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Number of spin I states for bosons*

ZHANG Li-Hua(张丽华)[1;1)] ZHAO Yu-Min(赵玉民)[1,2,3]

1 (Department of Physics, Shanghai Jiao Tong University, Shanghai 200240, China)
2 (Center of Theoretical Nuclear Physics, National Laboratory of Heavy Ion Accelerator, Lanzhou 730000, China)
3 (CCAST, World Laboratory, P.O. Box 8730, Beijing 100080, China)

Abstract We study number of spin I states for bosons in this paper. We extend Talmi's recursion formulas for number of states with given spin I to boson systems, and we prove empirical formulas for five bosons by using these recursions. We give number of states with given spin I and isospin F for three and four bosons by using sum rules of six-j and nine-j symbols. We also present formulas of states with given spin I and given F-spin for three and four single-l bosons.

Key words boson, isospin, state, number

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1 Introduction

Recently there have been many efforts to obtain simple formulas of enumerating number of states with given spin. In Ref. [1], Ginocchio and Haxton obtained a simple formula of spin zero states for four particles. In Ref. [2] Zamick and Escuderos gave a much simpler proof for dimension of spin zero states of the j^4 configuration. In Ref. [3], two of present authors, Zhao and Arima, obtained empirical formulas for given spin I states with particle number $n=3$ and 4, and some for $n=5$. In Ref. [4], Talmi developed recursion relations for n, $n-1$ and $n-2$ fermions, and proved results of Ref. [3] for three fermions. In Ref. [5], we found a simple correspondence between number of given spin states of fermions and that of bosons, and proved results of Ref. [3] for $n=4$ by using reduction rule of d bosons. In Ref. [6], Zamick and Escuderos derived an interesting relation between dimension for isospin $T=0$ and spin I states and that for isospin $T=2$ and spin I states. In Ref. [7], formulas of dimension with given spin and isospin for three and four nucleons are derived by using sum rules of six-j and nine-j symbols of Refs. [8, 9]. However, most studies concentrated on fermions, it is therefore interesting to study boson systems as well. The purpose of this paper is to present formulas for spin-l bosons which have not been extensively discussed in previous studies.

This paper is organized as follows. In Sec. II we extend Talmi's recursions to boson systems and apply it to prove empirical results for $n=5$ in Ref. [3]. In Sec. III we present number of states with given spin I and isospin F for three and four bosons, by using sum rules of six-j and nine-j symbols derived in Ref. [8].

2 Number of states for five bosons

In this Section, we use notations of Talmi's paper[4] for bosons. We denote z-axis projection of total spin I of n spin-l bosons by $M=m_1+m_2+\cdots+m_n$, and the number of states with given M in the l^n configuration by $N(M,l,n)$. The number of states with given value of I in the l^n configuration will be denoted $D(I,l,n)$.

Then for $m_1<l$, the sum of number of states with z-axis projection M for n spin-l bosons is given by

$$N(M,l-1,n)+N(M+l,l-1,n-1)+$$
$$N(M+2l,l-1,n-2)+\cdots+N(M+(n-2)l,l-1,2)+$$
$$N(M+(n-1)l,l-1,1)\ ;$$

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1) E-mail: ymzha@sjtu.edu.cn

and for $m_1 = l$, the sum of number of states with z-axis projection M for n spin-l bosons is given by

$$\begin{aligned} & N(M,l,n-2)+N(M-l,l-1,n-1)+ \\ & \quad N(M-2l,l-1,n-2)+\cdots+ \\ & \quad N(M-(n-2)l,l-1,2)+ \\ & \quad N(M-(n-1)l,l-1,1)\ . \end{aligned}$$

According to the relationship of $D(I,l,n)$ and $N(I,l,n)$, One has following recursion relations for bosons. For $I\leqslant l-1$,

$$\begin{aligned} D(I,l,n) \ = \ & D(I,l,n-2)+D(I,l-1,n)+ \\ & D(I+l,l-1,n-1)+ \\ & D(I+2l,l-1,n-2)+\cdots+ \\ & D(I+(n-2)l,l-1,2)+ \\ & D(I+(n-1)l,l-1,1)- \\ & D(l-1-I,l-1,n-1)- \\ & D(2l-1-I,l-1,n-2)-\cdots- \\ & D((n-2)l-1-I,l-1,2)- \\ & D((n-1)l-1-I,l-1,1)\ . \end{aligned} \qquad (1)$$

