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Lie Algebraic Analysis for the Nonlinear Transport in Electrostatic Quadrupoles

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Abstract With Lie algebraic methods, we analysed the nonlinear transport of particle motions in electrostatic quadrupoles up to third order. The procedures are: first, set up the Hamiltonian for the electrostatic quadrupoles, then expand the Hamiltonian into a sum of homogeneous polynomials of different degrees, finally, calculate the particle's nonlinear trajectories up to third order. Higher orders could be obtained if necessary.

Key words Lie algebra, nonlinear, electrostatic quadrupole

1 Hamiltonian and its expanded homogeneous polynomial

The Hamiltonian in the cartesian coordinate system with time as an independent variable

$$H_t = (m_0^2c^4 + p_x^2c^2 + p_y^2c^2 + p_z^2c^2)^{\frac{1}{2}} + q\psi, \quad (1)$$

$$\psi = (x^2 \quad y^2)V/r_0^2, \quad (2)$$

here V is the voltage of the pole, r_0 the radius of inner circle electrostatic quadrupoles.

Suppose $p_t = -H_t$, and solve p_z from it, one can obtain

$$p_z = -K(t, x, y, p_t, p_x, p_y) = [(p_t + q\psi)^2/c^2 - p_x^2 - p_y^2 - m_0^2c^2]^{\frac{1}{2}}. \quad (3)$$

Eq.(3) is the Hamiltonian corresponding to the canonical variables x, p_x, y, p_y, t and p_t with z as the independent variable. Let

$$x = x,\ p_x = p_x,\ y = y,\ p_y = p_y, \quad \tau = t - z/v_0,\ p_\tau = p_t - p_t^0, \quad (4)$$

where v_0 is the velocity of the reference particle, p_t^0 the value of p_t for the reference particle. In the phase space $\zeta = (x, p_x, y, p_y, \tau, p_\tau)$, the coordinate of reference particle always keeps zero. According to Eq.(1), we can obtain

$$p_t^0 = -H_t\big|_{\text{reference orbit}} = -(m_0^2c^4 + p_s^2c^2)^{\frac{1}{2}} = -m_0\gamma_0c^2, \quad (5)$$

here p_s is the momentum of the reference particle, γ_0 the relativistic parameter of the reference particle, i.e., $\gamma_0 = [1 - (v_0/c)^2]^{-1/2}$

The generating function F_2 for the canonical transformation of Eq.(4) is

$$F_2 = \sum_{i=1}^{3} Q_i p_i = xp_x + yp_y + \tau p_t = xp_x + yp_y + (t - z/v_0)p_t. \quad (6)$$

The new Hamiltonian corresponding to the generating function F_2 is

$$H = K + \partial F_2/\partial z = -[(p_\tau - m_0\gamma_0c^2 + q\psi)^2/c^2 - p_x^2 - p_y^2 - (m_0c)^2]^{1/2} - (p_\tau - m_0\gamma_0c^2)/v_0 = -[(p_\tau - m_0\gamma_0c^2 + q(x^2 - y^2)V/r_0^2)^2/c^2 - p_x^2 - p_y^2 - (m_0c)^2]^{1/2} - (p_\tau - m_0\gamma_0c^2)/v_0. \quad (7)$$

Expand the new Hamiltonian H into Taylor series as

$$H = \sum_{n=0}^{\infty} H_n, \quad (8)$$

where H_n is the homogeneous polynomial of n-th order in the phase space $(x, p_x, y, p_y, \tau, p_\tau)$. The first five terms are

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$$H_0 = p_s(\beta_0^{-2} - 1) \quad (\text{here } p_s = m_0\gamma_0 v_0),$$
$$H_1 = 0,$$
$$H_2 = (p_x^2 + p_y^2)/(2m_0\gamma_0\beta_0 c) + qV(x^2 - y^2)/(\beta_0 r_0^2 c) + p_\tau^2/(2m_0\gamma_0^3\beta_0^3 c^3),$$
$$H_3 = (p_x^2 + p_y^2)p_\tau/(2m_0^2\gamma_0^2\beta_0^3 c^3) + qV(x^2 - y^2)p_\tau/(m_0\gamma_0^3\beta_0^3 c^3 r_0^2) + p_\tau^3/(2m_0^2\gamma_0^4\beta_0^5 c^5),$$
$$H_4 = (p_x^4 + p_y^4)/(8m_0^3\gamma_0^3\beta_0^3 c^3) + q^2V^2(x^4 + y^4)/(2m_0\gamma_0^3\beta_0^3 c^3 r_0^4) + p_\tau^4(15 - 3\beta_0^2)/(24m_0^3\beta_0^7\gamma_0^5 c^7) +$$
$$3qVp_\tau^2(x^2 - y^2)/(2m_0^2\beta_0^5\gamma_0^4 c^5 r_0^2) + p_\tau^2(p_x^2 + p_y^2)(3 - \beta_0^2)/(4m_0^3\beta_0^5\gamma_0^3 c^5) +$$
$$qV(p_x^2 + p_y^2)(x^2 - y^2)/(2m_0^2\beta_0^3\gamma_0^2 c^3 r_0^2) - q^2V^2x^2y^2/(m_0\beta_0^3\gamma_0^3 c^3 r_0^4) + p_x^2p_y^2/(4m_0^3\beta_0^3\gamma_0^3 c^3).$$

