^x\sigma_{i+2}^z$ and represented by the matrices

$$\left\{A_1=\begin{bmatrix}0&0\\1&1\end{bmatrix},\ \ A_2=\begin{bmatrix}1&-1\\0&0\end{bmatrix}\right\}$$

[5], and the exact matrix product ground state of the Affleck-Kennedy-Lieb-Tasaki model specified by $\{A_i\}=\{\sigma_z,\ \sqrt{2}\sigma_+,\ -\sqrt{2}\sigma_-\}$ [5]. MPSs are therefore undoubtedly a new powerful and convenient playground for studying one-dimensional spin lattice model theory, especially for quantum phase transitions, by use of the quantum information approach [2, 6–11].

Here it is stressed that the transition point is defined as any discontinuity in an observable quantity in a wider sense than usual [9–11]. In this paper we present a new model of quantum phase transitions in matrix product

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systems of one-dimensional spin-1 chains and study the phases coexistence phenomenon. We find that in the thermodynamic limit the specified system has three different quantum phases and by adjusting the control parameters we are able to realize any phase, any two phases equal coexistence and the three phases equal coexistence. At every critical point the physical quantities including the entanglement are not discontinuous and the matrix product system has long-range correlation and N-spin maximal entanglement.

2 Model and method

Let us begin with the one-dimensional translation invariant MPS:

$$|\Psi\rangle=\frac{1}{\sqrt{\mathcal{N}}}\sum_{i_1,\cdots,i_N=1}^{d}\mathrm{Tr}(A^{i_1}\cdots A^{i_N})|i_1,\cdots,i_N\rangle, \qquad (1)$$

where d is the dimension of Hilbert space of one site in the spin chain, and a set of $D\times D$ matrices $\{A^i, i=1,\cdots,d\}$ parameterize the N-spin state with the dimension $D\leqslant d^{N/2}$ [3]. $E=\sum_{i=1}^{d}\bar{A}^i\otimes A^i$ contained in the normalization factor $\mathcal{N}=\mathrm{Tr}E^N$, is the so-called transfer matrix and the symbol bar denotes complex conjugation.

2.1 The concrete model

Here we present the MPS $|\Psi\rangle$ with

$$A^1=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad A^2=\begin{bmatrix}0&0&0\\0&\gamma_1&0\\0&0&0\end{bmatrix},\quad A^3=\begin{bmatrix}0&0&0\\0&0&0\\0&0&\gamma_2\end{bmatrix}, \qquad (2)$$

where γ_1, $\gamma_2>0$. It is shown that the transfer matrix E has nonzero eigenvalues $\{\lambda_{\mathrm{a}}\equiv1,\ \lambda_{\mathrm{b}}\equiv{\gamma_1}^2,\ \lambda_{\mathrm{c}}\equiv{\gamma_2}^2\}$ and then $E=|\lambda_{\mathrm{a}}^{\mathrm{R}}\rangle\langle\lambda_{\mathrm{a}}^{\mathrm{L}}|+{\gamma_1}^2|\lambda_{\mathrm{b}}^{\mathrm{R}}\rangle\langle\lambda_{\mathrm{b}}^{\mathrm{L}}|+{\gamma_2}^2|\lambda_{\mathrm{c}}^{\mathrm{R}}\rangle\langle\lambda_{\mathrm{c}}^{\mathrm{L}}|$ where the normalized right (left) eigenvector $|\lambda_{\mathrm{i}}^{\mathrm{R(L)}}\rangle(\mathrm{i}=\mathrm{a,b,c})$ corresponding to the nonzero eigenvalue λ_{i}. Obviously for $0<\gamma_1,\ \gamma_2<1$ the largest absolute eigenvalue is $\lambda_{\max}=\lambda_{\mathrm{a}}$, for $\gamma_1>1,\ \gamma_2,\ \lambda_{\max}=\lambda_{\mathrm{b}}$ and for $\gamma_2>1,\ \gamma_1,\ \lambda_{\max}=\lambda_{\mathrm{c}}$, which indicts that in the thermodynamic limit the proposed system varying with the parameters γ_1 and γ_2 is shown in Fig. 1. For $0<\gamma_1,\ \gamma_2<1$; $\gamma_1>1,\ \gamma_2$ and $\gamma_2>1,\ \gamma_1$ the proposed system is respectively in the region of phase $|\Psi_{\mathrm{a}}\rangle=|1\cdots1\rangle$, $|\Psi_{\mathrm{b}}\rangle=|0\cdots0\rangle$ and $|\Psi_{\mathrm{c}}\rangle=|{-1}\cdots{-1}\rangle$; hence the specified system has three different two-phase transition lines $\gamma_1=1>\gamma_2(\mathrm{ab})$, $\gamma_2=1>\gamma_1(\mathrm{ac})$ and $\gamma_2=\gamma_1>1(\mathrm{bc})$ i.e., $\mathrm{ij}(\mathrm{i,j}=\mathrm{a,b,c,i}\neq\mathrm{j})$ where the two phases $|\Psi_{\mathrm{i}}\rangle$ and $|\Psi_{\mathrm{j}}\rangle$ coexist equally, and the three-phase coexisting point $\gamma_2=\gamma_1=1$ i.e., the point abc at which the three phases coexist equally. In the following we will investigate in detail the properties of the kind of MPS QPT by the aforementioned quantum information approach.

