Black hole tunneling in loop quantum gravity*

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Abstract: In this paper, we investigate the Hawking radiation of the quantum Oppenheimer-Snyde black hole with the tunneling scheme by Parikh and Wilczek. We calculate the emission rate of massless scalar particles. Compared to the traditional results within the framework of General Relativity, our findings include quantum correction terms arising from loop quantum gravity effects. Following the approach in [1, 2], we establish the entropy of the black hole. This entropy includes a logarithmic correction, which arises from quantum gravity effects. Our result is consistent with the well-known result in the context of quantum gravity.

Keywords: black hole entropy, Hawking radiation, quantum tunneling, emission rate

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I. INTRODUCTION

As an important prediction of General Relativity (GR), black hole (BH) have drawn considerable attentions and undergone widely study (see e.g., [3−7]). Both the gravitational wave detections [8−12] and the observations of supermassive BHs by the Event Horizon Telescope [13−18] provide the strong evidences of the existence of BH. In the future, more gravitational wave and electromagnetic wave detection will provide stronger tests of the BH paradigm [19−24]. Despite the remarkable success in the classical paradigm of BH, there still some problems remain, e.g., the BH singularity and information paradox (see e.g., [25−31]), where quantum gravity potentially plays a significant role.

Loop quantum gravity (LQG), a background-independent and non-perturbative quantum gravity, is one candidate resolution of the problems mentioned previously [32−39]. By taking into account of the LQG effects, the BH singularity is replaced by a bounce, thus solving the singularity problem [40, 41]. Another advancement is in exploring the BH entropy and information paradox. LQG successfully reproduces BH entropy by counting the number of spacetime microstates [42, 43]. Recently, the concept of a black-to-white hole transition proposal is regraded as a promising candidate as the solution to the BH information paradox [44−52]. The addressing of these issues will be accompanied by effects beyond the standard BH paradigm. Therefore, LQG is expected to introduce phenomenological corrections to BHs within the framework of GR.

To obtain this modified model, studying the quantum corrections to the spherically symmetric self-gravitational collapse problem provides a promising starting point. The spherically symmetric gravitational collapse plays a crucial role in understanding dynamical formation of BH, from both classical and quantum perspectives. In classical theory, the first collapse model is proposed by Oppenheimer and Snyder [53], known as the Oppenheimer-Snyder (OS) model. This model assumes that the matter field in the interior region is a pressure-less dust field. Thus, the interior dynamics can be described by the standard Friedmann equations. The exterior region is characterized by Schwarzschild solution according to Birkhoff's theorem. Due to its simplicity, this model can be solved exactly, offering valuable insights into the nature of self-gravitational collapse.

In recent developments [54], LQG effects have been incorporated into the classical OS model, i.e., quantum OS model in literature. It can be viewed as an effective model of LQG-modified BH. It is formed as the intermediate product of the self-gravitational collapse process. Later, the collapsing phase transits to an expanding phase, resulting in a black-to-white hole transition. This BH has been extensively studied from various aspects,

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such as shadows and quasinormal modes [55–61]. In this framework, the interior region is treated using the Ashtekar-Pawlowski-Singh (APS) model, where the classical Friedmann equation is modified by quantum corrections arising from loop quantum cosmology (LQC) effects. Appropriate junction conditions are then introduced to derive the spacetime metric in the exterior region. Then the spacetime metric reads [1)]

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{\alpha M^2}{r^4}\right)dt^2+\left(1-\frac{2M}{r}+\frac{\alpha M^2}{r^4}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{1}$$

where M is the ADM energy of the spacetime, and $\alpha = 16\sqrt{3}\pi\gamma^3\ell_p^2$ with $\ell_p = \sqrt{\hbar}$ is the Planck length and γ is the Barbero-Immirzi parameter. Note that γ is dimensionally less, $[\alpha]=[M]^2=[r]^2$.[2)] Compared to the classical Schwarzschild solution, a quantum correction arises from LQG effects. We will discus this with more details later.

For simplicity, this paper focuses on the scenarios where the matter distribution in the interior spacetime region is assumed to be homogeneous and isotropic, in accordance with the cosmological principle. However, to better reflect realism, an anisotropic cosmological background for the interior spacetime region should also be considered. In particular, the Kantowski-Sachs spacetime is of critical importance in this context [62–65]. This model describes the anisotropic collapse of the matter field, where expansion occurs in one direction while contraction takes place in the other directions. We will investigate these topics in our future research, which may potentially provide new insights into the study of quantum gravity.

