Gravitational losses for the binary systems induced by the next-to-leading spin-orbit coupling effects*

Hao Zhang (张豪)[1,2] Wei Gao (高玮)[1,2] Guansheng He (贺观圣)[1†] Siming Liu (刘四明)[2‡] Huanyu Jia (贾焕玉)[2§] Wenbin Lin (林文斌)[1,2♮]

[1]School of Mathematics and Physics, University of South China, Hengyang 421001, China
[2]School of Physical Science and Technology, Southwest Jiaotong University, Chengdu 610031, China

Abstract: Compact binary systems lose orbital energy and momentum because of gravitational radiation. Based on the mass and mass-current multipole moments of the binary system with the spin vector defined by Bohé *et al.* [Class. Quantum Grav. **30**, 075017 (2013)], we calculate the loss rates of energy, angular momentum and linear momentum induced by the next-to-leading spin-orbit effects. For the case of circular orbit, the formulations the these losses are formulated in terms of orbital frequency.

Keywords: gravitational wave radiation, gravitational losses, spin-orbit coupling, post-Newtonian approximation

DOI: 10.1088/1674-1137/adc2db **CSTR:** 32044.14.ChinesePhysicsC.49075103

I. INTRODUCTION

Compact binary systems such as black hole-black hole pairs, black hole-dense star (neutron star, white dwarf, or others) pairs, or double dense stars, are the best candidates for gravitational-wave sources, which have been detected by LIGO/VIRGO [1−4]. They are also expected to be detected by the space-based gravitational-wave detectors such as LISA, TianQin, and Taiji in the near future [5−9].

Gravitational radiation leads to a loss of orbital energy, angular momentum, and linear momentum of the compact binary systems. The analytic calculations for the gravitational losses can only be realized via post-Newtonian (PN) approximations, which have been extensively studied in the literature. The references where these results were first presented are summarized in the following tables.

Table 1 presents the references for the loss rates of the non-spinning binary system's energy E, angular momentum $\boldsymbol{J}$, and linear momentum $\boldsymbol{P}$ to different PN orders including the tail contributions.

For the case of the spinning binary systems, the situation is somewhat more complicated as the definitions of the spin vector and supplementary spin condition (SSC) are not unique. Although they describe the same physical phenomena, the formulations of the gravitational loss rates are dependent on these definitions. Table 2 provides the reference for the formulations of gravitational loss rates of the spinning binary system's energy, angular momentum, and linear momentum induced by the spin-orbit (SO) coupling based on the spin vector defined by Barker and O'Connell [23] under the SSC given by Pirani [24].

Table 3 presents the references for the formulations of the gravitational losses of the spinning binary system's energy, angular momentum, and linear momentum due to SO contributions and spin-spin (SS) contributions based on the spin vector defined by Faye, Blanchet and Buonanno [26] under Tulczyjew's SSC [27]. Table 4 provides the references for the case of the spin vector defined by Bohé *et al.* [28].

In this work, we utilize the 2.5PN dynamic equation of the center-of-mass and 1PN precession equations of the spin vector given by Bohé *et al.* [28] and the mass moment given by Marsat *et al.* [29], to calculate the gravitational loss of the spinning binary systems induced by the next-to-leading SO coupling effects.

The remainder of this paper is organized as follows. Section II introduces the 2.5PN acceleration for the relative motion of the spinning binary systems, which is used in later derivations. In Section III we provide the formulas for calculating gravitational losses. In Section IV, we derive the gravitational losses induced by the next-to-

Received 9 January 2025; Accepted 19 March 2025; Published online 20 March 2025

* Supported in part by the National Natural Science Foundation of China (12475057, U1931204, 12147208)

† E-mail: hgs@usc.edu.cn
‡ E-mail: liusm@swjtu.edu.cn
§ E-mail: hyjia@swjtu.edu.cn
♮ E-mail: lwb@usc.edu.cn

Table 1. References for the gravitational loss rates of the non-spinning binary systems in the Newtonian and post-Newtonian 1PN, 2PN, 2.5PN, 3PN and 3.5PN approximations. The superscript "·" denotes the time derivative.

gravitational loss rates	circular orbit	general orbit
$\dot{E}_{\rm N}$	Peters [10]	Peters [10]
$\dot{E}_{\rm 1PN}$	Will & Wagoner [11]	Will & Wagoner [11]
$\dot{E}_{\rm 2PN}$	Blanchet, Damour & Iyer [12]	Will & Wiseman [13]
$\dot{E}_{\rm 2.5tail}$	Blanchet [14]	Arun *et al.* [15]
$\dot{E}_{\rm 3PN}$	Blanchet, Iyer & Joguet [16]	Arun *et al.* [15]
$\dot{E}_{\rm 3.5tail}$	Blanchet [17]	N/A
$\dot{\boldsymbol{J}}_{\rm N}$, $\dot{\boldsymbol{J}}_{\rm 1PN}$	Junker & Schäfer [18]	Junker & Schäfer [18]
$\dot{\boldsymbol{J}}_{\rm 2PN}$	Gopakumar & Iyer [19]	Gopakumar & Iyer [19]
$\dot{\boldsymbol{J}}_{\rm 2.5tail}$, $\dot{\boldsymbol{J}}_{\rm 3PN}$	Arun *et al.* [20]	Arun *et al.* [20]
$\dot{\boldsymbol{P}}_{\rm N}$	Junker & Schäfer [18]	Junker & Schäfer [18]
$\dot{\boldsymbol{P}}_{\rm 1PN}$, $\dot{\boldsymbol{P}}_{\rm 2PN}$	Racine, Buonanno & Kidder [21]	Racine, Buonanno & Kidder [21]
$\dot{\boldsymbol{P}}_{\rm 2.5tail}$	Kastha [22]	Kastha [22]

Table 2. Reference for the gravitational loss rates of the spinning binary systems with the spin vector defined by Barker and O'Connell under Pirani's SSC.

gravitational loss rates	circular orbit	general orbit
$\dot{E}_{\rm 1.5SO}$, $\dot{\boldsymbol{J}}_{\rm 1.5SO}$, $\dot{\boldsymbol{P}}_{\rm 1.5SO}$	Kidder [25]	Kidder [25]

Table 3. References for the gravitational loss rates of the spinning binary system based on the spin vector defined by Faye, Blanchet and Buonanno under Tulczyjew's SSC.

gravitational loss rates	circular orbit	general orbit
$\dot{E}_{\rm 1.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{E}_{\rm 2.5SO}$	Blanchet, Buonanno & Faye [30]	Blanchet, Buonanno & Faye [30]
$\dot{\boldsymbol{J}}_{\rm 1.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{\boldsymbol{J}}_{\rm 2.5SO}$	This work	This work
$\dot{\boldsymbol{P}}_{\rm 0.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{E}_{\rm 2SS}$, $\dot{\boldsymbol{J}}_{\rm 2SS}$, $\dot{\boldsymbol{P}}_{\rm 1.5SO}$, $\dot{\boldsymbol{P}}_{\rm 2SS}$	Racine, Buonanno & Kidder [21]	Racine, Buonanno & Kidder [21]

Table 4. References for the gravitational loss rates of the spinning binary systems based on the spin vector defined by Bohé et al. under Tulczyjew's SSC.

gravitational loss rates	circular orbit	general orbit
$\dot{E}_{\rm 1.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{E}_{\rm 2.5SO}$	Marsat *et al.* [29]	This work
$\dot{E}_{\rm 3SOtail}$, $\dot{E}_{\rm 3.5SO}$, $\dot{E}_{\rm 4SOtail}$	Marsat *et al.* [29]	N/A
$\dot{E}_{\rm 2SS}$, $\dot{E}_{\rm 3SS}$	Bohé *et al.* [31]	Bohé *et al.* [31]
$\dot{\boldsymbol{J}}_{\rm 1.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{\boldsymbol{J}}_{\rm 2.5SO}$	This work	This work
$\dot{\boldsymbol{P}}_{\rm 0.5SO}$	Same as Kidder [25]	Same as Kidder [25]
$\dot{\boldsymbol{P}}_{\rm 1.5SO}$	This work	This work

leading spin-orbit coupling effects. For completeness and to enable comparisons, we present in Appendic B the 2.5PN angular momentum loss in terms of the spin vector defined by Faye, Blanchet and Buonanno [26] . Section V provides these loss rates for the circular orbit, which are further summarized in Section VI. In this article, small Greek alphabet represents 0, 1, 2, 3, and small letter represents 1, 2, 3.