2 Lie map

In the phase space $\zeta = (x, p_x, y, p_y, \tau, p_\tau)$, the original point ζ^{in} is mapped to the final point ζ^{fin} after the electrostatic quadrupoles. The Lie transformation related to the Hamiltonian H is

$$\boldsymbol{M} = \exp\left[-:\int_{z_0}^{z_f} H\mathrm{d}z:\right] = \cdots \boldsymbol{M}_4\boldsymbol{M}_3\boldsymbol{M}_2 =$$
$$\cdots\left(1 + :f_4: + \frac{1}{2}:f_4:^2 + \cdots\right)$$
$$\left(1 + :f_3: + \frac{1}{2}:f_3:^2 + \cdots\right)\boldsymbol{M}_2 =$$
$$\boldsymbol{M}_2 + :f_3:\boldsymbol{M}_2 + \left(:f_4: + \frac{1}{2}:f_3:^2\right)\boldsymbol{M}_2. \quad (10)$$

In the expression of Eq.(10) the factorization theorem[1,2] is used. That is

$$\boldsymbol{M}_2 = \exp(:f_2:),\ \boldsymbol{M}_3 = \exp(:f_3:),$$
$$\boldsymbol{M}_4 = \exp(:f_4:),$$

here

$$f_2 = -\int_{z_0}^{z_f} H_2\mathrm{d}z,$$
$$f_3 = -\int_{z_0}^{z_f} h_3^{int}\mathrm{d}z,$$
$$f_4 = -\int_{z_0}^{z_f} h_4^{int}\mathrm{d}z + \frac{1}{2}\int_{z_0}^{z_f}\mathrm{d}z_1\int_{z_0}^{z_1}\mathrm{d}z_2$$
$$[-h_3^{int}(z_2), -h_3^{int}(z_1)],$$

where

$$h_n^{int}(z) = \boldsymbol{M}_2 H_n.$$

Let the map $\boldsymbol{M}$ act on the canonical coordinate ζ^{in}, and the subscript denote the order of approximation, we have

$$\zeta_1 = \exp(:f_2:)\zeta^{in}, \qquad (\text{first order})$$
$$\zeta_2 = :f_3:\zeta_1, \qquad (\text{second order}) \quad (14)$$
$$\zeta_3 = :f_4:\zeta_1 + \frac{1}{2}:f_3:^2\zeta_1. \quad (\text{third order})$$

In the following, we give the results of particle trajectories of first order, second order and third order approximation, respectively

3 Particle trajectory calculations

3.1 First order

The definition of the Lie transformation is[3]

$$\exp(:f:)g = g + [f,g] + [f,[f,g]]/2!$$

Because H_2 in the Eq.(8) does not depend on z, the first Eq.(12) can be written as

$$f_2 = -lH_2.$$

Let

$$k^2 = \frac{2qV}{m_0\beta_0^2\gamma_0 c^2 r_0^2}, \quad (17)$$
$$p_s = m_0\gamma_0\beta_0 c,$$

according to the first expression of Eq.(12) and Eq.(14), we can obtain the transformation of the first order approximation:

$$\begin{bmatrix} x_1 \\ p_{x_1} \\ y_1 \\ p_{y_1} \\ \tau_1 \\ p_{\tau_1} \end{bmatrix} = \begin{bmatrix} \cos(kl) & \frac{1}{p_s k}\sin(kl) & 0 & 0 & 0 & 0 \\ -p_s k\sin(kl) & \cos(kl) & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh(kl) & \frac{1}{p_s k}\sinh(kl) & 0 & 0 \\ 0 & 0 & p_s k\sinh(kl) & \cosh(kl) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{l}{m_0\beta_0^3\gamma_0^3 c^3} \\ 0 & 0 & 0 & 0 & 0 & l \end{bmatrix} \begin{bmatrix} x \\ p_x \\ y \\ p_y \\ \tau \\ p_\tau \end{bmatrix}.$$