2.2 The properties of the kind of MPS QPT

2.2.1 The properties of local physical observables

First we turn to the properties of local physical observables. For a local observable of l adjacent spins, $O^{(1,l)}\equiv O_{i_1}^{[1]}\cdots O_{i_l}^{[l]}$ the expectation is expressed as

$$\langle\Psi|O^{(1,l)}|\Psi\rangle=\frac{\mathrm{Tr}(E_{O^{(1,l)}}E^{N-l})}{\mathrm{Tr}(E^N)}, \qquad (3)$$

where $E_{\boldsymbol{O}^{(1,l)}}=E_{O_{\mathrm{i}_1}}E_{O_{\mathrm{i}_2}}\cdots E_{O_{\mathrm{i}_l}}$ and

$$E_{O_k}\equiv\sum_{\mathrm{i,i'}}\langle\mathrm{i}|O_k|\mathrm{i'}\rangle\bar{A}^{\mathrm{i}}\otimes A^{\mathrm{i'}},$$

taking the thermodynamic limit $N\rightarrow\infty$, which reduces to

$$\langle O^{(1,l)}\rangle=\begin{cases}\langle O^{(1,l)}\rangle_{\mathrm{a}}=\dfrac{\langle\lambda_{\mathrm{a}}^{\mathrm{L}}|E_{O^{(1,l)}}|\lambda_{\mathrm{a}}^{\mathrm{R}}\rangle}{(\lambda_{\mathrm{a}})^l} & \gamma_1,\gamma_2<1,\\[2ex] \langle O^{(1,l)}\rangle_{\mathrm{b}}=\dfrac{\langle\lambda_{\mathrm{b}}^{\mathrm{L}}|E_{O^{(1,l)}}|\lambda_{\mathrm{b}}^{\mathrm{R}}\rangle}{(\lambda_{\mathrm{b}})^l} & \gamma_1>\gamma_2,1,\\[2ex] \langle O^{(1,l)}\rangle_{\mathrm{c}}=\dfrac{\langle\lambda_{\mathrm{c}}^{\mathrm{L}}|E_{O^{(1,l)}}|\lambda_{\mathrm{c}}^{\mathrm{R}}\rangle}{(\lambda_{\mathrm{c}})^l} & \gamma_2>,\gamma_1,1,\\[2ex] \langle O^{(1,l)}\rangle_{\mathrm{ab}}=\dfrac{1}{2}(\langle O^{(1,l)}\rangle_{\mathrm{a}}+\langle O^{(1,l)}\rangle_{\mathrm{b}}) & \gamma_1=1>\gamma_2,\\[2ex] \langle O^{(1,l)}\rangle_{\mathrm{ac}}=\dfrac{1}{2}(\langle O^{(1,l)}\rangle_{\mathrm{a}}+\langle O^{(1,l)}\rangle_{\mathrm{c}}) & \gamma_2=1>\gamma_1,\\[2ex] \langle O^{(1,l)}\rangle_{\mathrm{bc}}=\dfrac{1}{2}(\langle O^{(1,l)}\rangle_{\mathrm{b}}+\langle O^{(1,l)}\rangle_{\mathrm{c}}) & \gamma_1=\gamma_2>1,\\[2ex] \langle O^{(1,l)}\rangle_{\mathrm{abc}}=\dfrac{1}{3}(\langle O^{(1,l)}\rangle_{\mathrm{a}}+\langle O^{(1,l)}\rangle_{\mathrm{b}}+\langle O^{(1,l)}\rangle_{\mathrm{c}}) & \gamma_1=\gamma_2=1.\end{cases} \qquad (4)$$