II. MOTION FOR THE BINARY SYSTEM IN THE 2.5PN APPROXIMATION

Assuming the spinning compact binary has masses M_1 and M_2, the position vectors of the bodies are $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$, with corresponding velocities $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ and spins of the two bodies $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$, respectively. The precession equations of the spinning binary systems can be written as [28]

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t}=\frac{GM}{c^2R^2}\eta\Big\{\boldsymbol{V}\Big[\frac{7}{2}(\boldsymbol{n}\cdot\boldsymbol{S})+\frac{3}{2}\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big] -\boldsymbol{n}\Big[\frac{7}{2}(\boldsymbol{V}\cdot\boldsymbol{S})+\frac{3}{2}\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big]\Big\}\,, \tag{1}$$

$$\frac{\mathrm{d}\boldsymbol{\Delta}}{\mathrm{d}t}=\frac{GM}{c^2R^2}\Big\{\boldsymbol{V}\Big[\frac{3}{2}\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{S})+\Big(\frac{3}{2}-\frac{5}{2}\eta\Big)(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big] -\boldsymbol{n}\Big[\frac{3}{2}\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{S})-\Big(\frac{3}{2}-\frac{5}{2}\eta\Big)(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big]\Big\}, \tag{2}$$

where $M\equiv M_1+M_2$ denotes the total mass of the system, $\delta M\equiv M_1-M_2$, and $\eta\equiv M_1M_2/M^2$. $\boldsymbol{S}\equiv\boldsymbol{S}_1+\boldsymbol{S}_2$ denotes the total spin. $\boldsymbol{\Delta}=M(\boldsymbol{S}_2/M_2-\boldsymbol{S}_1/M_1)$. $\boldsymbol{V}\equiv\boldsymbol{V}_1-\boldsymbol{V}_2$ is the binary system's relative velocity. $\boldsymbol{n}\equiv\boldsymbol{R}/R$ is the unit vector, with $\boldsymbol{R}=\boldsymbol{X}_1-\boldsymbol{X}_2$ denoting the separation vector between the two bodies and $R\equiv|\boldsymbol{R}|$. We also need the 2.5PN acceleration of the spinning binary system's relative motion, which can be written as [25, 28]

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}=\boldsymbol{A}_N+\boldsymbol{A}_{1\mathrm{PN}}+\boldsymbol{A}_{1.5\mathrm{SO}}+\boldsymbol{A}_{2\mathrm{PN}}+\boldsymbol{A}_{2.5\mathrm{SO}}\,, \tag{3}$$

where

$$\boldsymbol{A}_{\mathrm{N}}=-\frac{GM}{R^2}\boldsymbol{n}\,, \tag{4}$$

$$\boldsymbol{A}_{1\mathrm{PN}}=-\frac{GM}{c^2R^2}\Big\{\boldsymbol{n}\Big[(1+3\eta)V^2-(4+2\eta)\frac{GM}{R}-\frac{3}{2}\eta\dot{R}^2\Big]-2(2-\eta)\dot{R}\boldsymbol{V}\Big\}\,, \tag{5}$$

$$\boldsymbol{A}_{1.5\mathrm{SO}}=\frac{G}{c^3R^3}\Big\{6\boldsymbol{n}\Big[2(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\frac{\delta m}{m}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big]-\Big[7(\boldsymbol{V}\times\boldsymbol{S})+3\frac{\delta m}{m}(\boldsymbol{V}\times\boldsymbol{\Delta})\Big]+3\dot{R}\Big(3\boldsymbol{n}\times\boldsymbol{S}+\frac{\delta m}{m}\boldsymbol{n}\times\boldsymbol{\Delta}\Big)\Big\}\,, \tag{6}$$

$$\begin{aligned}\boldsymbol{A}_{2\mathrm{PN}}=-\frac{GM}{c^4R^2}\Big\{&\boldsymbol{n}\Big[\frac{3}{4}(12+29\eta)\frac{(GM)^2}{R^2}+\eta(3-4\eta)V^4+\frac{15}{8}\eta(1-3\eta)\dot{R}^4-\frac{3}{2}\eta(3-4\eta)V^2\dot{R}^2-\frac{1}{2}\eta(13-4\eta)\frac{GM}{R}V^2\\&-(2+25\eta+2\eta^2)\frac{GM}{R}\dot{R}^2\Big]-\frac{1}{2}\dot{R}\boldsymbol{V}\Big[\eta(15+4\eta)V^2-(4+41\eta+8\eta^2)\frac{GM}{R}-3\eta(3+2\eta)\dot{R}^2\Big]\Big\}\,,\end{aligned} \tag{7}$$

$$\begin{aligned}\boldsymbol{A}_{2.5\mathrm{SO}}=\frac{G}{c^5R^3}\Big\{&\Big[(24+19\eta)\frac{GM}{R}+\frac{3}{2}(1+10\eta)\dot{R}^2-14\eta V^2\Big](\boldsymbol{V}\times\boldsymbol{S})-\Big[(28+29\eta)\frac{GM}{R}+\frac{45}{2}\eta\dot{R}^2+\frac{3}{2}(1-15\eta)V^2\Big]\dot{R}(\boldsymbol{n}\times\boldsymbol{S})\\&+\Big[\frac{1}{2}(24+19\eta)\frac{GM}{R}+\frac{3}{2}(1+6\eta)\dot{R}^2-7\eta V^2\Big]\frac{\delta M}{M}(\boldsymbol{V}\times\boldsymbol{\Delta})-\Big[\frac{1}{2}(24+31\eta)\frac{GM}{R}+15\eta\dot{R}^2+\frac{3}{2}(1-8\eta)V^2\Big]\frac{\delta M}{M}\dot{R}^2(\boldsymbol{n}\times\boldsymbol{\Delta})\\&-\boldsymbol{n}\Big[(44+33\eta)\frac{GM}{R}+30\eta\dot{R}^2-24\eta V^2\Big](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}-\boldsymbol{n}\Big[\frac{1}{2}(48+37\eta)\frac{GM}{R}+15\eta\dot{R}^2-12\eta V^2\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\\&-\Big[\frac{21}{2}(1-\eta)(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\frac{3}{2}(3-4\eta)\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big]\dot{R}\boldsymbol{V}\Big\}\,,\end{aligned} \tag{8}$$

where the Tulczyjew's SSC has been employed, $\dot{R}=\boldsymbol{n}\cdot\boldsymbol{V}$.

III. FORMULAS FOR CALCULATING THE GRAVITATIONAL LOSSES

The losses in an isolated system's energy, angular momentum, and linear momentum due to the gravitational-wave radiation can be expressed in terms of the symmetric and tracefree (STF)-multipole moments as follows [32]:

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\frac{G}{c^5}\sum_{l=2}^{\infty}\Big\{\Big(\frac{1}{c}\Big)^{2(l-2)}\frac{(l+1)(l+2)}{l(l-1)l!(2l+1)!!}\overset{(l+1)}{\mathcal{I}}_{A_l}\overset{(l+1)}{\mathcal{I}}_{A_l}+\Big(\frac{1}{c}\Big)^{2(l-1)}\frac{4l(l+2)}{(l-1)(l+1)!(2l+1)!!}\overset{(l+1)}{\mathcal{J}}_{A_l}\overset{(l+1)}{\mathcal{J}}_{A_l}\Big\}\,, \tag{9}$$

$$\frac{\mathrm{d}J_j}{\mathrm{d}t}=-\frac{G}{c^5}\sum_{l=2}^{\infty}\Big\{\Big(\frac{1}{c}\Big)^{2(l-2)}\frac{(l+1)(l+2)}{(l-1)l!(2l+1)!!}\epsilon_{jpq}\overset{(l)}{\mathcal{I}}_{pA_{l-1}}\overset{(l+1)}{\mathcal{I}}_{qA_{l-1}}+\Big(\frac{1}{c}\Big)^{2(l-1)}\frac{4l^2(l+2)}{(l-1)(l+1)!(2l+1)!!}\epsilon_{jpq}\overset{(l)}{\mathcal{J}}_{pA_{l-1}}\overset{(l+1)}{\mathcal{J}}_{qA_{l-1}}\Big\}\,, \tag{10}$$

$$\begin{aligned}\frac{\mathrm{d}P_j}{\mathrm{d}t}=&-\frac{G}{c^7}\sum_{l=2}^{\infty}\Big\{\Big(\frac{1}{c}\Big)^{2(l-2)}\frac{2(l+2)(l+3)}{l(l+1)!(2l+3)!!}\overset{(l+2)}{\mathcal{I}}_{jA_l}\overset{(l+1)}{\mathcal{I}}_{A_l}+\Big(\frac{1}{c}\Big)^{2(l-1)}\frac{8(l+3)}{(l+1)!(2l+3)!!}\overset{(l+2)}{\mathcal{J}}_{jA_l}\overset{(l+1)}{\mathcal{J}}_{jA_l}\\&+\Big(\frac{1}{c}\Big)^{2(l-2)}\frac{8(l+2)}{(l-1)(l+1)!(2l+1)!!}\epsilon_{jpq}\overset{(l+1)}{\mathcal{I}}_{pA_{l-1}}\overset{(l+1)}{\mathcal{J}}_{qA_{l-1}}\Big\}\,,\end{aligned} \tag{11}$$

where E, J_j and P_j are the orbital energy, angular momentum, and linear momentum of the system. $\mathcal{I}_{A_l}$ and $\mathcal{J}_{A_l}$ are the STF radiative mass and current multipole moments, respectively. A_l denotes a multi-index length l, *i.e.* $A_l=a_1a_2\cdots a_l$, where a_i with $1\le i\le l$ takes indices $1,2,3$. $\overset{(l)}{\mathcal{I}}\equiv d^l\mathcal{I}/dt^l$ and $\overset{(l)}{\mathcal{J}}\equiv d^l\mathcal{J}/dt^l$.