3.2 Second order

According to the second expression of Eq.(11) and the expression of Eq.(12), one can obtain

$$f_3 = -\int_0^l h_3^{\text{int}}(\zeta, z_1)\,\mathrm{d}z_1 = -\int_0^l M_2 H_3(\zeta, z_1)\,\mathrm{d}z_1 . \tag{19}$$

According to the second expression of the Eq.(12) and the second expression of the Eq.(14), one can obtain the final coordinates of second order:

$$\begin{aligned} x_2 = &\left\{\frac{\sin(kl)[\beta_0^2\gamma_0^2\sin(2kl) - 2kl(1+\gamma_0^2)]}{4\gamma_0^2\beta_0 c p_s} - \frac{\beta_0\cos(kl)\sin^2(kl)}{2p_s c}\right\} x p_\tau \\ &\left\{\frac{\cos(kl)[2kl(1+\gamma_0^2) + \beta_0^2\gamma_0^2\sin(2kl)]}{4p_s^2 k\gamma_0^2\beta_0 c} - \frac{\beta_0\sin^3(kl)}{2kp_s^2 c}\right\} p_x p_\tau , \end{aligned} \tag{20}$$

$$\begin{aligned} p_{x_2} = &\left\{\frac{\beta_0\cos(kl)\sin^2(kl)}{2p_s c} - \frac{\sin(kl)[2kl(1+\gamma_0^2) + \beta_0^2\gamma_0^2\sin(2kl)]}{4p_s\gamma_0^2\beta_0 c}\right\} p_x p_\tau \\ &\left\{\frac{k\beta_0\sin^3(kl)}{2c} - \frac{k\cos(kl)[2kl(1+\gamma_0^2) - \beta_0^2\gamma_0^2\sin(2kl)]}{4\gamma_0^2\beta_0 c}\right\} x p_\tau , \end{aligned} \tag{21}$$

$$\begin{aligned} y_2 = &\left\{-\frac{\beta_0\sinh^3(kl)}{2kp_s^2 c} + \frac{\cosh(kl)[2kl(1+\gamma_0^2) + \beta_0^2\gamma_0^2\sinh(2kl)]}{4p_s^2 k\gamma_0^2\beta_0 c}\right\} p_y p_\tau + \\ &\left\{\frac{\beta_0\cosh(kl)\sinh^2(kl)}{2p_s c} + \frac{\sinh(kl)[2kl(1+\gamma_0^2) - \beta_0^2\gamma_0^2\sinh(2kl)]}{4p_s\gamma_0^2\beta_0 c}\right\} y p_\tau , \end{aligned} \tag{22}$$

$$\begin{aligned} p_{y_2} = &-\left\{\frac{\beta_0\cosh(kl)\sinh^2(kl)}{2p_s c} - \frac{\sinh(kl)[2kl(1+\gamma_0^2) + \beta_0^2\gamma_0^2\sinh(2kl)]}{4p_s\gamma_0^2\beta_0 c}\right\} p_y p_\tau + \\ &\left\{\frac{k\cosh(kl)[2kl(1+\gamma_0^2) - \beta_0^2\gamma_0^2\sinh^2(2kl)]}{4\gamma_0^2\beta_0 c} + \frac{k\beta_0\sinh^3(kl)}{2c}\right\} y p_\tau , \end{aligned} \tag{23}$$

$$\begin{aligned} \tau_2 = &-\frac{\sin^2(kl)\beta_0}{2p_s c} x p_x + \frac{\sinh^2(kl)\beta_0}{2p_s c} y p_y + \frac{3lp_\tau^2}{2p_s^2\gamma_0^2\beta_0^3 c^3} + \\ &\frac{[2kl(1+\gamma_0^2) + \beta_0^2\gamma_0^2\sinh(2kl)]p_y^2}{8p_s^2 k\gamma_0^2\beta_0 c} + \frac{k[2kl(1+\gamma_0^2) - \beta_0^2\gamma_0^2\sin(2kl)]x^2}{8\gamma_0^2\beta_0 c} - \\ &\frac{k[2kl(1+\gamma_0^2) - \beta_0^2\gamma_0^2\sinh(2kl)]y^2}{8\gamma_0^2\beta_0 c} + \frac{[2kl(1+\gamma_0^2) + \beta_0^2\gamma_0^2\sin(2kl)]p_x^2}{8p_s^2 k\gamma_0^2\beta_0 c} , \end{aligned} \tag{24}$$