For a given BH, one can calculate its Hawking radiation and entropy. Historically, various methods have been developed for this purpose (see e.g., [66–74]). One of them is the tunneling approach, a semiclassical method introduced by Parikh and Wilczek (PW) in [68, 69], and further developed in [1, 75, 76]. In this approach, Hawking radiation is regarded as tunneling effects across the BH horizon, with considerations for back-reaction effects of emitted particles and energy conservation of the system. This leads to a spectrum for emitted particles that is not purely thermal but includes a modification term. Additionally, the BH entropy can be derived from this spectrum.

In this work, we apply the PW approach to the quantum OS BH, focusing on massless scalar outgoing particles. We compute the emission rate Γ of the outgoing particles. Comparing to the original result in classical GR [68], our results reveal additional quantum correction terms, which are interpreted as arising from quantum gravity effects. Furthermore, by following the arguments in [1, 2], we extract the BH entropy from the emission rate. Compared to the classical results, quantum correction terms also appear in the entropy formula for the quantum OS BH. By considering the quantum gravity effects, quantum corrections contribute to the BH entropy, where a logarithmic term appears (see e.g., [77–89])

$$\tilde{S} = S + a\log\left(\frac{\mathscr{A}}{l_p^2}\right) + O(1). \tag{2}$$

Here a is a constant. $\mathscr{A}$ is the area of the BH horizon. S is the classical BH entropy, given by $S=\frac{\mathscr{A}}{4l_p^2}$. As the main result of our work, in the case of the quantum OS BH, we extract the BH entropy from the emission spectrum, which also includes a logarithmic correction

$$\tilde{S} = S + \frac{\sqrt{2}\pi\alpha}{l_p^2}\log\left(\frac{\mathscr{A}_{\rm Sch}}{l_p^2}\right) + O(\alpha^2). \tag{3}$$

Here $\mathscr{A}_{\rm Sch}$ is the area of the Schwarzschild BH horizon. Our finding agrees with the traditional results (2). Besides, we obtain this result with a semiclassical approach, providing evidence for the validity of PW method.

This paper is organized as following: In Sec. 2, we provide a brief review of quantum OS model. In Sec. 3, we compute the emission rate of the massless scalar particles created near the BH horizon and analyze the BH entropy. In Sec. 4, we provide conclusions and outlooks of this work.

II. A BRIEF REVIEW OF QUANTUM OPPENHEIMER-SNYDER MODEL

Recently, the quantum OS model is investigated in [54–60]. The classical OS model is the first model which describes the collapse a pressure-less dust field and the formation of a BH in a spherically symmetric background [4, 53]. In this model, space is divided into two regions: the interior region $\mathcal{M}^-$, which is filled with the matter field, and the exterior $\mathcal{M}^+$, which is vacuum. The geometry of the interior region is described by the usual FRW metric

$$ds_{\rm APS}^2 = -d\tau^2 + a(\tau)^2\left(d\tilde{r}^2 + \tilde{r}^2\, d\Omega^2\right), \tag{4}$$

where $(\tau, \tilde{r}, \theta, \varphi)$ denotes a coordinate system. $a(\tau)$ is the

1) Here we set $C=G=1$. In this article, the metric has the signature $(-,+,+,+)$.
2) $[A]$. denotes the dimension of the quantity [A].

scalar factor, which satisfies the classical Friedmann equation

$$H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho, \tag{5}$$

with H is the Hubble constant. The exterior region is vacuum, which is described by the Schwarzschild metric. Ref. [54] considers the quantum version of the OS model. The authors assume that the matter in the interior region satisfies a LQC-deformed Friedmann equation (see e.g., [90−92])

$$H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho\left(1-\frac{\rho}{\rho_c}\right), \rho = \frac{M}{\frac{4}{3}\pi\tilde{r}_0^3 a^3}. \tag{6}$$

