Relativistic particle scattering states with tensor potential and spatially-dependent mass

M. Eshghi[1;1)] M. R. Abdi[2]

[1] Department of Physics, Faculty of Science, Imam Hossein Comprehensive University, Tehran, Iran
[2] Department of Physics, Faculty of Science, University of Isfahan, Isfahan 81747-73441, Iran

Abstract: We investigate the relativistic equation for particles with spin 1/2 in the q-parameter modified Pöschl-Teller potential, including Coulomb-like tensor interaction with spatially-dependent mass for the D-dimension. We present approximate solutions of the Dirac equation with these potentials for any spin-orbit quantum number κ under spin symmetry. The normalized wave functions are expressed in terms of the hyper-geometric series of the scattering states on the $k/2\pi$ scale. We also give the formula for the phase shifts, and use the Nikiforov-Uvarov method to obtain the energy eigen-values equation.

Key words: Dirac equation, relativistic scattering states, modified Pöschl-Teller potential, spin symmetry, tensor interaction

PACS: 03.65.Pm, 03.65Nk, 03.65.Ge **DOI:** 10.1088/1674-1137/37/5/053103

1 Introduction

The problem of the Dirac equation has been the subject of much discussion for decades because of its importance in the prediction of antiparticles, and also for the description of particle spin and magnetic moments. In fact, exact solutions of the wave equations permit a better understanding of the quantum behavior of relativistic particles such as the nucleus in nuclei, in the presence of external fields [1]. Solutions for the wave equations have recently become interesting in view of the spatially-dependent mass (SDM). There has been increasing interest in searching for analytic solutions of relativistic equations with SDM [2–4]. Extensive applications of this formalism have been performed in different areas of physics such as condensed matter physics and materials science like compositionally graded crystals [5], in describing the transport properties of semiconductors and quantum dots [6, 7], and it also gives interesting results in quantum liquids [8], the energy density many-body problem [9], ^{3}He clusters [10], metal clusters [11], and full/partial Gaussian wave packet revival inside an infinite potential [12].

The concept of SDM formalism comes from the effective-mass approximation [13, 14], which is a useful tool for studying the motion of carrier electrons in pure crystals and for the virtual-crystal approximation in the treatment of homogeneous alloys (where the actual potential is approximated by a periodic potential), as well as graded mixed semiconductors (where the potential is not periodic). In this field, there has been an extraordinary development in crystal-growth techniques like molecular beam epitaxy, which allow the production of nonuniform semiconductor specimens with abrupt heterojunctions [7]. In these mesoscopic materials, the effective mass of the charge carrier is position dependent. In addition, the effective mass of an electron/hole in the thin layered quantum wells varies with the composition rate. In such systems, the mass of the electron may change with the composition rate, which depends on the position, and to external potentials, where a particle propagating in solid state systems can face a situation where its effective mass changes in space. Therefore, the corresponding Schrodinger equation should be formulated in a correct form, in order to comply with hermiticity and flux conservation. If the particles are described by the Dirac equation, the information on the material properties is encoded both in the Fermi velocity [15] and in the mass. We can then imagine the possibility of producing a heterostructure where the Fermi velocity, $\nu_{\mathrm{F}}(r)$, and the mass change with the position of the particle. The effective mass is also an important parameter in Landau's Fermi liquid theory that deals with low-level excited states of strongly interacting systems in a very appealing single particle approximation [16]. In addition, to its practical applicability side, conceptual problems of delicate nature erupt in the study of quantum mechanical systems with position-dependent mass (e.g. the

Received 3 July 2012, Revised 29 November 2012

1) Corresponding author. E-mail: eshgi54@gmail.com; kpeshghi@ihu.ac.ir

momentum operator does not commute with $m(r)$, the uniqueness of the kinetic energy operator, etc). Comprehensive discussion on such issues can be found in, e.g. [17–20, and related references therein]. Thus, one leads to the study of quantum mechanical problems with position-dependent effective mass, and the solution of the Dirac equation under the circumstance where the mass depends on the position of electrons will be of interest in studying materials containing heavy elements. However, such treatment encounters a nontrivial problem related to ordering ambiguity in the quantization of the momentum and mass operators in the kinetic energy term of the effective Hamiltonian.

On the other hand, in order to get complete information about quantum mechanical systems, one should study the bound and scattering states in the presence of an external potential. Therefore, the scattering problem has become an interesting topic in relativistic/non-relativistic quantum mechanics, and the scattering of a Dirac particle by a potential can be treated exactly by finding the continuum solutions of the Dirac equation. In recent years, the scattering problem has been extended to the case where the mass depends on the spatial coordinate [21–23].

There has also been continuous interest in studying the solutions of scattering states within the framework of non-relativistic and relativistic quantum mechanics for central and non-central potentials [24–26]. Prof. Dong and Lozada-Gassou studied the scattering of the two-dimensional Dirac particle by the Coulomb potential [27]. Arda et al. obtained the scattering solutions of the one-dimensional Schrödinger equation for the Wood-Saxon potential within the position-dependent mass formalism [28]. In Ref. [29], the authors investigated the relativistic scattering state for the Klein-Gordon equation with Makarov potential, and discussed the analytical properties of the scattering amplitude. Alhaidari [30] investigated the scattering state for the three-dimensional Schrödinger equation for a large class of non-central potentials.