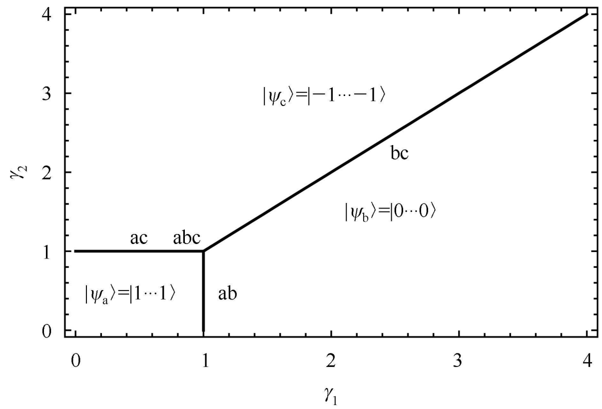

Fig. 1. The MPS $|\Psi\rangle$ as a function of the dimensionless parameters γ_1 and γ_2 in the thermodynamic limit. For $0<\gamma_1,\ \gamma_2<1$; $\gamma_1>1,\gamma_2$ and $\gamma_2>1,\ \gamma_1$ the proposed system is respectively in the region of phase $|\Psi_\mathrm{a}\rangle=|1\cdots1\rangle$, $|\Psi_\mathrm{b}\rangle=|0\cdots0\rangle$ and $|\Psi_\mathrm{c}\rangle=|-1\cdots-1\rangle$; hence, the specified system has three different phase transition lines $\gamma_1=1>\gamma_2(\mathrm{ab})$, $\gamma_2=1>\gamma_1(\mathrm{ac})$ and $\gamma_2=\gamma_1>1(\mathrm{bc})$ i.e., ij(i,j = a,b,c,i ≠ j) where the two phases $|\Psi_\mathrm{i}\rangle$ and $|\Psi_\mathrm{j}\rangle$ coexist equally, and the three-phase point $\gamma_2=\gamma_1=1$ i.e., the point abc at which the three phases coexist equally.

For simplicity, let us study the properties of the operator J_z. The physical quantity $\langle J_z\rangle$ is obtained as

$$\langle J_z\rangle=\frac{1-{\gamma_2}^{2N}}{1+{\gamma_1}^{2N}+{\gamma_2}^{2N}}. \tag{5}$$

Under the thermodynamic limit it takes a discrete form,

$$\langle J_z\rangle=\begin{cases}1 & 0<\gamma_1,\gamma_2<1,\\ 0 & \gamma_1>\gamma_2,1,\\ -1 & \gamma_2>\gamma_1,1\\ \dfrac{1}{2} & \gamma_1=1>\gamma_2,\\ 0 & \gamma_2=1>\gamma_1,\\ -\dfrac{1}{2} & \gamma_2=\gamma_1>1,\\ 0 & \gamma_1=\gamma_2=1,\end{cases} \tag{6}$$

which shows that only in the thermodynamic limit $\langle J_z\rangle$ turns out to be discontinuous, at the two-phase transition lines, $\gamma_1=1,\ \gamma_2<1,\ \gamma_2=1,\ \gamma_1<1$ and $\gamma_2=\gamma_1>1$, and at the three-phase coexisting point where $\gamma_1=1$ and $\gamma_2=1$. It follows that the quantum phase transition can take place only in the thermodynamic limit and is clearly manifested by the singularity of the above physical quantity.