For $I=1$ and $n=5$, we obtain

$$\begin{aligned} D(1,l,5) \ = \ & D(1,l,3)+D(1,l-1,5)+ \\ & D(l+1,l-1,4)+D(2l+1,l-1,3)- \\ & D(l-2,l-1,4)-D(2l-2,l-1,3)\ . \end{aligned} \qquad (2)$$

In Ref. [3], an empirical formula for $I = 1$ and $n = 5$ was given by $D(1,l,5) = (Q+1)(Q+1+q)$, where

$$\begin{cases} Q=\left[\dfrac{l}{4}\right], \quad q=(l \bmod 4-1)/2, \quad \text{if } l \bmod 2=1, \\ Q=\left[\dfrac{l-3}{4}\right], \ q=((l-3) \bmod 4-1)/2, \\ \qquad\qquad\qquad \text{if } l \bmod 2=0, \end{cases}$$

and [ ] means to take the largest integer not exceeding the value inside.

Now we prove the formula of $D(1,l,5)$ by induction with respect to l, namely, we assume that it holds for spin $l-1$ bosons and prove it holds also for spin l bosons (it was shown to hold for lower spins up to $l=99$ in Ref. [3]). For convenience, we first take cases with even $l=6k$ (k is an odd number here; cases with even k can be shown similarly). Cases of other even $l=6k+2$ and $6k+4$ can be solved in the same way. We also note without details that one can repeat this process while proving the formula of $D(1,l,5)$ in Ref. [3] for odd l, and that the formula of $I=0$ and $n=5$ for spin-l bosons can be proved via the same procedure.

Using Eq. (1) of Ref. [3], we obtain

$$\begin{aligned} & D(1,6k,3)=0\ , \\ & D(12k+1,6k-1,3)=k, \\ & D(12k-2,6k-1,3)=k. \end{aligned} \qquad (3)$$

Using Eq. (5) of Ref. [5], we obtain

$$D(6k+1,6k-1,4)=3k^2-k+3\left[\frac{k}{2}\right]^2+4\left[\frac{k}{2}\right]+1\ ,$$

$$D(6k-2,6k-1,4)=3k^2-k+3\left[\frac{k}{2}\right]^2+7\left[\frac{k}{2}\right]+3\ . \qquad (4)$$

Here we used following identities: for odd k, $\left[\frac{6k-1}{3}\right]=2k-1$, $\left[\frac{k-1}{2}\right]=\left[\frac{k}{2}\right]$, $\left[\frac{6k+2}{4}\right]=3\left[\frac{k}{2}\right]+2$, $(6k-1) \bmod 3=2$, and $(k-1) \bmod 2=0$. According to our assumption,

$$\begin{aligned} D(1,6k-1,5)= & \left(\left[\frac{6k-1}{4}\right]+1\right)\left(\left[\frac{6k-1}{4}\right]+1+\right. \\ & \left.((6k-1) \bmod 4-1)/2\right)=\left(3\left[\frac{\mathrm{k}}{2}\right]+2\right)^2 . \end{aligned} \qquad (5)$$

Here we note that $\left[\frac{6k-1}{4}\right]=3\left[\frac{k}{2}\right]+1$, $(6k-1)$ mod $4 = 1$.

Substituting Eqs. (3—5) into Eq. (2), we obtain that

$$D(1,6k,5)=\left(3\left[\frac{k}{2}\right]+1\right)\left(3\left[\frac{k}{2}\right]+2\right). \qquad (6)$$

For odd k, $3\left[\frac{k}{2}\right]=\left[\frac{6k-3}{4}\right]=\left[\frac{l-3}{4}\right]$, $((6k-3)$ mod $4-1)/2 = ((l-3)$ mod $4-1)/2 = 1$, we obtain

$$\begin{aligned} D(1,6k,5) \ = \ & \left(\left[\frac{l-3}{4}\right]+1\right)\left(\left[\frac{l-3}{4}\right]+1+\right. \\ & \left.((l-3) \bmod 4-1)/2\right)\ . \end{aligned} \qquad (7)$$

This is indeed identical to $D(1,l,5)$ result of Ref. [3].

We shall not go to cases with $l = 6k+2$ (or $l = 6k+4$) and k is odd, cases with $l = 6k+1$ (or $l=6k+3$, $l=6k+5$), but point out the procedure is exactly the same as above.

3 Number of states with given spin and F spin for bosons in a single-l shell

In this Section we apply the method of Ref. [9], in which we obtained number of states with given spin and isospin for nucleons in a single-j orbit, to obtain number of states with given spin and F-spin for three and four spin-l bosons.