On the other hand, the spin and pseudo-spin symmetries have attracted many theoretical investigations during the past 15 years because of their successful description of observed experimental phenomena [31–34]. Concepts of spin and pseudo-spin symmetries and a tensor potential have found interesting applications in the field of nuclear physics [31, 33, 34]. Tensor potentials were introduced into the Dirac equation with the substitution $\vec{p}\rightarrow\vec{p}-\mathrm{i}m\omega\beta\cdot\hat{r}U(r)$ [32, 35]. In this way, a spin-orbit coupling term is added to the Dirac Hamiltonian. Recently, tensor couplings have been used widely in the studies of nuclear properties. In this regard, see [36–41]. Ginocchio pointed out that the spin and pseudo-spin symmetries may explain the degeneracies in some heavy meson spectra (spin symmetry), or in single-particle energy levels in nuclei (pseudo-spin symmetry), when these physical systems are described by relativistic mean-field theories with scalar and vector potentials [31].

According to the report that was given in the research, the SDM of the form [4] is

$$M(r)=m_0+4V_0\frac{\mathrm{e}^{-2\alpha r}}{(1+q\mathrm{e}^{-2\alpha r})^2}, \tag{1}$$

if $r\rightarrow 0$, then $M(r)=M_0+2V_0$, if $r\rightarrow\infty$, then $M(r)=M_0$, and the modified Pöschl-Teller potential [4] is as follows

$$V(r)=-\frac{V_0}{\cosh_q^2\alpha r}, \tag{2}$$

where V_0 is the depth of the well, α is related to the range of the potential, and r is the relative distance from the equilibrium position. This potential has been frequently used in nuclear physics, molecular physics, chemical physics and has undergone considerable theoretical investigations [42–45]. It is also a short-range model potential and used to describe bending molecular vibrations [46, 47].

In addition, according to the report that was given in Refs. [2, 36–41], the tensor potential Coulomb-like is

$$U(r)=-\frac{H}{r},\ H=\frac{Z_{\mathrm{a}}Z_{\mathrm{b}}e^2}{4\pi\varepsilon_0},\ r\geqslant R_{\mathrm{c}}, \tag{3}$$

where $R_{\mathrm{c}}=7.78$ fm is the Coulomb radius, and Z_{a} and Z_{b} denote the charges of projectile a and target nuclei b, respectively.

In the non-relativistic regime, different methodologies have been applied to solve the problem [48–50] and the consequences have been analyzed [51–53]. Within the present study, due to the recent interest in higher dimensions, we intend to work on the case of SDM distribution (1) for the Dirac equation in the spin symmetry under the above potential. We are going to consider the Dirac equation for the modified Pöschl-Teller potential, including the Coulomb-like tensor interaction under the spin symmetry. In this work we also investigate the scattering state and obtain the normalized wave functions.

Our stages are as follows. We first review the Dirac equation in the D-dimension. Next, we review the spin symmetry limit of the Dirac equation for two potentials. Also, to get rid of the centrifugal term, we use a physical approximation and therefore provide radial solutions. We obtain the energy eigenvalues equation by using the Nikiforov-Uvarov [54] technique. Then, we investigate the scattering state in this problem.

2 The Dirac equation in *D*-dimensions

The extension of physical problems to higher dimensional spaces plays an important role in various areas of physics. Many physical systems of interest in quantum mechanics have been thoroughly studied in D-dimensional space [1, 55–58]. The Dirac equation in natural units $\hbar=c=1$ in D-dimensional space with a scalar potential $S(r)$, a vector potential $V(r)$, a tensor potential $U(r)$ and position-dependent mass $M(r)$ can be written as [55, 59]

$$H\Psi(r)=E_{n_r\kappa}\Psi(r), \tag{4}$$

where the r is a D-dimensional position vector with Cartesian components $r_1,\ r_2,\cdots,r_D$ and

$$H=\sum_{j=1}^{D}\hat{\alpha}_j p_j+\hat{\beta}[M(r)+S(r)]-\mathrm{i}\hat{\beta}\hat{\alpha}_j.\hat{r}U(r)+V(r), \tag{5}$$

where $E_{n_r\kappa}$ is the relativistic energy, and $\{\hat{\alpha}_j\}$ and $\hat{\beta}$ are the Dirac matrices, which satisfy the anti-commutation relations

$$\hat{\alpha}_j\hat{\alpha}_k+\hat{\alpha}_k\hat{\alpha}_j=2\delta_{jk}1,\quad \hat{\alpha}_j\hat{\beta}+\hat{\beta}\hat{\alpha}_j=0,\quad \hat{\alpha}_j^2=\hat{\beta}^2=1, \tag{6}$$

and

$$p_j=-\mathrm{i}\partial_j=-\mathrm{i}\frac{\partial}{\partial x_j}\quad j\in[1,D]. \tag{7}$$

A set of the total angular momentum operators J_{jk} are defined as:

$$L_{jk} = -L_{kj}=\mathrm{i}x_j\partial_k-\mathrm{i}x_k\partial_j, \tag{8}$$

$$S_{jk} = -S_{kj}=\mathrm{i}\hat{\alpha}_j\hat{\alpha}_k/2,\quad J_{jk}=L_{jk}+S_{jk}. \tag{9}$$

$$\begin{aligned} L^2 &= \sum_{j<k=2}^{D}L_{jk}^2=-\left\{\frac{1}{\sin^{j-1}x_j}\frac{\partial}{\partial x_j}\left(\sin^{j-1}x_j\frac{\partial}{\partial x_j}\right)\right. \\ &\quad \left.-\frac{L_{j-1}^2}{\sin^2 x_j}\right\}, \end{aligned} \tag{10}$$

$$S^2 = \sum_{j<k=2}^{D}S_{jk}^2,\quad J^2=\sum_{j<k=2}^{D}J_{jk}^2, \tag{11}$$

where L_{jk} are the orbital angular momentum operators and S_{jk} are the spinor operators.