2.2.2 The property of the correlation

The properties of the correlation at the three different two-phase coexisting lines and the three-phase coexisting point are discussed below. Firstly, the correlation function of two local blocks is

$$C_n[O^{(1,l)}]\equiv\langle\Psi|O^{(1,l)}O^{(n+1,n+l)}|\Psi\rangle-\langle\Psi|O^{(1,l)}|\Psi\rangle^2. \tag{7}$$

In the thermodynamic limit, for large distances $n\gg1$ and at the two-phase coexisting lines, this formula reduces to

$$C_\infty[O^{(1,l)}]=\frac{1}{4}(\langle O^{(1,l)}\rangle_\mathrm{i}^\mathrm{pt}-\langle O^{(1,l)}\rangle_\mathrm{j}^\mathrm{pt})^2(\mathrm{i,j=a,b,c,i\neq j}). \tag{8}$$

At the three-phase coexisting point, the long-range correlation is expressed as

$$\begin{aligned}C_\infty[O^{(1,l)}] = \frac{1}{9}[(\langle O^{(1,l)}\rangle_\mathrm{a}^\mathrm{pt}-\langle O^{(1,l)}\rangle_\mathrm{b}^\mathrm{pt})^2+(\langle O^{(1,l)}\rangle_\mathrm{a}^\mathrm{pt}\\ -\langle O^{(1,l)}\rangle_\mathrm{c}^\mathrm{pt})^2+(\langle O^{(1,l)}\rangle_\mathrm{b}^\mathrm{pt}-\langle O^{(1,l)}\rangle_\mathrm{c}^\mathrm{pt})^2].\end{aligned} \tag{9}$$

For the physical observable J_z, the long-range correlation is

$$C_\infty[J_z)]=\begin{cases}\dfrac{1}{4} & \gamma_1=1>\gamma_2,\\ 1 & \gamma_2=1>\gamma_1,\\ \dfrac{1}{4} & \gamma_2=\gamma_1>1,\\ \dfrac{2}{3} & \gamma_1=\gamma_2=1.\end{cases} \tag{10}$$

It follows that the proposed MPS $|\Psi\rangle$ has a long-range correlation at the critical points.

2.2.3 The entanglement property

Now, let us study the entanglement property of the MPS, the key quantity of quantum information theory [12–15], in detail. About measures of entanglement there are many kinds of methods, such as the quantification characteristic function of quantum nonlocality [16], Bell inequality [17, 18], quantum discord [19], averaged entropy [20] and so on. Considering our system, we shall adopt the von Neumann entropy which [21–28] according to bipartition parameterization by the adjacent spin number n of a $\mathcal{B}_n$ spin block is,

$$S_n=-\mathrm{Tr}(\rho_n\log_2\rho_n), \tag{11}$$

where $\rho_n=\mathrm{Tr}_{\bar{\mathcal{B}}_n}\rho$ is the reduced density matrix for the $\mathcal{B}_n$ block of n adjacent spins. The n-spin entanglement entropy S_n as a function of the parameters γ_1 and γ_2 is obtained as

$$S_n=\begin{cases}0 & 0<\gamma_1,\gamma_2<1,\\ 0 & \gamma_1>\gamma_2,1,\\ 0 & \gamma_2>\gamma_1,1\\ 1 & \gamma_1=1>\gamma_2,\\ 1 & \gamma_2=1>\gamma_1,\\ 1 & \gamma_2=\gamma_1>1,\\ \log_2 3 & \gamma_1=\gamma_2=1,\end{cases} \tag{12}$$

which is independent of the adjacent spin number n. Obviously no matter how the system is in any region of phase $|\Psi_{\rm a}\rangle$, $|\Psi_{\rm b}\rangle$ or $|\Psi_{\rm c}\rangle$ the n-spin entanglement entropy is zero; no matter how the system is in any line of any two-phase coexisting, the n-spin entanglement entropy takes the larger value of 1; while the system is at the three-phase coexisting point, the n-spin entanglement entropy takes the largest value of $\log_2 3$. It is worth pointing out that whether the n-spin are consecutive or not the n-spin entanglement entropy S_n is exactly the same. It follows that at the critical points the MPS $|\Psi\rangle$ have a larger entanglement entropy due to its coherent and collective properties.