Following the calculation method in [18], all possible values for the parameter l in Eqs. (9)−(11) were considered to ensure the accuracy of $\frac{\mathrm{d}E}{\mathrm{d}t}$ and $\frac{\mathrm{d}J_j}{\mathrm{d}t}$ to the 2.5PN order, and that of $\frac{\mathrm{d}P_j}{\mathrm{d}t}$ to the 1.5PN order, as follows:

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\frac{G}{c^5}\Big[\frac{1}{5}\overset{(3)}{\mathcal{I}}_{kl}\overset{(3)}{\mathcal{I}}_{kl}+\frac{1}{c^2}\frac{16}{45}\overset{(3)}{\mathcal{J}}_{kl}\overset{(3)}{\mathcal{J}}_{kl}+\frac{1}{c^2}\frac{1}{189}\overset{(4)}{\mathcal{I}}_{klm}\overset{(4)}{\mathcal{I}}_{klm}+\frac{1}{c^4}\frac{1}{84}\overset{(4)}{\mathcal{J}}_{klm}\overset{(4)}{\mathcal{J}}_{klm}+\frac{1}{c^4}\frac{1}{9072}\overset{(5)}{\mathcal{I}}_{klmn}\overset{(5)}{\mathcal{I}}_{klmn}\Big]\,, \tag{12}$$

$$\frac{\mathrm{d}J_j}{\mathrm{d}t}=-\frac{G}{c^5}\epsilon_{jpq}\Big[\frac{2}{5}\overset{(2)}{\mathcal{I}}_{pk}\overset{(3)}{\mathcal{I}}_{qk}+\frac{1}{c^2}\frac{32}{45}\overset{(2)}{\mathcal{J}}_{pk}\overset{(3)}{\mathcal{J}}_{qk}+\frac{1}{c^2}\frac{1}{63}\overset{(3)}{\mathcal{I}}_{pkl}\overset{(4)}{\mathcal{I}}_{qkl}+\frac{1}{c^4}\frac{1}{28}\overset{(3)}{\mathcal{J}}_{pkl}\overset{(4)}{\mathcal{J}}_{qkl}+\frac{1}{c^4}\frac{1}{2268}\overset{(4)}{\mathcal{I}}_{pklm}\overset{(4)}{\mathcal{I}}_{qklm}\Big], \tag{13}$$

$$\frac{\mathrm{d}P_j}{\mathrm{d}t}=-\frac{G}{c^7}\Big[\frac{2}{63}\overset{(4)}{\mathcal{I}}_{jkl}\overset{(3)}{\mathcal{I}}_{kl}+\frac{1}{c^2}\frac{4}{63}\overset{(4)}{\mathcal{J}}_{jkl}\overset{(3)}{\mathcal{J}}_{kl}+\frac{16}{45}\epsilon_{jpq}\overset{(3)}{\mathcal{I}}_{pk}\overset{(3)}{\mathcal{J}}_{qk}+\frac{1}{c^2}\frac{1}{1134}\overset{(5)}{\mathcal{I}}_{jklm}\overset{(4)}{\mathcal{I}}_{klm}+\frac{1}{c^2}\frac{1}{126}\epsilon_{jpq}\overset{(4)}{\mathcal{I}}_{pkl}\overset{(4)}{\mathcal{J}}_{qkl}\Big]\,. \tag{14}$$

The non-spin mass and mass-current multipole moments for the binary system can be written as [13,33]

$$\begin{aligned}\underset{\mathrm{NS}}{\mathcal{I}_{ij}}=&\,\mu R^2\Big\{1+\frac{1}{c^2}\Big[\frac{29}{42}(1-3\eta)V^2-\frac{1}{7}(5-8\eta)\frac{GM}{R}\Big]+\frac{1}{c^4}\Big[\frac{1}{756}(2021-5947\eta-4883\eta^2)\frac{GM}{R}V^2\\&-\frac{1}{252}(355+1906\eta-337\eta^2)\frac{(GM)^2}{R^2}-\frac{1}{756}(131-907\eta+1273\eta^2)\frac{GM}{R}\dot{R}^2+\frac{1}{504}(253-1835\eta+3545\eta^2)V^4\Big]\Big\}n_{<ij>}\\&+\mu R^2\Big\{\frac{11}{21c^2}(1-3\eta)+\frac{1}{c^4}\Big[\frac{1}{189}(742-335\eta-985\eta^2)\frac{GM}{R}+\frac{1}{126}(41-337\eta+733\eta^2)V^2+\frac{5}{63}(1-5\eta+5\eta^2)\dot{R}^2\Big]\Big\}V_{<ij>}\\&-\mu\dot{R}R^2\Big\{\frac{4}{7c^2}(1-3\eta)+\frac{2}{c^4}\Big[\frac{1}{63}(13-101\eta+209\eta^2)V^2+\frac{1}{756}(1085-4057\eta-1463\eta^2)\frac{GM}{R}\Big]\Big\}n_{<i}V_{j>}\,,\end{aligned} \tag{15}$$

$$\underset{\mathrm{NS}}{\mathcal{J}_{ij}}=\mu R^2\frac{\delta M}{M}\Big\{1+\frac{1}{c^2}\Big[\frac{3}{14}(9+10\eta)\frac{GM}{R}+\frac{1}{28}(13-68\eta)V^2\Big]\Big\}\epsilon_{pq<i}n_{j>}n_pV_q-\mu\frac{\delta M}{M}\frac{1}{c^2}\frac{5}{28}(1-2\eta)\dot{R}R^2\epsilon_{pq<i}V_{j>}n_pV_q\,, \tag{16}$$

$$\underset{\rm NS}{\mathcal{I}_{ijk}}=\mu R^3\frac{\delta M}{M}\Big\{-1+\frac{1}{c^2}\Big[\frac{1}{6}(5-13\eta)\frac{GM}{R}-\frac{1}{6}(5-19\eta)V^2\Big]\Big\}n_{<ijk>}+\mu\frac{\delta M}{M}\frac{1}{c^2}(1-2\eta)\dot{R}R^3 n_{<ij}V_{k>}$$
$$-\mu\frac{\delta M}{M}\frac{1}{c^2}(1-2\eta)R^3 n_{<i}V_{jk>}\,, \tag{17}$$

$$\underset{\rm NS}{\mathcal{J}_{ijk}}=\mu(1-3\eta)R^3\epsilon_{pq<i}n_{jk>}n_pV_q\,, \tag{18}$$

$$\underset{\rm NS}{\mathcal{I}_{ijkl}}=\mu(1-3\eta)R^4 n_{<ijkl>}\,, \tag{19}$$

where n_i and V_i denote component i of $\boldsymbol{n}$ and $\boldsymbol{V}$. $\mu=\eta M$ is the reduced mass of the binary system. $n_{<ij>}=n_{<i}n_{j>}$ denotes the STF of the tensors, as in [18].

The spin mass and mass-current multipole moments for the binary system using the spin vector defined by Bohé *et al.* can be written as [29]

$$\underset{S}{\mathcal{I}_{ij}}=\frac{R\eta}{c^3}\Big\{\frac{8}{3}(\boldsymbol{V}\times\boldsymbol{S})_{<i}n_{j>}+\frac{8}{3}\frac{\delta M}{M}(\boldsymbol{V}\times\boldsymbol{\Delta})_{<i}n_{j>}-\frac{4}{3}(\boldsymbol{n}\times\boldsymbol{S})_{<i}V_{j>}-\frac{4}{3}\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{\Delta})_{<i}V_{j>}\Big\}+\frac{R\eta}{c^5}\Big\{\Big[\frac{1}{21}(33+125\eta)\frac{GM}{R}$$
$$+\frac{26}{21}(1-3\eta)V^2\Big](\boldsymbol{V}\times\boldsymbol{S})_{<i}n_{j>}+\Big[\frac{1}{3}(1+16\eta)\frac{GM}{R}+\frac{1}{21}(26-116\eta)V^2\Big]\frac{\delta M}{M}(\boldsymbol{V}\times\boldsymbol{\Delta})_{<i}n_{j>}$$
$$+\Big[-\frac{1}{3}(22+10\eta)\frac{GM}{R}-\frac{4}{21}(1-3\eta)V^2\Big](\boldsymbol{n}\times\boldsymbol{S})_{<i}V_{j>}+\Big[-\frac{1}{21}(56+34\eta)\frac{GM}{R}$$
$$+\frac{1}{21}(-4+36\eta)V^2\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{\Delta})_{<i}V_{j>}-\frac{4}{21}(1-3\eta)\dot{R}(\boldsymbol{V}\times\boldsymbol{S})_{<i}V_{j>}-\frac{4}{21}(1-5\eta)\dot{R}\frac{\delta M}{M}(\boldsymbol{V}\times\boldsymbol{\Delta})_{<i}V_{j>}$$
$$+\Big[\frac{3}{7}(1-3\eta)(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\frac{1}{21}(9-40\eta)\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big]V_{<i}V_{j>}+\frac{GM}{R}\Big[\frac{38}{21}(1+12\eta)(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}$$
$$+\frac{2}{21}(24-13\eta)\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big]n_{<i}n_{j>}+\frac{GM}{R}\Big[\frac{1}{21}(17+61\eta)\dot{R}(\boldsymbol{n}\times\boldsymbol{S})_{<i}n_{j>}+\frac{1}{21}(21+34\eta)\dot{R}\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{\Delta})_{<i}n_{j>}\Big]$$
$$+\frac{GM}{R}\Big[-\frac{1}{3}(6-10\eta)(\boldsymbol{n}\cdot\boldsymbol{S})-\frac{2}{3}(3-2\eta)\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big](\boldsymbol{n}\times\boldsymbol{V})_{<i}n_{j>}\Big\}\,, \tag{20}$$