For a spherically symmetric potential, the total angular momentum operator J_{jk} and the spin-orbit operator $\hat{K}=-\hat{\beta}\left(\sum_{j<k}\mathrm{i}\hat{\alpha}_j\hat{\alpha}_k L_{jk}+(D-1)/2\right)=-\hat{\beta}(J^2-L^2-S^2+(D-1)/2)$ commutate with the Dirac Hamiltonian [5]. For a given total angular momentum j, the eigenvalues of $\hat{K}$, the quantum number κ is related to the quantum numbers for spin symmetry ℓ and pseudo-spin symmetry $\tilde{\ell}$ as

$$\kappa=\begin{cases} -(\ell+1)=-\left(j+\dfrac{D-2}{2}\right)\ (s_{1/2},p_{3/2},\text{etc.}) \\ \qquad j=\ell+\dfrac{1}{2},\ \text{alined spin}(\kappa<0) \\ +\ell=+\left(j+\dfrac{D-2}{2}\right)\ \ (p_{1/2},d_{3/2},\text{etc.}) \\ \quad j=\ell-\dfrac{1}{2},\ \text{unaligned spin}(\kappa>0) \end{cases}, \tag{12a}$$

and the quasi-degenerate doublet structure can be expressed in terms of a pseudo-spin angular momentum $\tilde{s}=1/2$ and pseudo-orbital angular momentum $\tilde{\ell}$, which is defined as

$$\kappa=\begin{cases} -\tilde{\ell}=-\left(j+\dfrac{D-2}{2}\right)\ (s_{1/2},p_{3/2},\text{etc.}) \\ \quad j=\tilde{\ell}-\dfrac{1}{2},\ \text{aligned pseudospin}(\kappa<0) \\ +(\tilde{\ell}+1)=+\left(j+\dfrac{D-2}{2}\right)\ \ (d_{3/2},f_{5/2},\text{etc.}) \\ \qquad j=\tilde{\ell}+\dfrac{1}{2},\ \text{unaligned spin}(\kappa>0) \end{cases}, \tag{12b}$$

where $\kappa=\pm1,\ \pm2,\cdots$. For example, $(1s_{1/2},\ 0d_{3/2})$ and $(1p_{3/2},\ 0f_{5/2})$ can be considered as pseudo-spin doublets.

Also, since $V(r)$ is spherically symmetric, the group of the system is the SO(D) group. Thus, the relations between the Cartesian coordinates x_i and the hyperspherical coordinates in D-dimensional space are defined by [55, 60]

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_\alpha &= r\sin\theta_1\cdots\sin\theta_{\alpha-1}\cos\phi\quad \alpha\in[2,D-1], \\ x_D &= r\sin\theta_1\cdots\sin\theta_{D-1}\sin\phi, \end{aligned} \tag{13}$$

The unit vector $\hat{x}$ along x is usually denoted by $\hat{x}=x/r$. The sum of the squares of Eq. (10) gives $r^2=\sum_{i=1}^{D}x_i^2$, thus, r is the radius of a D-dimensional sphere where the volume element of the configuration space is given by

$$\prod_{j=1}^{D}\mathrm{d}x_j=r^{D-1}\mathrm{d}r\mathrm{d}\Omega,\quad \mathrm{d}\Omega=\prod_{j=1}^{D-1}(\sin\theta_j)^{j-1}\mathrm{d}\theta_j, \tag{14}$$

where $r\in[0,\infty)$, $\theta_k\in[0,\pi]$, $k=1,2,\cdots,D$–2, $\phi\in[0,2\pi]$,, and such spinor wave-functions can be classified according to the hyper-radial quantum number n_r and the spin-orbit quantum number κ and can be written using the Pauli-

Dirac representation [55, 60]

$$\begin{aligned}
&\Psi_{n_r\kappa}(r,\Omega_D)\\
&=\begin{pmatrix} f_{n_r\kappa}(r,\Omega_D)\\ g_{n_r\kappa}(r,\Omega_D)\end{pmatrix}=r^{-\frac{D-1}{2}}\begin{pmatrix} F_{n_r\kappa}(r)Y^{\ell}_{jm}(\Omega_D)\\ \mathrm{i}G_{n_r\kappa}(r)Y^{\tilde{\ell}}_{jm}(\Omega_D)\end{pmatrix}\\
&=r^{-\frac{D-1}{2}}\begin{pmatrix} F_{n_r\kappa}(r)Y^{\ell}_{jm}(\theta_1\cdots\theta_{D-1})\\ \mathrm{i}G_{n_r\kappa}(r)Y^{\tilde{\ell}}_{jm}(\theta_1\cdots\theta_{D-1})\end{pmatrix}\\
&=r^{-\frac{D-1}{2}}\begin{pmatrix} F_{n_r\kappa}(r)\Phi(\theta_1=\phi)H(\theta_2\ldots\theta_{D-1})\\ \mathrm{i}G_{n_r\kappa}(r)\Phi(\theta_1=\phi)H(\theta_2\ldots\theta_{D-1})\end{pmatrix},
\end{aligned}\tag{15}$$