2.2.4 The dynamics of the specified system

Here we undertake the study of the Hamiltonian of the specified system. In general, the condition of the k-spin reduced density matrix having null space is that $d^k > D^2$. Here it only needs $k \geqslant 2$. The MPS $|\Psi\rangle$ is the GS of any Hamiltonian which is a sum of local positive operators supported in that null-space. Thinking along this line we can always construct a local Hamiltonian such that a given MPS is its GS. Without loss of generality such a Hamiltonian is mathematically expressed as

$$H=\sum_i u_i(P_k), \tag{13}$$

with P_k being the projector onto the null-space of ρ_k and $u_i>0$ its translation to site i. In terms of the proposed system in the thermodynamic limit, the Hamiltonian is described by

$$H=\begin{cases} H_{\rm a}=\sum\limits_{i=1}^{N} I-\frac{1}{4}(J_{iz}^2J_{(i+1)z}^2+J_{iz}J_{(i+1)z}+J_{iz}^2J_{(i+1)z}+J_{iz}J_{(i+1)z}^2) & \gamma_1,\gamma_2<1,\\ H_{\rm b}=\sum\limits_{i=1}^{N} J_{iz}^2+J_{(i+1)z}^2-\frac{1}{2}J_{iz}^2J_{(i+1)z}^2 & \gamma_1>\gamma_2,1,\\ H_{\rm c}=\sum\limits_{i=1}^{N} I-\frac{1}{4}(J_{iz}^2J_{(i+1)z}^2+J_{iz}J_{(i+1)z}-J_{iz}^2J_{(i+1)z}-J_{iz}J_{(i+1)z}^2) & \gamma_2>1,\gamma_1,\\ H_{\rm ab}=\sum\limits_{i=1}^{N} J_{iz}^2+J_{(i+1)z}^2-\frac{1}{4}(5J_{iz}^2J_{(i+1)z}^2+J_{iz}J_{(i+1)z}+J_{iz}^2J_{(i+1)z}+J_{iz}J_{(i+1)z}^2) & \gamma_1=1>\gamma_2,\\ H_{\rm ac}=\sum\limits_{i=1}^{N} I-\frac{1}{2}J_{iz}^2J_{(i+1)z}^2-\frac{1}{2}J_{iz}J_{(i+1)z} & \gamma_2=1>\gamma_1,\\ H_{\rm bc}=\sum\limits_{i=1}^{N} J_{iz}^2+J_{(i+1)z}^2-\frac{1}{4}(5J_{iz}^2J_{(i+1)z}^2+J_{iz}J_{(i+1)z}-J_{iz}^2J_{(i+1)z}-J_{iz}J_{(i+1)z}^2) & \gamma_1=\gamma_2>1,\\ H_{\rm abc}=\sum\limits_{i=1}^{N} J_{iz}^2+J_{(i+1)z}^2-\frac{3}{2}J_{iz}^2J_{(i+1)z}^2-\frac{1}{2}J_{iz}J_{(i+1)z} & \gamma_1=\gamma_1=1. \end{cases} \tag{14}$$

By construction the GS energy is always zero, i.e., it is evidently analytic in γ and moreover $|\Psi\rangle$ is its unique GS for either side of the critical point discussed in Refs. [2, 9, 29]. The analyticity of the Hamiltonian ground state energy and the uniqueness of its GS for either side of the critical point immediately imply that a nonanalyticity in the physical quantities can only be caused by a vanishing energy gap at the transition points.