We first discuss the case of four spin-l bosons. Similarly to Eq. (2) of Ref. [9], we obtain that the trace of H_{IF} matrix is given by summing

$$\langle 0|[A^{(JF_2)}A^{(KF_2')}]^{(IF)}_{MM_F}[A^{(JF_2)\dagger}A^{(KF_2')\dagger}]^{(IF)}_{MM_F}|0\rangle = 1+(-)^{I+F}\delta_{JK}+ 4(2J+1)(2K+1)(2F_2+1)(2F_2'+1)\times \left\{\begin{array}{ccc} l & l & J \\ l & l & K \\ J & K & I \end{array}\right\}\left\{\begin{array}{ccc} 1/2 & 1/2 & F_2 \\ 1/2 & 1/2 & F_2' \\ F_2 & F_2' & F \end{array}\right\}, \quad (8)$$

over K, F_2, and F_2'. Here $F_2(F_2')$ and F are F spin for two and four bosons, respectively. Similar to Eq. (3) of Ref. [9], one sees

$$\sum_J\sum_\alpha \langle j^4\alpha IF|H_J|j^4\alpha IF\rangle = \sum_{JKF_2F_2'}\langle 0|[A^{(JF_2)}A^{(KF_2')}]^{(IF)}_{MM_F} [A^{(JF_2)\dagger}A^{KF_2'\dagger}]^{(IF)}_{MM_F}|0\rangle = 6D(I,l,4,F)\ . \quad (9)$$

$D(I,l,n,F)$ refer to number of states l^n bosons with given spin I and F spin.

The same as D_{IT} with $T=T_{\max}$ in Ref. [9], $D(I,l,n,F)$ with $F=F_{\max}$ here must equals D_I of Refs. [3, 5], and we shall not discuss this case in the present paper.

For convenience we define

$$S_I(l^4,\text{condition } X \text{ on } J \text{ and } K) = \sum_X \left\{\begin{array}{ccc} l & l & J \\ l & l & K \\ J & K & I \end{array}\right\}\ . \quad (10)$$

Now we discuss the case of $n=4$ and $F=1$. Here (F_2, F_2') can take the following values: (1,0), (0,1), (1,1). Because of the symmetry of the wave functions of bosons, corresponding requirements for (J,K) are ($J=$even,$K=$odd), ($J=$odd, $K=$even), or ($J=$even, $K=$even). Thus we obtain

$$6D(I,l,4,1) = \sum_{\text{even } J \text{ even } K}[1-(-)^I\delta_{JK}] + 2\sum^{JKI \text{ forms a triangle}}_{\text{even } J \text{ odd } K} 1 + S(l^4,\text{even } J \text{ odd } K) \quad (11)$$

for $F=1$.

When $n=4$ and $F=1$, $I_{\max}$ equals $4l$. For $I\geqslant 2l$, let us define $I=I_{\max}-2I_0-1$ for odd I and $I_{\max}-2I_0-2$ for even I. Using (33) of Ref. [8], we obtain

$$D(I,l,4,1) = \left(\left[\frac{I_0}{2}\right]+1\right)\left(\left[\frac{I_0}{2}\right]+1+(I_0 \bmod 2)\right)\ . \quad (12)$$

Now we come to case with $n=4$ and $I\leqslant 2l-1$. We define $I_0=(I-1)/2$ for odd I, and obtain

$$D(I,l,4,1) = (I_0+1)(l+\frac{1}{2}) - \left(1+4\left[\frac{I_0}{2}\right]+ 6\left[\frac{I_0}{2}\right]^2 + (I_0 \bmod 2)\left(6\left[\frac{I_0}{2}\right]+3\right)\right)/2\ ; \quad (13)$$

we define $I_0=I/2$ for even I, and obtain

$$D(I,l,4,1) = (I_0+1)\left(l+\frac{1}{2}\right) - (l-I_0) - \left(1+4\left[\frac{I_0}{2}\right]+6\left[\frac{I_0}{2}\right]^2+(I_0 \bmod 2)\left(6\left[\frac{I_0}{2}\right]+3\right)\right)/2\ . \quad (14)$$

For the case of $F=0$ and $n=4$, and the case of $F=1/2$ and $n=3$, we can get the similar results via the same procedure.

4 Summary and discussion

To summarize, in this paper we studied number of spin-I boson states for l^n configurations (denoted by $D(I,l,n)$). First, we extended Talmi's recursion relations to bosons and proved number of states with $I=1$ and $n=5$, which was constructed empirically in Ref. [3]. The same procedure is readily used to prove other formulas for bosons. Second, we derived number of states for three and four spin-l bosons with total angular momentum I and F spin, by using sum-rules of six-j and nine-j symbols given in Ref. [9].

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