where $f_{n_r\kappa}(r,\,\Omega_D)$ is the upper (large) component and $g_{n_r\kappa}(r,\Omega_D)$ is the lower (small) component of the Dirac spinors. Using the D-dimensional polar co-ordinates with polar variables r is the hyper radius and the angular momentum variables $\theta_1\cdots\theta_{D-1}$, ϕ are the hyper angle. $Y^{\ell}_{jm}(\Omega_D)$ and $Y^{\tilde{\ell}}_{jm}(\Omega_D)$ are the generalized spherical harmonic functions coupled with the total angular momentum j. The orbital and pseudo-orbital angular momentum quantum numbers for spin symmetry ℓ and pseudo-spin symmetry $\tilde{\ell}$ refer to the upper- and lower-component, respectively.

Substituting (15) into (4), and separating the variables, we obtain the following coupled radial Dirac equation for the spinor components:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)F_{n_r\kappa}(r)=[E_{n_r\kappa}+M(r)-\Delta(r)]G_{n_r\kappa}(r),\tag{16}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)G_{n_r\kappa}(r)=[M(r)-E_{n_r\kappa}+\Sigma(r)]F_{n_r\kappa}(r),\tag{17}$$

where $\Delta(r)=V(r)-S(r)$, $\Sigma(r)=V(r)+S(r)$ and $\kappa=\pm(2\ell+D-1)/2$. Eliminating $G_{n_rk}(r)$ and $F_{n_rk}(r)$ from (16) and (17), we obtain the following two Schrödinger-like differential equations for the upper and lower components, respectively

$$\begin{aligned}
&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+(E_{n_r\kappa}+M(r)-\Delta(r))(E_{n_r\kappa}-M(r)\right.\\
&-\Sigma(r))+\frac{2\kappa}{r}U(r)-\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)\\
&\left.+\frac{\dfrac{\mathrm{d}M(r)}{\mathrm{d}r}-\dfrac{\mathrm{d}\Delta(r)}{\mathrm{d}r}}{(M(r)+E_{n_r\kappa}-\Delta(r))}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)\right\}F_{n_r\kappa}(r)=0,
\end{aligned}\tag{18}$$

$$\begin{aligned}
&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+(E_{n_r\kappa}+M(r)-\Delta(r))(E_{n_r\kappa}-M(r)\right.\\
&-\Sigma(r))+\frac{2\kappa}{r}U(r)+\frac{\mathrm{d}U(r)}{\mathrm{d}r}-U^2(r)\\
&\left.+\frac{\dfrac{\mathrm{d}M(r)}{\mathrm{d}r}+\dfrac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}}{(M(r)-E_{n_r\kappa}+\Sigma(r))}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)\right\}G_{n_r\kappa}(r)=0.
\end{aligned}\tag{19}$$

We note that the energy eigen-values in these equations depend on the angular momentum quantum number ℓ and dimension D.

3 Exact solutions for the spin symmetry limit

Equations (18) and (19) can not be solved analytically because of the last term in the equations. It is convenient to solve the mathematical relation $(\mathrm{d}M(r)/\mathrm{d}r)=(\mathrm{d}V(r)/\mathrm{d}r)$ [59]. By using this relation, we can exactly solve Eq. (18). Substituting (1), (2) and (3) into (18), considering spin symmetry, taking $\Sigma(r)$ as the modified Pöschl-Teller potential and $\Delta(r)=C_{\mathrm{s}}$=const.($\mathrm{d}\Delta(r)/\mathrm{d}r$=0), i.e. [61, 62], the equation obtained for the upper component of the Dirac spinor $F_{nk}(r)$ becomes

$$\begin{aligned}
&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{(\kappa+1)(\kappa+H+1)}{r^2}\right.\\
&+(E_{n_r\kappa}-M_0)(M_0+E_{n_r\kappa}-C_{\mathrm{s}})\\
&\left.+(E_{n_r\kappa}-M_0)\frac{V_0}{\cosh^2_q\alpha r}\right\}F_{n_r\kappa}(r)=0,
\end{aligned}\tag{20}$$

where $(\kappa+H)(\kappa+H+1)/r^2$ is known as the centrifugal term. Obviously, equation (20) cannot be solved exactly due to the centrifugal term. To obtain a quasi-analytical solution to the above equation, we must use an approximation for the centrifugal term. We have taken the following approximation [63–66]

$$\frac{1}{r^2}\approx4\alpha^2\left[C_0+\frac{\mathrm{e}^{-2\alpha r}}{(1-q\mathrm{e}^{-2\alpha r})^2}\right],\tag{21}$$

where the parameter $C_0=1/12$ is a dimensionless constant and a good approximation for small values of parameter α. However, when $C_0=0$ then the new improved approximation scheme becomes the conventional approximation scheme suggested by Greene and Aldrich [67]. Therefore, to see the accuracy of our approximation, we plotted the pseudo-centrifugal term, $1/r^2$, and its approximation with parameter α=0.6, in Fig. 1.