2.2.5 The long-wavelength behavior of the specified system

In order to have a comprehensive and deeper understanding of the kind of MPS QPT, we study below the scaling property. Specifically, we resort to the renormalization group approach to characterize the long-wavelength behavior of the specified system. Similar to the standard Kadanof Blocking scheme, the coarse-graining procedure for matrix product states could be achieved by merging the representative matrices of neighboring sites as $A\rightarrow A^{\rm (pq)}\equiv A^{\rm p}A^{\rm q}$ and subsequently performing a fine-grained transformation $A\rightarrow A'$ to select out new representatives [30]. The transfer matrix in every step transforms as $E\rightarrow E'\equiv E^2$ and an iterative process hence leads to a fixed point $E^{\infty}\equiv E^{\rm fp}$ in which only the vector(s) of the largest eigenvalue(s) can survive. In terms of the MPS $|\Psi\rangle$ under consideration, for $0<\gamma_1,\gamma_2<1$, the normalized transfer operator of the fixed point is $E^{\rm fp}=|\lambda_{\rm a}^{\rm R}\rangle\langle\lambda_{\rm a}^{\rm L}|$, and the corresponding representative matrices of the fixed point state $|\Psi_{\rm a}\rangle$ are obtained as

$$\{A^{\rm i}_{\rm a(fp)}\}=\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right\}, \tag{15}$$

which represents all particles spin up. For $\gamma_1>\gamma_2,1$, the normalized transfer operator of the fixed point is $E^{\rm fp}=|\lambda_{\rm b}^{\rm R}\rangle\langle\lambda_{\rm b}^{\rm L}|$, and the corresponding representative matrices of the fixed point state $|\Psi_{\rm b}\rangle$ are obtained as

$$\{A^{\rm i}_{\rm b(fp)}\}=\left\{\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right\},\qquad(16)$$

which represents that the spin of every particle is zero. For $\gamma_2>\gamma_1,1$, the normalized transfer operator of the fixed point is $E^{\rm fp}=|\lambda^{\rm R}_{\rm c}\rangle\langle\lambda^{\rm L}_{\rm c}|$, and the corresponding representative matrices of the fixed point state $|\Psi_{\rm c}\rangle$ are obtained as

$$\{A^{\rm i}_{\rm c(fp)}\}=\left\{\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\},\qquad(17)$$

which represents all particles spin down. For $\gamma_1=1>\gamma_2$, the corresponding fixed point of the MPS $|\Psi\rangle$ is characterized by the normalized $E^{\rm fp}_{\rm ab}=|\lambda^{\rm R}_{\rm a}\rangle\langle\lambda^{\rm L}_{\rm a}|+|\lambda^{\rm R}_{\rm b}\rangle\langle\lambda^{\rm L}_{\rm b}|$ and the corresponding representative matrices of the fixed point state $|\Psi_{\rm ab}\rangle$ are

$$\{A^{\rm i}_{\rm ab(fp)}\}=\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right\},\qquad(18)$$

which represents the two phases $|\Psi_{\rm a}\rangle$ and $|\Psi_{\rm b}\rangle$ coexisting equally and that the system has the N-spin maximal entanglement. For $\gamma_2=1>\gamma_1$, the corresponding fixed point of the MPS $|\Psi\rangle$ is characterized by the normalized $E^{\rm fp}_{\rm ac}=|\lambda^{\rm R}_{\rm a}\rangle\langle\lambda^{\rm L}_{\rm a}|+|\lambda^{\rm R}_{\rm c}\rangle\langle\lambda^{\rm L}_{\rm c}|$ and the corresponding representative matrices of the fixed point state $|\Psi_{\rm ac}\rangle$ are

$$\{A^{\rm i}_{\rm ac(fp)}\}=\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\},\qquad(19)$$

which represents the two phases $|\Psi_{\rm a}\rangle$ and $|\Psi_{\rm c}\rangle$ coexisting equally and that the system has the N-spin maximal entanglement. For $\gamma_2=\gamma_1>1$, the corresponding fixed point of the MPS $|\Psi\rangle$ is characterized by the normalized $E^{\rm fp}_{\rm bc}=|\lambda^{\rm R}_{\rm b}\rangle\langle\lambda^{\rm L}_{\rm b}|+|\lambda^{\rm R}_{\rm c}\rangle\langle\lambda^{\rm L}_{\rm c}|$ and the corresponding representative matrices of the fixed point state $|\Psi_{\rm bc}\rangle$ are