$$\underset{S}{\mathcal{J}_{ij}}=-\frac{3}{2}\frac{R\eta}{c}n_{<i}\Delta_{j>}+\frac{R\eta}{c^3}\Big\{\Big[-\frac{2}{7}\frac{\delta M}{M}V^2+\frac{10}{7}\frac{\delta M}{M}\frac{GM}{R}\Big]n_{<i}S_{j>}+\Big[-\frac{1}{28}(29-143\eta)V^2+\frac{1}{28}(61-71)\frac{GM}{R}\Big]n_{<i}\Delta_{j>}$$
$$+\frac{1}{28}\Big[33\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{S})+(33-155\eta)(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big]n_{<i}V_{j>}+\frac{1}{7}\dot{R}\Big[3\frac{\delta M}{M}V_{<i}S_{j>}+(3-16\eta)V_{<i}\Delta_{j>}\Big]-\frac{1}{14}\Big[(11-47\eta)(\boldsymbol{n}\cdot\boldsymbol{\Delta})$$
$$+11\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{S})\Big]V_{<i}V_{j>}-\frac{1}{14}\frac{GM}{R}\Big[29\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{S})+(8-31\eta)(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big]n_{<i}n_{j>}\Big\}, \tag{21}$$

$$\underset{S}{\mathcal{I}_{ijk}}=\frac{R^2\eta}{c^3}\Big\{-\frac{9}{2}\frac{\delta M}{M}(\boldsymbol{V}\times\boldsymbol{S})_{<i}n_jn_{k>}-\frac{3}{2}(3-11\eta)(\boldsymbol{V}\times\boldsymbol{\Delta})_{<i}n_jn_{k>}+3\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{S})_{<i}n_jV_{k>}+3(1-3\eta)(\boldsymbol{n}\times\boldsymbol{\Delta})_{<i}n_jV_{k>}\Big\}\,, \tag{22}$$

$$\underset{S}{\mathcal{J}_{ijk}}=\frac{R^2\eta}{c}\Big[2n_{<ij}S_{k>}+2\frac{\delta M}{M}n_{<ij}\Delta_{k>}\Big]\,. \tag{23}$$

The STF tensors used in our derivations are provided in the Appendix A.

IV. GRAVITATIONAL LOSSES INDUCED BY THE NEXT-TO-LEADING SPIN-ORBIT COUPLING EFFECTS

Substituting Eqs. (15)−(23) into Eqs. (12)−(14) and utilizing Eqs. (4)−(8) the time derivative of velocity was calculated. By keeping the results as contributions of the next-to-leading spin-orbit coupling effects, we can obtain the loss rates of the orbital energy, angular momentum, and linear momentum, as follows:

$$\frac{\mathrm{d}E}{\mathrm{d}t}=\dot{E}_{\mathrm{N}}+\dot{E}_{1\mathrm{PN}}+\dot{E}_{1.5\mathrm{SO}}+\dot{E}_{2\mathrm{PN}}+\dot{E}_{2.5\mathrm{SO}}\,, \tag{24}$$

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}=\dot{\boldsymbol{J}}_{\mathrm{N}}+\dot{\boldsymbol{J}}_{1\mathrm{PN}}+\dot{\boldsymbol{J}}_{1.5\mathrm{SO}}+\dot{\boldsymbol{J}}_{2\mathrm{PN}}+\dot{\boldsymbol{J}}_{2.5\mathrm{SO}}\,, \tag{25}$$

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}=\dot{\boldsymbol{P}}_{\mathrm{N}}+\dot{\boldsymbol{P}}_{0.5\mathrm{SO}}+\dot{\boldsymbol{P}}_{1\mathrm{PN}}+\dot{\boldsymbol{P}}_{1.5\mathrm{SO}}\,, \tag{26}$$

where

$$\dot{E}_{\mathrm{N}}=-\frac{8}{15}\frac{G^3M^2\mu^2}{c^5R^4}\left[12V^2-11\dot{R}^2\right], \tag{27}$$

$$\begin{aligned}\dot{E}_{1\mathrm{PN}}=&-\frac{2}{105}\frac{G^3M^2\mu^2}{c^7R^4}\Big[(785-825\eta)V^4-2(1487-1392\eta)V^2\dot{R}^2+3(687-620\eta)\dot{R}^4-160(17-\eta)\frac{GM}{R}V^2\\&+8(367-15\eta)\frac{GM}{R}\dot{R}^2+16(1-4\eta)\frac{(GM)^2}{R^2}\Big]\,,\end{aligned} \tag{28}$$

$$\dot{E}_{1.5\mathrm{SO}}=-\frac{8}{15}\frac{G^3M\mu^2}{c^8R^5}\left\{\left[78\dot{R}^2-8\frac{GM}{R}-80V^2\right](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\left[4\frac{GM}{R}-43V^2+51\dot{R}^2\right]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\right\}\,, \tag{29}$$

$$\begin{aligned}\dot{E}_{2\mathrm{PN}}=&-\frac{2}{2835}\frac{G^3M^2\mu^2}{c^9R^4}\Big\{\Big[18(1692-5497\eta+4430\eta^2)V^6-24(253-1026\eta+56\eta^2)\frac{(GM)^3}{R^3}-54(1719-10278\eta\\&+6292\eta^2)V^4\dot{R}^2+108(4987-8513\eta+2165\eta^2)\frac{GM}{R}V^2\dot{R}^2-3(106319+9798\eta+5376\eta^2)\frac{(GM)^2}{R^2}\dot{R}^2\\&+54(2018-15207\eta+7572\eta^2)V^2\dot{R}^4-12(33510-60971\eta+14290\eta^2)\frac{GM}{R}\dot{R}^4-36(4446-5237\eta+1393\eta^2)\frac{GM}{R}V^4\\&-18(2501-20234\eta+8404\eta^2)\dot{R}^6+(281473+81828\eta+4368\eta^2)\frac{(GM)^2}{R^2}V^2\Big]\Big\}\,,\end{aligned} \tag{30}$$

$$\begin{aligned}\dot{E}_{2.5\mathrm{SO}}=&-\frac{2}{105}\frac{G^3M\mu^2}{c^{10}R^5}\Big\{\Big[(3776+1560\eta)\frac{G^2M^2}{R^2}+(15220-896\eta)\frac{GM}{R}V^2-(12892-2024\eta)\frac{GM}{R}\dot{R}^2-(4828-7240\eta)V^4\\&+(14076-20016\eta)V^2\dot{R}^2-(8976-12576\eta)\dot{R}^4\Big](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\Big[(10774-1988\eta)\frac{GM}{R}V^2\\&-(14654-4796\eta)\frac{GM}{R}\dot{R}^2-(2603-4160\eta)V^4+(9456-14484\eta)V^2\dot{R}^2-(548-952\eta)\frac{G^2M^2}{R^2}\\&-(7941-10704\eta)\dot{R}^4\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big\}\,,\end{aligned} \tag{31}$$

$$\dot{\boldsymbol{J}}_{\mathrm{N}}=-\frac{8}{5}\frac{G^2M\mu^2}{c^5R^2}(\boldsymbol{n}\times\boldsymbol{V})\left\{2V^2-3\dot{R}^2+2\frac{GM}{R}\right\}\,, \tag{32}$$

$$\begin{aligned}\dot{\boldsymbol{J}}_{1\mathrm{PN}}=&-\frac{2}{105}\frac{G^2M\mu^2}{c^7R^2}(\boldsymbol{n}\times\boldsymbol{V})\Big[(307-548\eta)V^4-6(74-277\eta)V^2\dot{R}^2+15(19-72\eta)\dot{R}^4-4(58+95\eta)\frac{GM}{R}V^2\\&+2(372+197\eta)\frac{GM}{R}\dot{R}^2-2(745-2\eta)\frac{(GM)^2}{R^2}\Big]\,,\end{aligned} \tag{33}$$