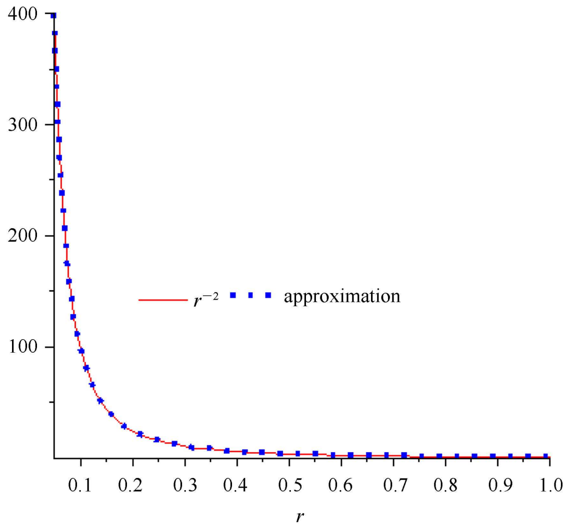

Fig. 1. (color online) The pseudo-centrifugal term $1/r^2$ (red curve) and its approximation Eq. (21) (dotted blue curve).

By using a transformation of the form $s=\tanh^2\alpha r(r\in[0,\infty),\ s\in[0,1])$, we rewrite it as follows

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{1-3s}{2s(1-s)}\frac{\mathrm{d}}{\mathrm{d}s}+\frac{1}{[2s(1-s)]^2}[-b_3\tilde{V}_0s^2+(4C_0qb_1+b_2b_3+b_3\tilde{V}_0)s-4C_0qb_1]\right\}F_{n_r\kappa}(s)=0, \quad (22)$$

where

$$b_1=\frac{(\kappa+1)(\kappa+H+1)}{q},\quad b_2=\frac{E_{n_r\kappa}+M_0-C_s}{\alpha^2},\quad b_3=E_{n_r\kappa}-M_0,\quad \tilde{V}_0=\frac{V_0}{q\alpha^2}. \quad (23)$$

In the above two expressions, we applied the deformed hyperbolic functions introduced for the first time by Arai [68].

$$\sinh_q x=\frac{\mathrm{e}^x-q\mathrm{e}^{-x}}{2},\quad \cosh_q x=\frac{\mathrm{e}^x+q\mathrm{e}^{-x}}{2},\quad \tanh_q x=\frac{\sinh_q x}{\cosh_q x},\quad \mathrm{sech}_q x=\frac{1}{\cosh_q x}, \quad (24)$$

where q is a real parameter and $q>0$.

For the scattering states, $E>0$ and $k>0$. In the $3D$ case, the boundary conditions of the scattering states of a short-range potential are

$$F(r)\xrightarrow{r\to0}0,\ \text{and}\ F(r)\xrightarrow{r\to\infty}2\sin(kr+\delta_l-l\pi/2). \quad (25a)$$

Using Refs. [51, 69], we have

$$F(r)\xrightarrow{r\to0}0,\ \text{and}\ F(r)\xrightarrow{r\to\infty}2\sin\left[kr+\delta_{J_{D-2}}-\pi\left(J_{D-2}+\frac{D-3}{2}\right)\Big/2\right], \quad (25b)$$

to ensure that the radial wave functions of the scattering states for a short-range potential are also normalized on the $k/2\pi$ scale, where $\delta_{J_{N-2}}$ represents the phase shift.

Now, according to the asymptotic behavior of the radial wave functions of the continuous states at $r\to0$, we take the wave function in the form of

$$F(s)=s^{\frac{\left[1+\sqrt{1+4C_0qb_1}\right]}{4}}(1-s)^{-\mathrm{i}k/2\alpha}f(s). \quad (26)$$

Substituting Eq. (26) into Eq. (22), we can obtain the following second-order differential equation as

$$s(1-s)\frac{\mathrm{d}^2f(s)}{\mathrm{d}s^2}+\left[\frac{\left[2+\sqrt{1+4C_0qb_1}\right]}{2}-\left(\frac{\left[4+\sqrt{1+4C_0qb_1}\right]}{2}-\frac{\mathrm{i}k}{2\alpha}\right)s\right]\frac{\mathrm{d}f(s)}{\mathrm{d}s}+\frac{1}{4}\left[b_3\tilde{V}_0-4C_0qb_1+b_2b_3+\frac{\mathrm{i}k}{\alpha}\left(\left[2+\sqrt{1+4C_0qb_1}\right]\right)\right]f(s)=0, \quad (27)$$

which is the hyper-geometric differential equation [70–72]. Thus, the analytical solution at $s=0(r\to0)$is the hyper-geometric function

$$f(s)={}_2F_1(a,b;c;s). \quad (28)$$

And the parameters are

$$a=\left(\frac{\left[2+\sqrt{1+4C_0qb_1}-\sqrt{1+4b_3\tilde{V}_0}\right]}{2}\right)/2-\mathrm{i}k/2\alpha,$$
$$b=\left(\frac{\left[2+\sqrt{1+4C_0qb_1}+\sqrt{1+4b_3\tilde{V}_0}\right]}{2}\right)/2-\mathrm{i}k/2\alpha,$$
$$c=\frac{\left[2+\sqrt{1+4C_0qb_1}\right]}{2}. \quad (29)$$

From the above equations, we have

$$c-a-b=\mathrm{i}k/\alpha=(a+b-c)^*,$$
$$c-a=\left(\frac{\left[2+\sqrt{1+4C_0qb_1}+\sqrt{1+4b_3\tilde{V}_0}\right]}{2}\right)/2+\mathrm{i}k/\alpha=b^*,$$
$$c-b=\left(\frac{\left[2+\sqrt{1+4C_0qb_1}-\sqrt{1+4b_3\tilde{V}_0}\right]}{2}\right)/2+\mathrm{i}k/\alpha=a^*, \quad (30)$$

where $k=\alpha\sqrt{b_2b_3}$.