Here, M is the ADM mass of the matter field. The critical density is given by $\rho_c = \sqrt{3}/\left(32\pi^2\gamma^3\hbar\right)$. The corrected term $-\dfrac{8\pi\rho^2}{3\rho_c}$ arises from LQC effects. In this work, we consider the mass of the matter to be on the scale of solar mass. The metric of the exterior region is given by applying appropriate junction conditions to both the reduced metric and the extrinsic curvature of the junction surface Σ[1)]

$$h^+_{ab}|_\Sigma = h^-_{ab}|_\Sigma, \quad K^+_{ab}|_\Sigma = K^-_{ab}|_\Sigma. \tag{7}$$

Here the "+" sign denotes the exterior region while the "−" sign denotes the interior region. Σ is a timelike hypersurface, which connects the exterior and the interior regions. h_{ab} is the reduced metric, while K_{ab} is the extrinsic curvature of Σ. In classical GR, the Israel junction conditions indicate that K_{ab} exhibits a jump at Σ only if the matter density is distributional [93], meaning the matter possesses a surface density. However, in the quantum OS model, where the matter does not have a surface density, K_{ab} is assumed to be continuous, as discussed in [84]. The metric of the exterior region reads

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}+\frac{\alpha M^2}{r^4}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}+\frac{\alpha M^2}{r^4}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{8}$$

Compared to the Schwarzschild solution, there is a quantum-corrected term $\dfrac{\alpha M^2}{r^4}$ in eq. (8). This term is interpreted as a contribution of LQG. The position of the BH horizon is given by

$$f(r) = 1-\frac{2M}{r}+\frac{\alpha M^2}{r^4} = 0. \tag{9}$$

There are several roots in eq. (9). However, as demonstrated in [54], there are only two real roots when $M > M_{\min}$ Here $M_{\min}$ is the lower bound of the mass required for BH formation, given by $M_{\min} := \dfrac{4}{3\sqrt{3}}\sqrt{\alpha}$. We focus only on the outer horizon, which reads

$$r_h = \frac{M}{2}+\frac{1}{2}\sqrt{h}+\frac{1}{2}\sqrt{k}, \tag{10}$$

with

$$h = M^2+\frac{1}{3}M\left(\frac{2\cdot 6^{2/3}\alpha}{\left(9M\alpha+\sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}}+\left(54M\alpha+6\sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}\right), \tag{11}$$

and

$$k = 2M^2-\frac{M\left(18M\alpha+2\sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}}{3^{2/3}}-\frac{2\cdot 2^{2/3}M\alpha}{\left(27M\alpha+3\sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}} + \frac{2M^3}{\sqrt{M^2+\frac{1}{3}M\left(\frac{2\cdot 6^{2/3}\alpha}{\left(9Ma+\sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}}+\left(54M\alpha+6\sqrt{3}\sqrt{(27M^2-16\alpha)\alpha^2}\right)^{1/3}\right)}}. \tag{12}$$

1) The Latin letters a, b, c are the abstract indices.

Note that $M^2 \gg \alpha > \frac{16}{24}\alpha$, we find the reality of the square roots above is preserved. Also, we can employ Taylor expansion to r_h, which yields

$$r_h = 2M - \frac{\alpha}{8M} + O(\alpha^2) < 2M. \tag{13}$$

The term $-\frac{\alpha}{8M}$ is the quantum correction of LQG.

III. HAWKING RADIATION AND BLACK HOLE ENTROPY

In this section, we investigate the Hawking radiation of quantum OS BH with the PW approach. First, we rewrite the metric (8) in Painlevé-Gullstrand coordinates [94] and then compute the emission rate of the massless scalar particles near the horizon. Finally, the BH entropy is discussed.

A. Quantum Oppenheimer-Snyder black hole in Painlevé-Gullstrand coordinates

To express the line element (8) in Painlevé-Gullstrand coordinates, we introduce the following coordinate transformation

$$\tilde{t} = t + F(r), \tag{14}$$

where

$$F(r) = \int_0^r \sqrt{\frac{Mr'^4(2r'^3 - M\alpha)}{(-2Mr'^3 + r'^4 + M^2\alpha)^2}}\,\mathrm{d}r'. \tag{15}$$

The explicit expression is quiet complicated, but it is not important in our discussions. Then we find

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{\alpha M^2}{r^4}\right)\mathrm{d}\tilde{t}^2 + 2\sqrt{\frac{2M}{r} - \frac{\alpha M^2}{r^4}}\,\mathrm{d}\tilde{t}\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{16}$$

A brief review of Painlevé-Gullstrand coordinates and a detailed derivation of eq. (16) can be found in Appendix A. We observe that there is no coordinate singularity at the horizon in (16), enabling us to investigate the tunneling process across the horizon. Furthermore, the timeslice of (16) is Euclidean, allowing traditional Schrödinger equation is remain valid.