$$p_{\tau_2} = 0. \tag{25}$$

3.3 Third order

In the same way, according the third expression of Eq.(12) and the third expression of Eq.(14), one can obtain the final coordinates of third order:

$$\begin{aligned} x_3 = &\sinh(kl)[\cos(kl)\sinh(kl) - \cosh(kl)\sin(kl)]\frac{\beta_0^2 y p_x p_y}{8p_s^2} - \frac{\sin(kl)\{16kl - [6kl + \sin(2kl)]\beta_0^2\}}{32}k^2 x^3 \\ &\frac{32kl\cos(kl) + [\sin(3kl) + 9\sin(kl) - 12\cos(kl)]\beta_0^2}{64p_s^3 k}p_x^3 + \\ &\frac{32kl\cos(kl) - 3[4kl\cos(kl) - 7\sin(kl) + \sin(3kl)]\beta_0^2}{64p_s}kx^2 p_x - \\ &\frac{\sin(kl)\{16kl - 3[2kl - \sin(2kl)]\beta_0^2\}}{32p_s^2}xp_x^2 - \{\cos(kl)[16kl - (4kl + \sinh(2kl))\beta_0^2] + \\ &[5 + \cosh(2kl)]\sin(kl)\beta_0^2\}\frac{ky^2 p_x}{32p_s} + \frac{\cosh(2kl)[2\sin(kl) + \sin(3kl)] - 3\sin(kl) - \cos(3kl)\sinh(2kl)}{16p_s} \end{aligned}$$

$$k\beta_0^2 xyp_y - \{4k^2 l^2\cos(kl)(1+2\gamma_0^2+\gamma_0^4) - 2kl\sin(kl)\gamma_0^2$$
$$[1+\cos(2kl) - (25+\cos(2kl) - (17+\cos(2kl)\beta_0^2)\gamma_0^2] - 4\cos(kl)\sin^2(kl)\beta_0^4\gamma_0^4\}\frac{xp_\tau^2}{32p_s^2\gamma_0^4\beta_0^2c^2} +$$
$$\{4kl\cos(kl)(4-\beta_0^2) + [3\sin(kl) - \cosh(2kl)\sin(kl) + \cos(kl)\sinh(2kl)]\beta_0^2\}\frac{p_xp_y^2}{32p_s^3k} +$$
$$\left[\frac{4kl\cos(kl)(2+\gamma_0^2)}{\gamma_0^2\beta_0^2c^2} - \frac{k^2l^2\sin(kl)(1+2\gamma_0^2+\gamma_0^4)}{\gamma_0^4\beta_0^2c^2} + \frac{\sin^3(kl)\beta_0^2}{c^2} + \frac{4\sin(kl)}{c^2}\right]\frac{p_xp_\tau^2}{8p_s^3k} -$$
$$\frac{16kl\sin(kl) - \{\cos(kl)[1-\cosh(2kl)] + [4kl - \sinh(2kl)]\sin(kl)\}\beta_0^2}{32p_s^2}xp_y^2 +$$
$$\frac{16kl\sin(kl) + \{\cos(kl)[1-\cosh(2kl)] - \sin(kl)[4kl+\sinh(2kl)]\}\beta_0^2}{32}k^2xy^2 \qquad (26)$$

Because of the page limit, the third terms of p_{x_3}, y_3, p_{y_3}, τ_3 and p_{τ_3} are not listed here.

4 Conclusion

Now we have obtained the third order solutions of the particle trajectories in electrostatic quadrupoles. If we give the initial canonical coordinates in six-dimensional phase space, we can directly obtain the finial canonical coordinates after the electrostatic quadrupoles. We can see that, however, the solutions are too complex to be used by hands. So, we will put them into a computer program.

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静电四极透镜中非线性传输的Lie代数分析

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摘要 用Lie代数方法分析了带电粒子在静电四极透镜中的非线性传输,结果为三级近似. 分析过程为:首先建立粒子在静电四极透镜中的运动的Hamilton函数,然后将Hamilton函数在平衡轨道附近展开成幂级数,最后计算粒子的非线性轨迹到三级近似. 根据需要,还可以扩展到更高级近似解.

关键词 Lie代数 非线性 静电四极透镜

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