Therefore, the radial wave function of the scattering state is

$$F_{n_r\kappa}(s)=A_{n_rJ_{D-2}}s^{\frac{\left[1+\sqrt{1+4C_0qb_1}\right]}{4}}(1-s)^{-\mathrm{i}\frac{k}{2\alpha}}{}_2F_1(a,b;c;s), \tag{31}$$

where form $s=\tanh^2(\alpha r)$. We now study the asymptotic form of the above expression for large r, and calculate the normalization constant $A_{n_rJ_{D-2}}$ of the radial wave function and phase shift $\delta_{J_{D-2}}$. By using the transformation formulas to the hyper-geometric function [63–65]

$$\begin{aligned}{}_2F_1\left(a,b;c;\tanh^2(\alpha r)\right)=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1\left(a,b;a+b-c+1;1-\tanh^2(\alpha r)\right)\\&+(1-\tanh^2(\alpha r))^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}{}_2F_1\left(c-a,c-b;c-a-b+1;1-\tanh^2(\alpha r)\right).\end{aligned} \tag{32}$$

By using the

$$1-\tanh^2(\alpha r)=1/\cosh^2(\alpha r)=4/(\mathrm{e}^{\alpha r}+\mathrm{e}^{-\alpha r})^2\overset{r\to\infty}{\longrightarrow}4\mathrm{e}^{-2\alpha r}, \tag{33}$$

we have

$$\left[1-\tanh^2(\alpha r)\right]^{-\mathrm{i}k/2\alpha}\overset{r\to\infty}{=\!\!=\!\!\longrightarrow}2^{-\mathrm{i}k/\alpha}\mathrm{e}^{-\mathrm{i}k\alpha},\ \text{and}\ \left[1-\tanh^2(\alpha r)\right]^{c-a-b}\overset{r\to\infty}{=\!\!=\!\!\longrightarrow}2^{2\mathrm{i}k/\alpha}\mathrm{e}^{-2\mathrm{i}k\alpha}. \tag{34}$$

Substituting Eqs. (32)–(34) into Eq. (31) and using ${}_2F_1(a,b;c;1)=1$, we have

$$\begin{aligned}F(s)=&N_{kJ_{D-2}}\left[\tanh^2(\alpha r)\right]^{\frac{\left[1+\sqrt{1+4C_0qb_1}\right]}{4}}\left[1-\tanh^2(\alpha r)\right]_2^{-\mathrm{i}\frac{k}{2\alpha}}F_1\left(a,b;c;\tanh^2(\alpha r)\right)\\&\overset{r\to\infty}{\longrightarrow}N_{kJ_{D-2}}2^{-\mathrm{i}\frac{k}{\alpha}}\mathrm{e}^{\mathrm{i}kr}\Gamma(c)\left[\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}+2^{2\mathrm{i}\frac{k}{\alpha}}\mathrm{e}^{-2\mathrm{i}kr}\frac{\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\right]\\&\overset{r\to\infty}{\longrightarrow}N_{kJ_{D-2}}\Gamma(c)\left[\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}2^{-\mathrm{i}\frac{k}{\alpha}}\mathrm{e}^{\mathrm{i}kr}+2^{\mathrm{i}\frac{k}{\alpha}}\mathrm{e}^{-\mathrm{i}kr}\left(\frac{\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\right)^*\right].\end{aligned} \tag{35}$$

If we write

$$\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}=\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\mathrm{e}^{\mathrm{i}\delta}, \tag{36}$$

then

$$\left(\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right)^*=\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\mathrm{e}^{-\mathrm{i}\delta}, \tag{37}$$

where δ is a real number. Using the above equation, Eq. (35) becomes

$$\begin{aligned}F(s)&\overset{r\to\infty}{\longrightarrow}N_{kJ_{D-2}}\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\left[\mathrm{e}^{\mathrm{i}(kr+\delta-k\ln2/\alpha)}+\mathrm{e}^{-\mathrm{i}(kr+\delta-k\ln2/\alpha)}\right]\\&\overset{r\to\infty}{\longrightarrow}2N_{kJ_{D-2}}\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\cos(kr+\delta-k\ln2/\alpha)\\&\overset{r\to\infty}{\longrightarrow}2N_{kJ_{D-2}}\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\sin[kr+\delta-k\ln2/\alpha-\pi[J_{N-2}+(N-3)/2]/2+\pi[J_{N-2}+(N-1)/2]/2].\end{aligned} \tag{38}$$

Comparing Eq. (25b) with Eq. (38), we obtain the normalization constant of the scattering state as

$$\begin{aligned}N_{kl}\ =\ &\frac{1}{\Gamma(c)}\left|\frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c-a)\Gamma(c-b)}\right|=\frac{\left|\Gamma\left(\left[2+\sqrt{1+4C_0qb_1}-\sqrt{1+4b_3\tilde{V}_0}\right]/4+\mathrm{i}k/2\alpha\right)\right|}{\Gamma\left(\dfrac{\left[2+\sqrt{1+4C_0qb_1}\right]}{2}\right)}\\&\times\left|\frac{\Gamma\left[\left(2+\sqrt{1+4C_0qb_1}+\sqrt{1+4b_3\tilde{V}_0}\right)/4+\mathrm{i}k/2\alpha\right]}{\Gamma(\mathrm{i}k/\alpha)}\right|.\end{aligned} \tag{39}$$