B. Parikh-Wilczek approach of quantum Oppenheimer-Snyder black hole

In the PW approach [68], the Hawking radiation is interpreted as a quantum tunneling process. This approach takes into account the energy conservation condition and incorporates the backreaction of Hawking radiation.

We consider the emitted particles to be massless scalar particles. The radial null geodesic reads

$$\dot{r} = \pm 1 - \sqrt{\frac{2M}{r} - \frac{\alpha M^2}{r^4}}, \tag{17}$$

with $\dot{r}$ denotes the derivative of r with respective to $\tilde{t}$. The "+" sign corresponds to an outgoing particle, while the "−" sign corresponds to an ingoing particle. In the following discussion, we focus on the outgoing particle which tunnels across the outer BH horizon. For simplicity, we assume the wave corresponding to the emitted particle is an s-wave. In classical GR, Birkhoff's theorem state that any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat. Nevertheless, the case of quantum OS black hole is different, due to the contributions of LQG effects. Recent work [95] addresses this issue by generalizing Birkhoff's theorem within the context of Polymerized Einstein Field Equations (PEFE), which incorporates quantum gravity corrections. The authors demonstrate in this work that the exterior region of the quantum OS spacetime satisfies the generalized Birkhoff's theorem, with three distinct branches corresponding to $k = -1, 0$ and 1, representing open, flat and closed universes for the interior region, respectively. In this paper, we focus on the $k = 0$ case, as shown in (6) and (8). As demonstrated in [95], the exterior region of this type of quantum OS model is unique and characterized by a single parameter: the mass. Therefore, no graviton is emitted in this process. Suppose the energy of this particle is ω. According to the energy conservation, the energy of the BH changes as $M \to M - \omega$. Accordingly, the modified line element yields

$$\mathrm{d}s^2 = -\left(1 - \frac{2(M-\omega)}{r} + \frac{\alpha(M-\omega)^2}{r^4}\right)\mathrm{d}\tilde{t}^2 + 2\sqrt{\frac{2(M-\omega)}{r} - \frac{\alpha(M-\omega)^2}{r^4}}\,\mathrm{d}\tilde{t}\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{18}$$

and the modified radial null geodesic yields

$$\dot{r} = \pm 1 - \sqrt{\frac{2(M-\omega)}{r} - \frac{\alpha(M-\omega)^2}{r^4}}. \tag{19}$$

The tunneling process occurs very close to the horizon. The outgoing particle, as measured by the local observer near the horizon, experiences an ever-increasing blue shift. Consequently, the geometric optics approximation is suitable for describing the outgoing particles, and the WKB approximation is justified.

The outgoing particles are created inside the horizon. As they are emitted, they cross the horizon and travel to infinity. This process is classically forbidden but permitted through quantum tunneling. Next, we calculate the emission rate of these outgoing particles. With the WKB approximation, the emission rate is given by

$$\Gamma \sim e^{-\frac{2}{\hbar}\mathrm{Im}A}, \tag{20}$$

with A is the action of the particle. The imaginary part of the action is given by

$$\mathrm{Im}A = \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}} p_r \mathrm{d}r = \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}}\int_0^{p_r} \mathrm{d}p_r' \mathrm{d}r. \tag{21}$$

Here p_r is the conjugate momentum of r, r_{in} and r_{out} are the locations of the particle before and after the tunneling across the event horizon. Hamiltonian equations imply

$$\dot{r} = +\left.\frac{\mathrm{d}H}{\mathrm{d}p_r}\right|_r, \tag{22}$$

with the Hamiltonian is $H = M-\omega$. Then with eq. (19) and focusing on the outgoing particles, we find

$$\begin{aligned}\mathrm{Im}A &= \mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}}\int_0^{p_r} \mathrm{d}p_r' \mathrm{d}r \\ &= -\mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}}\int_0^{\omega} \frac{\mathrm{d}r}{1-\sqrt{\dfrac{2(M-\omega')}{r}-\dfrac{\alpha(M-\omega')^2}{r^4}}}\mathrm{d}\omega'.\end{aligned} \tag{23}$$