$$\{A^{\rm i}_{\rm bc(fp)}\}=\left\{\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\},\qquad(20)$$

which represents the two phases $|\Psi_{\rm b}\rangle$ and $|\Psi_{\rm c}\rangle$ coexisting equally and that the system has the N-spin maximal entanglement. For $\gamma_1=\gamma_2=1$, the corresponding fixed point of the MPS $|\Psi\rangle$ is characterized by the normalized $E^{\rm fp}_{\rm abc}=|\lambda^{\rm R}_{\rm a}\rangle\langle\lambda^{\rm L}_{\rm a}|+|\lambda^{\rm R}_{\rm b}\rangle\langle\lambda^{\rm L}_{\rm b}|+|\lambda^{\rm R}_{\rm c}\rangle\langle\lambda^{\rm L}_{\rm c}|$ and the corresponding representative matrices of the fixed point state $|\Psi_{\rm abc}\rangle$ are

$$\{A^{\rm i}_{\rm abc(fp)}\}=\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\},\qquad(21)$$

which represents the three phases coexisting equally and that the system has the N-spin maximal entanglement and stands distinctly for an isolated intermediate-coupling phase transition point. That is to say the fixed point state $|\Psi_{\rm fix}\rangle$ of the specified system is described by

$$|\Psi_{\rm fp}\rangle=\begin{cases}|\Psi_{\rm a}\rangle=|1\cdots1\rangle & \gamma_1,\gamma_2<1,\\ |\Psi_{\rm b}\rangle=|0\cdots0\rangle & \gamma_1>\gamma_2,1,\\ |\Psi_{\rm c}\rangle=|-1\cdots-1\rangle & \gamma_2>\gamma_1,1\\ |\Psi_{\rm ab}\rangle=\dfrac{\sqrt{2}}{2}(|1\cdots1\rangle+|0\cdots0\rangle) & \gamma_1=1>\gamma_2,\\ |\Psi_{\rm ac}\rangle=\dfrac{\sqrt{2}}{2}(|1\cdots1\rangle+|-1\cdots-1\rangle) & \gamma_2=1>\gamma_1,\\ |\Psi_{\rm bc}\rangle=\dfrac{\sqrt{2}}{2}(|0\cdots0\rangle+|-1\cdots-1\rangle) & \gamma_2=\gamma_1>1,\\ |\Psi_{\rm abc}\rangle=\dfrac{\sqrt{3}}{3}(|1\cdots1\rangle+|0\cdots0\rangle+|-1\cdots-1\rangle) & \gamma_1=\gamma_2=1.\end{cases}\qquad(22)$$

The results reconfirm the above conclusions about the kind of phase transition.

3 Conclusions

In conclusion, MPSs provide an effective tool for investigating novel types of quantum phase transitions. Here we present a new kind of quantum phase transitions in matrix product states of one-dimensional spin-1 chains and study the phases coexistence phenomenon. We find that in the thermodynamic limit the specified system has three different quantum phases, $|\Psi_{\rm a}\rangle=|1\cdots1\rangle$, $|\Psi_{\rm b}\rangle=|0\cdots0\rangle$ and $|\Psi_{\rm c}\rangle=|-1\cdots-1\rangle$, and by adjusting the control parameters we are able to realize any phase, any two phases equal coexistence and the three phases equal coexistence. At every critical point the physical quantities including the entanglement are not discontinuous and the matrix product system has long-range correlation and N-spin maximal entanglement. It is worth pointing out that in this vein that we construct the matrix product phase transition system of one-dimensional spin-1 chains we can construct the corresponding phase transition system of one-dimensional spin chains where the spin number is greater than 1. We believe that our work is helpful for having a comprehensive understanding of quantum phase transitions in matrix product states of one-dimensional spin chains and of certain directive significance to the preparation and control of one-dimensional spin lattice models with stable coherence and N-spin maximal entanglement.

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