$$
\begin{aligned}
\dot{\boldsymbol{J}}_{1.5\mathrm{SO}} =\; & \frac{4}{15}\frac{G^2\mu^2}{c^8R^3}(\boldsymbol{n}\times\boldsymbol{V})\Big\{\Big[163\frac{GM}{R}+111V^2-195\dot{R}^2\Big](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\Big[71\frac{GM}{R}+57V^2-105\dot{R}^2\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big\} \\
& +\frac{4}{15}\frac{G^2\mu^2}{c^8R^3}\boldsymbol{V}\Big\{\Big[12(\boldsymbol{n}\cdot\boldsymbol{S})+5\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big]\frac{GM}{M}\dot{R}+\Big[50\frac{GM}{R}+71V^2-108\dot{R}^2\Big](\boldsymbol{V}\cdot\boldsymbol{S}) \\
& +\Big[27\frac{GM}{R}+35V^2-54\dot{R}^2\Big]\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big\}+\frac{4}{15}\frac{G^2\mu^2}{c^8R^3}\boldsymbol{n}\Big\{\Big[41\frac{GM}{R}V^2-4\frac{(GM)^2}{R^2}-45\frac{GM}{R}\dot{R}^2\Big](\boldsymbol{n}\cdot\boldsymbol{S})+\Big[165\dot{R}^2-132V^2 \\
& -54\frac{GM}{R}\Big]\dot{R}(\boldsymbol{V}\cdot\boldsymbol{S})+\Big[2\frac{(GM)^2}{R^2}+24\frac{GM}{R}V^2-27\frac{GM}{R}\dot{R}^2\Big]\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{\Delta})-\Big[25\frac{GM}{R}+60V^2-75\dot{R}^2\Big]\frac{\delta M}{M}\dot{R}(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big\} \\
& +\frac{4}{15}\frac{G^2\mu^2}{c^8R^3}\Big\{\boldsymbol{S}\Big[4\frac{G^2M^2}{R^2}-91\frac{GM}{R}V^2-71V^4+87\frac{GM}{R}\dot{R}^2+240V^2\dot{R}^2-165\dot{R}^4\Big] \\
& +\boldsymbol{\Delta}\Big[2\frac{G^2M^2}{R^2}+49\frac{GM}{R}V^2+35V^4-45\frac{GM}{R}\dot{R}^2-114V^2\dot{R}^2+75\dot{R}^4\Big]\Big\},
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\dot{\boldsymbol{J}}_{2\mathrm{PN}} =\; & -\frac{1}{2835}\frac{G^2M\mu^2}{c^9R^2}(\boldsymbol{n}\times\boldsymbol{V})\Big[(340724+140922\eta+2772\eta^2)\frac{(GM)^3}{R^3}-(49140-205380\eta+122220\eta^2)\dot{R}^6 \\
& +(23985-111195\eta+116046\eta^2)V^6+(151848-451836\eta+82566\eta^2)\frac{(GM)^2}{R^2}\dot{R}^2-(200808-372582\eta \\
& +87255\eta^2)\frac{GM}{R}\dot{R}^4+(96525-453735\eta+423360\eta^2)V^2\dot{R}^4-(191718-183222\eta+61704\eta^2)\frac{(GM)^2}{R^2}V^2 \\
& +(196677-194427\eta+22959\eta^2)\frac{GM}{R}V^2\dot{R}^2-(60642-341631\eta+422199\eta^2)V^4\dot{R}^2 \\
& +(1485-4419\eta+36198\eta^2)\frac{GM}{R}V^4\Big],
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
\dot{\boldsymbol{J}}_{2.5\mathrm{SO}} =\; & -\frac{2}{315}\frac{G^2\mu^2}{c^{10}R^3}(\boldsymbol{n}\times\boldsymbol{V})\Big\{(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}\Big[(60751+4021\eta)\frac{(GM)^2}{R^2}-(3996-12501\eta)V^4+(11340-10710\eta)\dot{R}^4 \\
& +(5276+18000\eta)\frac{GM}{R}V^2-(8640+4995\eta)V^2\dot{R}^2-(23799+19353\eta)\frac{GM}{R}\dot{R}^2\Big]+\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta} \\
& \times\Big[(30042+1376\eta)\frac{G^2M^2}{R^2}+(6030+6041\eta)\frac{GM}{R}V^2-(17850+4164\eta)\frac{GM}{R}\dot{R}^2-(1638-7335\eta)V^4 \\
& -(8820+11160\eta)V^2\dot{R}^2+(11970+2205\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\boldsymbol{n}\Big\{(\boldsymbol{n}\cdot\boldsymbol{S})\Big[(6360+744\eta)\frac{(GM)^3}{R^3} \\
& -(4013+1388\eta)\frac{GM}{R}V^2\dot{R}^2-(35306+1742\eta)\frac{(GM)^2}{R^2}V^2+(61380+4956\eta)\frac{(GM)^2}{R^2}\dot{R}^2-(3024-5040\eta)V^6 \\
& +(12429+5166\eta)\frac{GM}{R}\dot{R}^4+(5645+1254\eta)\frac{GM}{R}V^4+(26460-44100\eta)V^4\dot{R}^2-(26460-44100\eta)V^2\dot{R}^4\Big] \\
& +\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{\Delta})\Big[(30942+5082\eta)\frac{(GM)^2}{R^2}\dot{R}^2-(17094+2194\eta)\frac{(GM)^2}{R^2}V^2-(720-420\eta)\frac{(GM)^3}{R^3} \\
& +(5058-247\eta)\frac{GM}{R}V^4-(18615+1338\eta)\frac{GM}{R}V^2\dot{R}^2+(7263-9\eta)\frac{GM}{R}\dot{R}^4-(3024-2016\eta)V^6 \\
& +(26460-17640\eta)V^4\dot{R}^2-(26460-17640\eta)V^2\dot{R}^4\Big]+(\boldsymbol{V}\cdot\boldsymbol{S})\dot{R}\Big[(51034+5742\eta)\frac{GM}{R}V^2 \\
& -(84174+8610\eta)\frac{GM}{R}\dot{R}^2-(17190-8460\eta)V^2\dot{R}^2-(3681-18324\eta)V^4+(37556+6938\eta)\frac{(GM)^2}{R^2} \\
& +(22365-34020\eta)\dot{R}^4\Big]+(\boldsymbol{V}\cdot\boldsymbol{\Delta})\dot{R}\Big[(14634+3748\eta)\frac{(GM)^2}{R^2}+(29595+5218\eta)\frac{GM}{R}V^2
\end{aligned}
$$

$$
\begin{aligned}
&-(41397+7176\eta)\frac{GM}{R}\dot{R}^2-(5913-7011\eta)V^4-(5850-6210\eta)V^2\dot{R}^2+(16065-16065\eta)\dot{R}^4\Big]\Big\}\\
&+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\boldsymbol{V}\Big\{(\boldsymbol{n}\cdot\boldsymbol{S})\dot{R}\Big[(3118+10176\eta)\frac{GM}{R}V^2-(27946+1222\eta)\frac{(GM)^2}{R^2}+(1902-8124\eta)\frac{GM}{R}\dot{R}^2\\
&+(3024-5040\eta)V^4-(26460-44100\eta)V^2\dot{R}^2+(26460-44100\eta)\dot{R}^4\Big]-(\boldsymbol{V}\cdot\boldsymbol{S})\Big[(20890+10614\eta)\frac{GM}{R}V^2\\
&+(33212+4178\eta)\frac{(GM)^2}{R^2}-(47262+10002\eta)\frac{GM}{R}\dot{R}^2-(2247-10974\eta)V^4-(22752-20448\eta)V^2\dot{R}^2\\
&+(25965-38970\eta)\dot{R}^4\Big]+\frac{\delta M}{M}\dot{R}(\boldsymbol{n}\cdot\boldsymbol{\Delta})\times\Big[(5313-2496\eta)\frac{GM}{R}\dot{R}^2-(26460-17640\eta)V^2\dot{R}^2+(885+3847\eta)\frac{GM}{R}V^2\\
&+(26460-17640\eta)\dot{R}^4-(13896+1136\eta)\frac{(GM)^2}{R^2}+(3024-2016\eta)V^4\Big]-\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big[(10995+6622\eta)\frac{GM}{R}V^2\\
&+(14970+1552\eta)\frac{(GM)^2}{R^2}-(20661+6012\eta)\frac{GM}{R}\dot{R}^2-(663-6147\eta)V^4-(19908-4122\eta)V^2\dot{R}^2\\
&+(24345-13545\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\Big\{\boldsymbol{S}\Big[(14731+14916\eta)\frac{GM}{R}V^4\\
&-(75236+6410\eta)\frac{(GM)^2}{R^2}\dot{R}^2+(72764+1658\eta)\frac{(GM)^2}{R^2}V^2-(80308+28326\eta)\frac{GM}{R}V^2\dot{R}^2-(6360+744\eta)\frac{(GM)^3}{R^3}\\
&+(73329+16746\eta)\frac{GM}{R}\dot{R}^4-(2247-10974\eta)V^6-(19071-2124\eta)V^4\dot{R}^2+(43155-47430\eta)V^2\dot{R}^4-(22365-34020\eta)\dot{R}^6\Big]\\
&+\frac{\delta M}{M}\boldsymbol{\Delta}\Big[(720-420\eta)\frac{(GM)^3}{R^3}+(36228+2050\eta)\frac{(GM)^2}{R^2}V^2-(35844+5998\eta)\frac{(GM)^2}{R^2}\dot{R}^2+(34089+11577\eta)\frac{GM}{R}\dot{R}^4\\
&+(6915+8983\eta)\frac{GM}{R}V^4-(38772+17776\eta)\frac{GM}{R}V^2\dot{R}^2-(663-6147\eta)V^6-(13995+2889\eta)V^4\dot{R}^2\\
&+(30195-19755\eta)V^2\dot{R}^4-(16065-16065\eta)\dot{R}^6\Big]\Big\},
\end{aligned}
\tag{36}
$$

$$
\dot{\boldsymbol{P}}_{\rm N}=-\frac{8}{105}\frac{G^3M^2\mu^2}{c^7R^4}(1-4\eta)^{\frac{1}{2}}\Big\{\boldsymbol{V}\Big[38\dot{R}^2-50V^2-8\frac{GM}{R}\Big]+\dot{R}\boldsymbol{n}\Big[55V^2+12\frac{GM}{R}-45\dot{R}^2\Big]\Big\}\,,
\tag{37}
$$

$$
\dot{\boldsymbol{P}}_{0.5\rm SO}=-\frac{8}{15}\frac{G^3M\mu^2}{c^8R^5}\Big\{4\dot{R}(\boldsymbol{V}\times\boldsymbol{\Delta})-2V^2(\boldsymbol{n}\times\boldsymbol{\Delta})-(\boldsymbol{n}\times\boldsymbol{V})\Big[3\dot{R}(\boldsymbol{n}\cdot\boldsymbol{\Delta})+2(\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big]\Big\}\,,
\tag{38}
$$