As assumed previously, $M^2 \gg \alpha$, we can apply Taylor expansion to the integrand of eq. (23)

$$\begin{aligned}\frac{1}{1-\sqrt{\dfrac{2(M-\omega')}{r}-\dfrac{\alpha(M-\omega')^2}{r^4}}} &= \frac{1}{1-\sqrt{\dfrac{2(M-\omega')}{r}}} \\ &- \frac{(M-\omega')\sqrt{\dfrac{(M-\omega')}{r}}}{2\sqrt{2}r^3\left(1-\sqrt{\dfrac{2(M-\omega')}{r}}\right)^2}\alpha + O\left(\alpha^2\right).\end{aligned} \tag{24}$$

Note that $[\alpha] = [M]^2$. Hence, the left-hand side of (24) is dimensionless, which implies that any term in the right hand side of (24) is also dimensionless. Especially, $\left[\dfrac{(M-\omega')\sqrt{\dfrac{(M-\omega')}{r}}}{2\sqrt{2}r^3\left(1-\sqrt{\dfrac{2(M-\omega')}{r}}\right)^2}\alpha\right] = 1$. The integrand is singular at the location of the classical horizon. To evaluate eq. (23), we apply Feynman's scheme. To ensure that the positive energy solutions decay over time, we deform the contour as $\omega \to \omega - i\epsilon$ with $\epsilon \to 0$. By deforming the contour, the singularity in eq. (24) is avoided, ensuring that the right-hand side of eq. (24) converges uniformly. Consequently, the integration and summation can be interchanged. Then we find

$$\mathrm{Im}A = -\lim_{\epsilon\to 0}\mathrm{Im}\int_{r_{\rm in}}^{r_{\rm out}}\int_0^{\omega}\left(\frac{1}{1-\sqrt{\dfrac{2(M-\omega'+i\epsilon)}{r}}} - \frac{(M-\omega'+i\epsilon)\sqrt{\dfrac{(M-\omega'+i\epsilon)}{r}}}{2\sqrt{2}r^3\left(1-\sqrt{\dfrac{2(M-\omega'+i\epsilon)}{r}}\right)^2}\alpha\right)\mathrm{d}r\mathrm{d}\omega' + O(\alpha^2). \tag{25}$$

Introducing the coordinate transformation $u = \sqrt{r}$, we find

$$\mathrm{Im}A = -\lim_{\epsilon\to 0}\mathrm{Im}\int_{u_{\rm in}}^{u_{\rm out}}\int_0^{\omega}\left(\frac{2u^2}{u-\sqrt{2(M-\omega'+i\epsilon)}} - \frac{(M-\omega'+i\epsilon)^{3/2}}{\sqrt{2}u^4\left(u-\sqrt{2(M-\omega'+i\epsilon)}\right)^2}\alpha\right)\mathrm{d}u\mathrm{d}\omega' + O(\alpha^2). \tag{26}$$

We focus on the particles created near the outer horizon. There is a pole at $u = \sqrt{2(M-\omega'+i\epsilon)}$ in eq. (26). Therefore, we find

$$\mathrm{Im}A = \lim_{\epsilon\to 0}\int_0^{\omega}\left(4\pi(M-\omega') + \mathrm{Im}\frac{\sqrt{2}\pi i}{M-\omega'+i\epsilon}\alpha\right)\mathrm{d}\omega' + O(\alpha^2). \tag{27}$$

For a solar mass black hole, we have $M \gg \omega$. Therefore,

$$\begin{aligned}\mathrm{Im}A &= 4\pi M\omega\left(1-\frac{\omega}{2M}\right)+\sqrt{2}\pi\alpha\left(\log\left(\frac{M}{l_p}\right)-\log\left(\frac{M-\omega}{l_p}\right)\right)+O\left(\alpha^2\right)\\ &= 4\pi M\omega\left(1-\frac{\omega}{2M}\right)+\frac{\sqrt{2}}{2}\pi\alpha\left(\log\left(\frac{\mathscr{A}_{\mathrm{Sch}}(M)}{l_p^2}\right)-\log\left(\frac{\mathscr{A}_{\mathrm{Sch}}(M-\omega)}{l_p^2}\right)\right)+O\left(\alpha^2\right).\end{aligned}\tag{28}$$