Substituting Eq. (39) into Eq. (31), we obtain the normalized wave functions of the continuous states with the modified Pöschl-Teller potential in the D-dimension on the $k/2\pi$ scale as

$$F_{n_r\kappa}(s) = \frac{1}{\Gamma\left(\frac{\left[2+\sqrt{1+4C_0qb_1}\right]}{2}\right)} \times \left|\frac{\Gamma\left[\left(\frac{\left[2+\sqrt{1+4C_0qb_1}+\sqrt{1+4b_3\tilde{V}_0}\right]}{2}\right)/2+\mathrm{i}\frac{k}{2\alpha}\right]\Gamma\left[\left(\frac{\left[2+\sqrt{1+4C_0qb_1}-\sqrt{1+4b_3\tilde{V}_0}\right]}{2}\right)/2+\mathrm{i}\frac{k}{2\alpha}\right]}{\Gamma(\mathrm{i}k/\alpha)}\right|$$
$$\times\left[\tanh^2(\alpha r)\right]^{\frac{\left[1+\sqrt{1+4C_0qb_1}\right]}{4}}\left[1-\tanh^2(\alpha r)\right]^{-\mathrm{i}\frac{k}{2\alpha}}{}_2F_1\left(a,b;c;\tanh^2(\alpha r)\right). \tag{40}$$

And phase shifts

$$\begin{aligned}\delta_{J_{D-2}} &= \pi\frac{[J_{D-2}+(D-1)/2]}{2}-k\ln2/\alpha+\arg\Gamma(c-a-b)-\arg\Gamma(c-a)-\arg\Gamma(c-b)\\ &= \pi\frac{[J_{D-2}+(D-1)/2]}{2}-k\ln2/\alpha+\arg\Gamma(c-a-b)-\arg\Gamma(b)-\arg\Gamma(a)\\ &= \pi\frac{[J_{D-2}+(D-1)/2]}{2}-k\ln2/\alpha+\arg\Gamma(\mathrm{i}k/\alpha)-\arg\Gamma\left(\frac{\left[2+\sqrt{1+4C_0qb_1}-\sqrt{1+4b_3\tilde{V}_0}\right]}{4}-\mathrm{i}k/2\alpha\right)\\ &\quad+\arg\Gamma\left(\frac{\left[2+\sqrt{1+4C_0qb_1}+\sqrt{1+4b_3\tilde{V}_0}\right]}{4}-\mathrm{i}k/2\alpha\right).\end{aligned} \tag{41}$$

Now, by using the Nikiforov-Uvarov (NU) method [54] and Eq. (22), we obtain the energy eigenvalues equation as follows

$$\begin{aligned}&-\frac{1}{2}\left[(-4C_0q+1)b_1-b_3\tilde{V}_0-b_2b_3\right]\\ &-\frac{1}{2}\sqrt{-b_2b_3\left[1+4(-4C_0q+1)b_1\right]}\\ &-\frac{1}{2}\left[2\sqrt{-b_2b_3}+\sqrt{1+4(-4C_0q+1)b_1}\right]-\frac{1}{2}\\ &= n_r\left(4+2\sqrt{-b_2b_3}+\sqrt{1+4(-4C_0q+1)b_1}\right)+2n_r(n_r-1),\end{aligned} \tag{42}$$

where b_1, b_2, b_3 and $\tilde{V}_0$ are given in formula (23).

4 Conclusions

In this work, we obtain approximate analytical solutions of the scattering states for the modified Pöschl-Teller and Coulomb-like tensor potential in the D-dimensional state in the position-dependent mass state under spin symmetry. The energy levels equation is obtained using the NU method. The phase shift is given in Eq. (41), and the normalized wave function expressed in terms of the hyper-geometric series of the scattering state on the $k/2\pi$ scale is given in Eq. (40).

We would like to thank the kind referee for positive suggestions, which have improved the manuscript.