$$
\begin{aligned}
\dot{\boldsymbol{P}}_{1\rm PN}=&-\frac{1}{945}\frac{G^3M^2\mu^2}{c^9R^4}(1-4\eta)^{\frac{1}{2}}\times\Big\{\boldsymbol{V}\Big[32(189+17\eta)\frac{(GM)^2}{R^2}-12(2663-1394\eta)\dot{R}^4+36(907-162\eta)\frac{GM}{R}V^2\\
&+120(392-257\eta)V^2\dot{R}^2-444(25-28\eta)V^4-12(2699+10\eta)\frac{GM}{R}\dot{R}^2\Big]+\dot{R}\boldsymbol{n}\Big[4(12301-1168\eta)\frac{GM}{R}\dot{R}^2\\
&+24(851-779\eta)V^4-24(2834-1877\eta)V^2\dot{R}^2-12(590-4\eta)\frac{(GM)^2}{R^2}+24(1843-1036\eta)\dot{R}^4\\
&-12(4385-956\eta)\frac{GM}{R}V^2\Big]\Big\}\,,
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\dot{\boldsymbol{P}}_{1.5\rm SO}=&\frac{4}{945}\frac{G^3M\mu^2}{c^{10}R^5}(\boldsymbol{n}\times\boldsymbol{V})\Big\{\Big[21852\dot{R}^2-19536V^2+2166\frac{GM}{R}\Big]\dot{R}\frac{\delta M}{M}(\boldsymbol{n}\cdot\boldsymbol{S})+\Big[-272\frac{GM}{R}-4857\dot{R}^2+3314V^2\Big]\\
&\times\dot{R}\frac{\delta M}{M}(\boldsymbol{V}\cdot\boldsymbol{S})+\Big[-(4902+2919\eta)\frac{GM}{R}+(6264-37062\eta)\dot{R}^2-(5172-35448\eta)V^2\Big]\dot{R}(\boldsymbol{n}\cdot\boldsymbol{\Delta})+\Big[-(572-647\eta)\frac{GM}{R}\\
&-(4281-8556\eta)\dot{R}^2+(2024-5621\eta)V^2\Big](\boldsymbol{V}\cdot\boldsymbol{\Delta})\Big\}+\frac{4}{945}\frac{G^3M\mu^2}{c^{10}R^5}\boldsymbol{n}\dot{R}\Big\{\Big[3180\frac{GM}{R}-17964\dot{R}^2+17592V^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}\Big\}+\Big[(1509-5451\eta)\frac{GM}{R}+(9492-35934\eta)V^2-(9396-35514\eta)\dot{R}^2\Big](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\\
&+\frac{4}{945}\frac{G^3M\mu^2}{c^{10}R^5}\boldsymbol{V}\Big\{-\Big[7985V^2+2176\frac{GM}{R}-8043\dot{R}^2\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{S}+\Big[(3822-14361\eta)\dot{R}^2-(3968-15017\eta)V^2\\
&-(697-2503\eta)\frac{GM}{R}\Big](\boldsymbol{n}\times\boldsymbol{V})\cdot\boldsymbol{\Delta}\Big\}+\frac{4}{945}\frac{G^3M\mu^2}{c^{10}R^5}\Big\{-\frac{\delta M}{M}\dot{R}\Big[608\frac{GM}{R}-5709\dot{R}^2+5431V^2\Big](\boldsymbol{V}\times\boldsymbol{S})\\
&-\Big[(3585+4506\eta)\frac{GM}{R}\dot{R}^2+(10287-9810\eta)\dot{R}^4+(1417-3484\eta)\frac{GM}{R}V^2+(10287-9810\eta)V^2\dot{R}^2\\
&+(1322-968\eta)V^4\Big](\boldsymbol{n}\times\boldsymbol{\Delta})-\dot{R}\Big[(5677-15868\eta)V^2-(8355-18048\eta)\dot{R}^2-(6478-184\eta)\frac{GM}{R}\Big](\boldsymbol{V}\times\boldsymbol{\Delta})\\
&+\Big[1812\frac{GM}{R}\dot{R}^2-6300\dot{R}^4-664\frac{GM}{R}V^2+7491V^2\dot{R}^2-1469V^4+72\frac{(GM)^2}{R^2}\Big]\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{S})\Big\}\,. \qquad (40)
\end{aligned}
$$

In the above results, the loss rate of the system's energy is given by Eqs. (27)−(30) obtained from [13]. The Newtonian and 1PN contributions to the loss rate of the system's angular momentum, are expressed by Eqs. (32) and (33) and the Newtonian contribution to the linear momentum loss rate in Eq. (37) were obtained from [18]. The 2PN contributions to the loss rate of the system's angular momentum and 1PN contributions the linear momentum loss rate in Eqs. (35) and (39), were obtained from [20, 21]. The 1.5PN SO contributions to the loss rates of the energy and angular momentum in Eqs. (29) and (34), as well as the 0.5PN SO contribution to the linear momentum loss rate in Eq. (38), were obtained from [25]. Here, we include them for completeness.

The core results of this work are the loss rates of the system's energy and angular momentum induced by the 2.5PN SO coupling effects, expressed by Eqs. (31) and (36), and the loss rate of the linear momentum induced by the 1.5PN SO coupling effect, as in Eq. (40). Note that all these losses can be regarded as the next-to-leading spin-orbit coupling effects, since the leading spin-orbit coupling effects to the system's energy and angular momentum are the 1.5PN SO coupling contribution, whereas the leading spin-orbit coupling effects to the system's linear momentum is the 0.5PN SO coupling contribution.

V. LOSS RATES OF GRAVITATIONAL RADIATION IN THE CASE OF CIRCULAR ORBIT

When the binary systems lose their orbital energy, angular momentum, and linear momentum due to gravitational-wave radiation, their orbit shrinks and their eccentricity decreases. In the final stage of binary inspiral, their orbit can be approximated as circular [25, 34]. In this case, we have $\dot{R}=0$. Following [16], we introduced the PN parameter

$$x=\Big(\frac{GM\omega}{c^3}\Big)^{2/3}, \qquad (41)$$

with ω being the orbital frequency

$$
\begin{aligned}
\omega^2=\frac{GM}{R^3}\Big\{&1-\frac{GM}{c^2R}(3-\eta)+\Big(\frac{GM}{c^2R}\Big)^2\Big(6+\frac{41}{4}\eta+\eta^2\Big)-\Big(\frac{GM}{c^2R}\Big)^{3/2}\Big(5\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}+3\frac{\delta M}{M}\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big)+\Big(\frac{GM}{c^2R}\Big)^{5/2}\Big[\Big(\frac{45}{2}-\frac{27}{2}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}\\
&+\Big(\frac{27}{2}-\frac{13}{2}\eta\Big)\frac{\delta M}{M}\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big]\Big\}\,, \qquad (42)
\end{aligned}
$$

where $\boldsymbol{l}\equiv\boldsymbol{n}\times\boldsymbol{\lambda}$ with $\boldsymbol{\lambda}\equiv\boldsymbol{V}/V$ denoting the unit vector for the angular momentum. Then, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t}=&-\frac{32}{5}\frac{c^5}{G}x^5\eta^2\Big\{1-x\Big(\frac{1247}{336}+\frac{35}{12}\eta\Big)-x^2\Big(\frac{44711}{9072}-\frac{9271}{504}\eta-\frac{65}{18}\eta^2\Big)-x^{3/2}\frac{1}{G}\Big(4\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}+\frac{5}{4}\frac{\delta M}{M}\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big)\\
&-x^{5/2}\frac{1}{G}\Big[\Big(\frac{9}{2}-\frac{272}{9}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}+\frac{\delta M}{M}\Big(\frac{13}{16}-\frac{43}{4}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big]\Big\}\,, \qquad (43)
\end{aligned}
$$

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}=-\frac{32}{5}Mc^2x^{\frac{7}{2}}\eta^2\Big\{1-x\Big(\frac{1247}{336}+\frac{35\eta}{12}\Big)-x^2\Big(\frac{44711}{9072}-\frac{9271}{504}\eta-\frac{65}{18}\eta^2\Big)-x^{\frac{3}{2}}\frac{1}{G}\Big(4\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}+\frac{\delta M}{M}\frac{5}{4}\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big)$$

$$
-x^{\frac{5}{2}}\frac{1}{G}\Big[\Big(\frac{9}{2}-\frac{272}{9}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{M^2}+\frac{\delta M}{M}\Big(\frac{13}{16}-\frac{43}{4}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{M^2}\Big]\Big\}\boldsymbol{l}-\frac{32}{5}Mc^2x^{\frac{7}{2}}\eta^2\Big\{x^{3/2}\Big(\frac{121}{24}\frac{\boldsymbol{n}\cdot\boldsymbol{S}}{GM^2}+\frac{5}{2}\frac{\delta M}{M}\frac{n\Delta}{GM^2}\Big)
$$

$$
-x^{5/2}\Big[\Big(\frac{2387}{96}+\frac{2057}{126}\eta\Big)\frac{\boldsymbol{n}\cdot\boldsymbol{S}}{GM^2}+\Big(\frac{545}{42}+\frac{5725}{672}\eta\Big)\frac{\delta M}{M}\frac{n\Delta}{GM^2}\Big]\Big\}\boldsymbol{n}-\frac{32}{5}Mc^2x^{\frac{7}{2}}\eta^2\Big\{x^{3/2}\Big(\frac{37}{24}\frac{\boldsymbol{\lambda}\cdot\boldsymbol{S}}{GM^2}+\frac{\delta M}{M}\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}}{GM^2}\Big)
$$