Here $\mathscr{A}_{\mathrm{Sch}}$ is the area of the Schwarzschild BH horizon with mass M. Compared with the original work [68], an additional term $\frac{\sqrt{2}}{2}\pi\alpha\left(\log\left(\frac{\mathscr{A}_{\mathrm{Sch}}(M)}{l_p^2}\right)-\log\left(\frac{\mathscr{A}_{\mathrm{Sch}}(M-\omega)}{l_p^2}\right)\right)$ appears. This term is interpreted as the quantum correction arising from LQG. Then the emisson rate of the particles yields

$$\Gamma\sim\exp(-\frac{2}{\hbar}\mathrm{Im}A)=\exp\left(-\frac{8\pi M\omega}{l_p^2}\left(1-\frac{\omega}{2M}\right)+\frac{\sqrt{2}\pi\alpha}{l_p^2}\left(\log\left(\frac{\mathscr{A}_{\mathrm{Sch}}(M-\omega)}{l_p^2}\right)-\log\left(\frac{\mathscr{A}_{\mathrm{Sch}}(M)}{l_p^2}\right)\right)+O\left(\alpha^2\right)\right)\tag{29}$$

C. The entropy of the quantum Oppenheimer-Snyde black hole

In classical GR, the emission rated can be expressed as

$$\Gamma\sim\exp\Delta S,\tag{30}$$

with ΔS is the difference of the black hole entropy between before and after the emission of the particle. It is stated that this result implies the unitary of black hole evaporation in GR [1, 2]. In quantum mechanics, the transition apmplitude between the initial and the final states is

$$\Gamma(i\to f)=|\mathcal{M}|^2\cdot(\text{ phase space factor}),\tag{31}$$

with $|\mathcal{M}|$ is the probabilty magnitude of this poccess. The phase space factor is given by

$$\text{phase space factor } =\frac{N_f}{N_i}=\frac{e^{S_f}}{e^{S_i}}=e^{\Delta S}.\tag{32}$$

Let's take the Schwarzschild BH as an example. For a Schwarzschild BH, its initial entropy is given by

$$S_i=\frac{4\pi M^2}{l_p^2}.\tag{33}$$

After the black hole emits particle with energy ω, the final entropy is

$$S_f=\frac{4\pi(M-\omega)^2}{l_p^2}.\tag{34}$$

Therefore,

$$\Delta S=S_f-S_i=-\frac{8\pi M\omega}{l_p^2}\left(1-\frac{\omega}{2M}\right)=-\frac{2}{\hbar}\mathrm{Im}A_{\mathrm{Sch}},\tag{35}$$

with A_{Sch} is the action of the particle created near the Schwarzschild BH horizon. Hence, the emission rate of this particle is

$$\Gamma_{\mathrm{Sch}}\sim\exp\Delta S.\tag{36}$$

We now turn back to the cases of quantum OS BH. Quantum correction terms appear in eq. (29). Hence, to express the emission rate Γ in the formula of eq.(30), we introduce the quantum-corrected entropy as

$$\tilde{S}=S+\frac{\sqrt{2}\pi\alpha}{l_p^2}\log\left(\frac{\mathscr{A}_{\mathrm{Sch}}}{l_p^2}\right)+O(\alpha^2),\tag{37}$$

Here S is the entropy of the Schwarzschild BH, given by $S=\frac{4\pi M^2}{l_p^2}$. Note that we are considering particles tunneling across the outer BH horizon. Therefore, $\tilde{S}$ is the entropy of the outer BH horizon, Our approach aligns with the scheme introduced in [68, 75, 76]. To compute the total BH entropy, a multi-horizon scenario should be considered [96, 97]. This approach is more intricate than it may initially appear, and we will explore this topic further in our future research. Finally, the emission rate reads

$$\Gamma\sim\exp\left[\tilde{S}_f-\tilde{S}_i\right]=\exp\Delta\tilde{S}.\tag{38}$$