References

1 Saad N, Hall R L, Ciftci H. Cent. Eur. J. Phys., 2008, **6**: 717
2 Eshghi M, Mehraban H. Few-Body Syst., 2012, **52**: 41
3 Jia C S, de Souza Dutra A. Ann. Phys., 2008, **323**: 566
4 Arda A, Sever R, Tezcan C. Chin. Phys. Lett., 2010, **27**: 010306
5 Galler M R, Kohn W. Phys. Rev. Lett., 1993, **70**: 3103
6 Serra L I, Lipparini E. Europhys. Lett., 1997, **40**: 667
7 Bastard G. Wave Mechanics Applied to Semiconductor Heterostructures, Les Editions de physique, Les Ulis, 1988
8 De Saavedra F A et al. Phys. Rev. B, 1994, **50**: 4248
9 Puente A, Gasas M. Comput. Mater. Sci., 1994, **2**: 441
10 Barranco M et al. Phys. Rev. B, 1997, **56**: 8997
11 Puente A, Serra L I, Casas M. Zeit. Phys. D, 1994, **31**: 283
12 Schmidt A G M. Phys. Lett. A, 2006, **353**: 459
13 Wannier G H. Phys. Rev., 1937, **52**: 191
14 Slater J C. Phys. Rev., 1949, **76**: 1592
15 DiVincenzo D P, Mele E J. Phys. Rev. B, 1984, **29**: 1685
16 Baym G, Pethick C J. Landau Fermi Liquid Theory: Concepts & Applications, New York: Wiley, 1991
17 Plastino A R, Casas M, Plastino A. Phys. Lett. A, 2001, **281**: 297
18 Tanaka T. J. Phys. A: Math. Gen., 2006, **39**: 219
19 Quesne C. Ann. Phys., 2006, **321**: 1221
20 GANG C. Phys. Lett. A, 2004, **329**: 22
21 Dekar L, Chetouani L, Hammann T F. J. Math. Phys., 1998, **39**: 2551
22 Alhaidari A D et al. Phys. Rev. A, 2007, **75**: 062711
23 Panella O, Biondini S, Arda A. J. Phys. A: Math. Theor., 2010, **43**: 325302
24 Dombey N, Kennedy P, Calogercos A. Phys. Rev. Lett., 2000, 85: 1787
25 Kennedy P. J. Phys. A, 2002, **35**: 689
26 Rojas C, Villalba V M. Phys. Rev. A, 2005, **71**: 052101-4
27 DONG S H, Lozada-Gassou M. Phys. Lett. A, 2004, **330**:168
28 Arda A, Aydogdu O, Sever R. J. Phys. A: Math. Theor., 2010, **43**: 425204
29 Movahedi M, Hamzavi M. Int. J. Phys. Sci., 2011, **6**: 891
30 Alhaidari A D. J. Phys. A: Math. Theor., 2005, **38**: 3409
31 Gincchio J N. Phys. Rep., 2005, **414**:165
32 MAO G. Phys. Rev. C, 2003, **67**: 044318
33 Alberto P et al. Phys. Rev. C, 2005, **71**: 034313
34 Furnstahl R J, Rusnak J J, Serot B D. Nucl. Phys. A, 1998, **632**: 607
35 Moshinsky M, Szczepaniak A. J. Phys. A: Math. Gen., 1989, **22**: L817
36 Eshghi M, Hamzavi M. Commun. Theor. Phys., 2012, **57**: 355
37 Akcay H, Tezcan C. Int. J. Mod. Phys. C, 2009, **20**: 931
38 Aydoğdu O, Sever R. Few-Boby Syst., 2010, **47**: 193
39 Eshghi M, Mehraban H. Chin. J. Phys., 2011, **50**: 533
40 Hamzavi M, Rajabi A A, Hassanabadi H. Few-Body Syst., 2010, **48**: 171
41 Ikhdair S M, Sever R. Appl. Math. Com., 2010, **216**: 911
42 Pöschl G, Teller E. Z. Physik, 1933, **83**: 143
43 Aktas M, Sever R. J. Mol. Struct. THEOCHEM, 2004, **710**: 223
44 Zuniga J et al. Int. J. Quntum. Chem., 1996, **57**: 43
45 De Rocha R, Capelas de Oliveira E. Rev. Mex. Fis., 2005, **51**: 1
46 Iachello F, Oss S. Chem. Phys. Lett., 1993, **205**: 285
47 Iachello F, Oss S. Chem. Phys. Lett., 1993, **99**: 7337
48 Landau L D, Lifshitz E M, Quantum Mechanics. third edition. New-York: Pergaman, 1965. 73
49 Agboola D. arXive: 0811.3613v3
50 Hassanabadi H, Yazarloo B, LU L L. Chin. Phys. Lett., 2012, **29**: 020303
51 JIA C S, SUN Y, LI Y. Phys. Lett. A, 2002, **305**: 231
52 Bender C M, Boettcher S. Phys. Rev. Lett., 1998, **80**: 5243
53 Mostafazadeh A. J. Math. Phys., 2002, **43**: 205
54 Nikiforov A F, Uvarov V B. Special Functions of Mathematical Physics, Birkhauser Verlag Basel, 1988
55 DONG S H. Wave Equations in Higher Dimensions, Springer, Dordrecht Heidelberg, London, New-York, 2011
56 ZENG J Y. Quantum Mechanis, third edition. Vol. II, Beijing: Science Press, 2000
57 Agboola D. Phys. Scr., 2009, **80**: 065304
58 CHEN C Y et al. Commun. Theor. Phys., 2011, **55**: 399
59 Agboola D. arXiv:1011.2368v1
60 DONG S H. Factorization Method in Quantum Mechanics, Springer Dordrecht, the Netherlands, 2007
61 MENG J et al. Phys. Rev. C, 1999, **59**: 154
62 MENG J et al. Phys. Rev. C, 1998, **58**: R628
63 JIA C S, CHEM T, GUI L G. Phys. Lett. A, 2009, **373**: 1621
64 ZHANG L H, LI X P, JIA C S. Phys. Scr., 2009, **80**: 035003
65 XU Y, HE S, JIA C S. Phys. Scr., 2010, **81**: 045001
66 JIA C S et al. Int. J. Mod. Phys. A, 2009, **24**: 4519
67 Greene R L, Adrich C. Phys. Rev. A, 1976, **14**: 2363
68 Arai A. J. math. Anal. Appl., 1991, **158**: 63
69 CHEN C Y, SUN D S, LU F L. Phys. Lett. A, 2004, **330**: 424
70 Gradshteyn I S, Ryzhik I M. Tables of Integrals, Series, and Products. five edition. New York: Academic Press, 1994
71 Abramovitz M, Stegun I A. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1970
72 WANG Z X, GUO D R. Introdaction to Special Functions, Beijing: Science Press, 1979