$$
-x^{\frac{5}{2}}\Big[\Big(\frac{2425}{224}+\frac{1387}{1008}\eta\Big)\frac{\boldsymbol{\lambda}\cdot\boldsymbol{S}}{GM^2}+\Big(\frac{2227}{336}+\frac{439}{224}\eta\Big)\frac{\delta M}{M}\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}}{GM^2}\Big]\Big\}\boldsymbol{\lambda}\,, \tag{44}
$$

$$
\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}=\frac{464}{105}\frac{c^4}{G}x^{11/2}\eta^2(1-4\eta)^{1/2}\Big\{1-\Big(\frac{452}{87}+\frac{1139}{522}\eta\Big)x-\frac{7}{29}\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{GM^2}x^{1/2}-\Big[\frac{470}{87}\frac{\delta M}{M}\frac{\boldsymbol{l}\cdot\boldsymbol{S}}{GM^2}+\Big(\frac{67}{58}-\frac{206}{29}\eta\Big)\frac{\boldsymbol{l}\cdot\boldsymbol{\Delta}}{GM^2}\Big]x^{3/2}\Big\}\boldsymbol{\lambda}
$$

$$
+\frac{464}{105}\frac{c^4}{G}x^6\eta^2(1-4\eta)^{1/2}\Big\{\frac{14}{29}\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}}{GM^2}+\Big[\frac{109}{116}\frac{\delta M}{M}\frac{\boldsymbol{\lambda}\cdot\boldsymbol{S}}{GM^2}+\Big(\frac{25}{116}-\frac{57}{58}\eta\Big)\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\Delta}}{GM^2}\Big]x\Big\}\boldsymbol{l}. \tag{45}
$$

VI. SUMMARY

Based on the spin vector defined by Bohé *et al.* and under Tulczyjew's SSC, we calculated the loss rates for the binary system's energy, angular momentum, and linear momentum induced by the next-to-leading spin-orbit coupling effects in the case of the general orbit. For comparison, we also adopted the spin vector defined by Faye, Blanchet, and Buonanno to calculate the loss rate of the angular momentum to the same PN order. For the case of the circular orbit, these gravitational losses were formulated in terms of the orbital frequency. The results are useful in determining the time change of the orbital parameters for the general motion when the spin vector defined by Bohé *et al.* is adopted.

APPENDIX A: THE STF TENSORS USED IN THE DERIVATIONS

For the readers' convenience, we provide the STF tensors used in the derivations of the compact binary systems' gravitational losses.

$$
A_{<i}B_{j>}=\frac{1}{2}(A_iB_j+B_iA_j)-\frac{1}{3}\delta_{ij}A_aB^a\,, \tag{A1}
$$

$$
A_{<i}B_jC_{k>}=\frac{1}{6}(A_iB_jC_k+A_iB_kC_j+A_jB_iC_k+A_jB_kC_i+A_kB_iC_j+A_kB_jC_i)-\frac{1}{15}[(\delta_{ij}C_k+\delta_{jk}C_i+\delta_{ik}C_j)A_aB^a
+(\delta_{ij}B_k+\delta_{jk}B_i+\delta_{ik}B_j)A_aC^a+(\delta_{ij}A_k+\delta_{jk}A_i+\delta_{ik}A_j)B_aC^a]\,, \tag{A2}
$$

$$
A_{<ijkl>}=A_iA_jA_kA_l-\frac{1}{7}A_aA^a(\delta_{ij}A_kA_l+\delta_{ik}A_jA_l+\delta_{il}A_jA_k+\delta_{jk}A_iA_l+\delta_{jl}A_iA_k+\delta_{kl}A_iA_j)
+\frac{1}{35}A_aA^aA_bA^b(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\,, \tag{A3}
$$

$$
\epsilon_{ab<i}A_{j>}B_aC_b=\frac{1}{2}(\epsilon_{abi}A_j+\epsilon_{abj}A_i)B_aC_b-\frac{1}{3}\delta_{ij}\epsilon_{abk}A_kB_aC_b\,, \tag{A4}
$$

$$
\epsilon_{ab<i}A_jB_{k>}C_aD_b=\Big\{\frac{1}{6}(\epsilon_{abi}A_jB_k+\epsilon_{abi}A_kB_j+\epsilon_{abj}A_iB_k+\epsilon_{abj}A_kB_i+\epsilon_{abk}A_iB_j+\epsilon_{abk}A_jB_i)-\frac{1}{15}[(\delta_{ij}B_k+\delta_{ik}B_j+\delta_{jk}B_i)\epsilon_{abl}A_l
+(\delta_{ij}A_k+\delta_{ik}A_j+\delta_{jk}A_i)\epsilon_{abl}B_l+(\delta_{ij}\epsilon_{abk}+\delta_{ik}\epsilon_{abj}+\delta_{jk}\epsilon_{abi})A_lB^l]\Big\}C_aD_b\,. \tag{A5}
$$

APPENDIX B: THE ANGULAR MOMENTUM LOSS INDUCED BY THE 2.5PN SO EFFECTS IN TERMS OF THE SPIN VECTOR DEFINED BY FAYE, BLANCHET AND BUONANNO

A new definition for the spin vector was provided by Faye, Blanchet and Buonanno who calculated the energy loss of the spinning binary systems to the 2.5PN order [26]. Later, Racine, Buonanno & Kidder calculated the linear momentum loss to the 1.5PN order [21] using the same definition. For the angular momentum loss, we checked whether the result to the 1.5PN order is the same as that for the spin vector defined by Barker and O'Connell [23], which is given by Kidder [25]. Note that among the different SSCs, the results are the same to the 1.5PN order.

For completeness and to enable comparisons with the angular momentum loss formulated in terms of the spin vector defined by Bohé *et al.* [28], here we derive the angular momentum loss in terms of the spin vector defined by Faye, Blanchet and Buonanno to the 2.5PN order.

The spins of the binary system were assumed to be $\tilde{\boldsymbol{S}}_1$ and $\tilde{\boldsymbol{S}}_2$. Let $\tilde{\boldsymbol{S}}=\tilde{\boldsymbol{S}}_1+\tilde{\boldsymbol{S}}_2$ and $\tilde{\boldsymbol{\Delta}}=M\left(\frac{\tilde{\boldsymbol{S}}_2}{M_2}-\frac{\tilde{\boldsymbol{S}}_1}{M_1}\right)$, to obtain the angular momentum loss induced by the 2.5PN SO effect for the general orbit, which can be written as

$$\begin{aligned}
\dot{\boldsymbol{J}}_{2.5\mathrm{SO}} = {} & \frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}(\boldsymbol{n}\times\boldsymbol{V})\Big\{(\boldsymbol{n}\times\boldsymbol{V})\cdot\tilde{\boldsymbol{S}}\Big[(1746-10800\eta)V^4-(106022+24242\eta)\frac{(GM)^2}{R^2}-(87570-163170\eta)\dot{R}^4 \\
& -(20458-7326\eta)\frac{GM}{R}V^2+(76320-121770\eta)V^2\dot{R}^2+(60612-33348\eta)\frac{GM}{R}\dot{R}^2\Big]+\frac{\delta M}{M}(\boldsymbol{n}\times\boldsymbol{V}) \\
& \cdot\tilde{\boldsymbol{\Delta}}\Big[(40326-39300\eta)\frac{GM}{R}\dot{R}^2-(16194-15062\eta)\frac{GM}{R}V^2-(48876+13480\eta)\frac{G^2M^2}{R^2} \\
& +(1674-6822\eta)V^4+(30240-49320\eta)V^2\dot{R}^2-(34650-71190\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\boldsymbol{n}\Big\{(\boldsymbol{n}\cdot\tilde{\boldsymbol{S}}) \\
& \times\Big[(5064+3792\eta)\frac{(GM)^3}{R^3}-(65118-101184\eta)\frac{GM}{R}V^2\dot{R}^2-(36782-12850\eta)\frac{(GM)^2}{R^2}V^2 \\
& +(62988-5160\eta)\frac{(GM)^2}{R^2}\dot{R}^2-(4032-8064\eta)V^6+(48123-101916\eta)\frac{GM}{R}\dot{R}^4+(13769-16314\eta) \\
& \times\frac{GM}{R}V^4+(35280-70560\eta)V^4\dot{R}^2-(35280-70560\eta)V^2\dot{R}^4\Big]+\frac{\delta M}{M}(\boldsymbol{n}\cdot\tilde{\boldsymbol{\Delta}})\Big[(25290+5346\eta)\frac{(GM)^2}{R^2}\dot{R}^2 \\
& -(15270-3170\eta)\frac{(GM)^2}{R^2}V^2-(2280-2940\eta)\frac{(GM)^3}{R^3}+(8034-7567\eta)\frac{GM}{R}V^4-(31143-44670\eta) \\
& \times\frac{GM}{R}V^2\dot{R}^2+(18009-46449\eta)\frac{GM}{R}\dot{R}^4-(4032-4032\eta)V^6+(35280-35280\eta)V^4\dot{R}^2-(35280-35280\eta)V^2\dot{R}^4\Big] \\
& +(\boldsymbol{V}\cdot\tilde{\boldsymbol{S}})\dot{R}\Big[(46570-4842\eta)\frac{GM}{R}V^2-(78072-8868\eta)\frac{GM}{R}\dot{R}^2+(57510-219420\eta)V^2\dot{R}^2 \\
& -(26289-89172\eta)V^4+(36434-1780\eta)\frac{(GM)^2}{R^2}-(31185-126630\eta)\dot{R}^4\Big]+\frac{\delta M}{M}(\boldsymbol{V}\cdot\tilde{\boldsymbol{\Delta}})\dot{R}\Big[(12192+760\eta) \\
& \times\frac{(GM)^2}{R^2}+(24123+610\eta)\frac{GM}{R}V^2-(30939+1560\eta)\frac{GM}{R}\dot{R}^2-(14265-40995\eta)V^4+(15390-93510\eta)V^2\dot{R}^2 \\
& +(4095+49455\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\boldsymbol{V}\Big\{(\boldsymbol{n}\cdot\tilde{\boldsymbol{S}})\dot{R}\Big[(25576-64002\eta)\frac{GM}{R}V^2-(29974-1166\eta)\frac{(GM)^2}{R^2} \\
& -(23046-80328\eta)\frac{GM}{R}\dot{R}^2+(4032-8064\eta)V^4-(35280-70560\eta)V^2\dot{R}^2+(35280-70560\eta)\dot{R}^4\Big]-(\boldsymbol{V}\cdot\tilde{\boldsymbol{S}}) \\
& \times\Big[(18730+504\eta)\frac{GM}{R}V^2+(33386-7516\eta)\frac{(GM)^2}{R^2}-(47052-18642\eta)\frac{GM}{R}\dot{R}^2-(8637-30144\eta)V^4 \\
& +(15066-89982\eta)V^2\dot{R}^2-(6345-54180\eta)\dot{R}^4\Big]-\frac{\delta M}{M}\dot{R}(\boldsymbol{n}\cdot\tilde{\boldsymbol{\Delta}})\Big[(6567-41892\eta)\frac{GM}{R}\dot{R}^2 \\
& +(35280-35280\eta)V^2\dot{R}^2-(11859-33338\eta)\frac{GM}{R}V^2-(35280-35280)\dot{R}^4+(12492+1772\eta)\frac{(GM)^2}{R^2} \\
& -(4032-3032\eta)V^4\Big]+\frac{\delta M}{M}(\boldsymbol{V}\cdot\tilde{\boldsymbol{\Delta}})\Big[(18075-9882\eta)\frac{GM}{R}\dot{R}^2-(14088-3956\eta)\frac{(GM)^2}{R^2}-(8835+400\eta)\frac{GM}{R}V^2
\end{aligned}$$

$$+(3885-15759\eta)V^4+(4462+45918\eta)V^2\dot{R}^2-(13815+25515\eta)\dot{R}^4\Big]\Big\}+\frac{1}{315}\frac{G^2\mu^2}{c^{10}R^3}\Big\{\tilde{\boldsymbol{S}}\Big[(3901+31614\eta)\frac{GM}{R}V^4$$
$$-(73736-5218\eta)\frac{(GM)^2}{R^2}\dot{R}^2-(5064+3792\eta)\frac{(GM)^3}{R^3}+(54591+37932\eta)\frac{GM}{R}\dot{R}^4-(8637-30144\eta)V^6$$
$$-(54796+53706\eta)\frac{GM}{R}V^2\dot{R}^2+(74456-19882\eta)\frac{(GM)^2}{R^2}V^2+(41355-179154\eta)V^4\dot{R}^2-(63855-273600\eta)V^2\dot{R}^4$$
$$+(31385-126630\eta)\dot{R}^6\Big]+\frac{\delta M}{M}\tilde{\boldsymbol{\Delta}}\Big[(2280-2940\eta)\frac{(GM)^3}{R^3}+(33564-6890\eta)\frac{(GM)^2}{R^2}V^2$$
$$-(29196+4570\eta)\frac{(GM)^2}{R^2}\dot{R}^2+(22857+20109\eta)\frac{GM}{R}\dot{R}^4+(1233+15751\eta)\frac{GM}{R}V^4-(26724+23836\eta)$$
$$\times\frac{GM}{R}V^2\dot{R}^2-(3885-15759\eta)V^6+(9603-86913\eta)V^4\dot{R}^2-(1575-119025\eta)V^2\dot{R}^4-(4095+49455\eta)\dot{R}^6\Big]\Big\}\,. \tag{B1}$$

For the case of the circular orbit, we have

$$\dot{\boldsymbol{J}}_{2.5\mathrm{SO}}=\frac{32}{5}\frac{Mc^2\eta^2}{GM^2}\tilde{x}^6\Big\{\Big[\Big(\frac{95}{28}+\frac{239}{63}\eta\Big)\boldsymbol{l}\cdot\tilde{\boldsymbol{S}}+\frac{\delta M}{M}\Big(\frac{31}{16}-\frac{109}{28}\eta\Big)\boldsymbol{l}\cdot\tilde{\Delta}\Big]\boldsymbol{l}+\Big[\Big(\frac{4471}{224}+\frac{5911}{252}\eta\Big)\boldsymbol{n}\cdot\tilde{\boldsymbol{S}}$$
$$+\Big(\frac{383}{42}+\frac{1175}{96}\eta\Big)\frac{\delta M}{M}\boldsymbol{n}\cdot\tilde{\Delta}\Big]\boldsymbol{n}+\Big[\Big(\frac{5323}{672}+\frac{149}{18}\eta\Big)\boldsymbol{\lambda}\cdot\tilde{\boldsymbol{S}}+\Big(\frac{229}{48}+\frac{1221}{224}\eta\Big)\frac{\delta M}{M}\boldsymbol{\lambda}\cdot\tilde{\boldsymbol{\Delta}}\Big]\boldsymbol{\lambda}\Big\}\,. \tag{B2}$$

where the parameter $\tilde{x}=\left(\frac{GM\tilde{\omega}}{c^3}\right)^{2/3}$ was obtained from [30].

References

[1] B. P. Abbott *et al.*, Phys. Rev. Lett. **116**(6), 061102 (2016)
[2] G. M. Harry, Class. Quant. Grav. **27**, 084006 (2010)
[3] J. Aasi *et al.*, Class. Quant. Grav. **32**, 074001 (2015)
[4] F. Acernese *et al.*, Class. Quant. Grav. **32**(2), 024001 (2015)
[5] K. Danzmann, Class. Quant. Grav. **14**, 1399 (1997)
[6] P. Amaro-Seoane *et al.*, (2017), arXiv: 1702.00786
[7] J. Luo *et al.*, Class. Quant. Grav. **33**(3), 035010 (2016)
[8] W. R. Hu and Y. L. Wu, Natl. Sci. Rev. **4**(5), 685 (2017)
[9] Y. Gong, J. Luo, and B. Wang, Nature Astron. **5**(9), 881 (2021)
[10] P. C. Peters and J. Mathews, Phys. Rev. **131**, 435 (1963)
[11] R. V. Wagoner and C. M. Will, Astrophys. J. **210**, 764 (1976)
[12] L. Blanchet, T. Damour, and B. R. Iyer, Phys. Rev. D **51**, 5360 (1995)
[13] C. M. Will and A. G. Wiseman, Phys. Rev. D **54**, 4813 (1996)
[14] L. Blanchet, Phys. Rev. D **54**, 1417 (1996)
[15] K. G. Arun, L. Blanchet, B. R. Iyer *et al.*, Phys. Rev. D **77**, 064035 (2008)
[16] L. Blanchet, G. Faye, B. R. Iyer *et al.*, Phys. Rev. D **65**, 061501 (2002)
[17] L. Blanchet, Phys. Rev. D. **15**, 113 (1998)
[18] W. Junker and G. Schäfer, Mon. Not. Roy. Astron. Soc. **254**(1), 146 (1992)
[19] A. Gopakumar and B. R. Iyer, Phys. Rev. D **56**, 7708 (1997)
[20] K. G. Arun, L. Blanchet, B. R. Iyer *et al.*, Phys. Rev. D **80**, 124018 (2009)
[21] E. Racine, A. Buonanno, and L. E. Kidder, Phys. Rev. D **80**, 044010 (2009)
[22] S. Kastha, Phys. Rev. D **105**(6), 064039 (2022)
[23] B. M. Barker and R. F. O'Connell, Gen. Relativ. Gravit. **5**, 539 (1974)
[24] F. A. E. Pirani, Acta Phys. Polon. **15**, 389 (1956)
[25] L. E. Kidder, Phys. Rev. D **52**, 821 (1995)
[26] G. Faye, L. Blanchet, and A. Buonanno, Phys. Rev. D **74**, 104033 (2006)
[27] W. Tulczyjew, Acta Phys. Pol. **18**, 393 (1959)
[28] A. Bohe, S. Marsat, G. Faye *et al.*, Class. Quant. Grav. **30**, 075017 (2013)
[29] S. Marsat, A. Bohé, L. Blanchet *et al.*, Class. Quant. Grav. **31**, 025023 (2014)
[30] L. Blanchet, A. Buonanno, and G. Faye, Phys. Rev. D **74**, 104034 (2006)
[31] A. Bohé, G. Faye, S. Marsat *et al.*, Class. Quant. Grav. **32**(19), 195010 (2015)
[32] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980)
[33] D. Bini, T. Damour, and A. Geralico, Phys. Rev. D **102**(2), 024061 (2020)
[34] C. W. Lincoln and C. M. Will, Phys. Rev. D **42**, 1123 (1990)