The pre-factor of the logarithmic correction of the BH entropy relates to the micro-states of BH horizon. In our findings, the sign of this pre-factor is positive. Although this differs from the results obtained within the framework of ordinary LQG [77–79], it is consistent with the entropy of the effective loop quantum BH computed by other approaches [83, 84]. Ref. [82] provides profound insights into this discrepancy. On the one hand, it states that the quantum correction to the BH entropy, arising from quantum gravity effects, reduces the entropy. On the other hand, it states that the logarithmic correction, which originates from thermal fluctuations, increases the entropy due to the enhanced uncertainty introduced by these fluctuations. Based on these insights, our finding (37) might suggest the logarithmic correction for quantum OS BH entropy incorporates the contributions from both quantum gravity effect and (effective) thermal fluctuations. For precise, let us denote a_q as the pre-factor for the logarithmic correction contributed by quantum gravity effect and a_f as the pre-factor for the logarithmic correction contributed by (effective) thermal fluctuation, then $a_q+a_f=\frac{\sqrt{2}\pi\alpha}{l_p^2}$. We will continue this interesting topic in our future researches. Moreover, it has been reported that the BH entropy is related to the dimension of the spacetime [98]. Our finding is consistent with the situation of 4-dimensional spacetime discussed in [98]. Unlike the arguments based on quantum gravity, we get this correction with semiclassical method. Therefore, our result supports the validity of the semiclassical method in the context of quantum gravity.

D. Comparison with other scenarios for computing black bole entropy

In addition to the previously introduced PW approach, several other methods for computing black hole entropy incorporate correction terms that modify the standard entropy formula. In this subsection, we compare these methods with our results.

• Noncommutative spacetime geometry approach

This approach is based on the assumption that the coordinates for the spacetime manifold do not commute to each other but instead satisfy the following relationship[1)]

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{39}$$

where $\theta^{\mu\nu}$ is an antisymmetric matrix that encodes the noncommutativity of spacetime [99−101]. The spherical and static solution within this framework reads [102, 103]

$$ds^2=-\left(1-\frac{2M_\theta}{r}\right)\mathrm{d}t^2+\left(1-\frac{2M_\theta}{r}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2. \tag{40}$$

Here, M_θ is the mass of the spacetime satisfying the Gaussian distribution of minimal width $\sqrt{\theta}$

$$M_\theta=\int_0^r\rho_\theta(r)4\pi r^2\mathrm{d}r=\frac{2M}{\sqrt{\pi}}\int_0^{\frac{r^2}{4\theta}}x^{\frac{1}{2}}e^{-x}\mathrm{d}x, \tag{41}$$

with M is a constant and the distribution function $\rho(r)$ reads

$$\rho_\theta(r)=\frac{M}{(4\pi\theta)^{\frac{3}{2}}}\exp\left(-\frac{r^2}{4\theta}\right). \tag{42}$$

One can show that

$$\lim_{\theta\to 0}M_\theta=M. \tag{43}$$

In this case, eq. (42) reduces to he ordinary Schwarzschild solution. Within this approach, the BH entropy entropy reads

$$S_{\rm NSG}=4\pi\frac{M^2}{l_p^2}\mathcal{E}\left(\frac{M}{l_p\sqrt{\theta}}\right)-6\pi\theta\mathcal{E}\left(\frac{M}{l_p\sqrt{\theta}}\right)+12\sqrt{\pi}\theta\frac{M}{l_p}\exp\left(-\frac{M^2}{l_p^2\theta}\right), \tag{44}$$

with $\epsilon(x)$ is the Gauss error function

$$\mathcal{E}(x)\equiv\frac{2}{\sqrt{\pi}}\int_0^x e^{-\lambda^2}\mathrm{d}\lambda. \tag{45}$$

Especially, in the large mass limit $\frac{M}{l_p\sqrt{\theta}}\gg 1$, one finds

$$S_{\rm NSG}=\frac{4\pi M^2}{l_p^2}+12\sqrt{\pi}\theta\frac{M}{l_p}\exp\left(-\frac{M^2}{l_p^2\theta}\right). \tag{46}$$

The correction in eq. (46) is exponential, differing from the logarithmic correction in our result (37). In fact, Ref. [103] demonstrates the BHs cannot evaporate completely in the noncommutative spacetime geometry framework. Instead, there is a lower bound for the mass, M_0, during the later stages of BH evaporation, and the information can be stored in the remnant, offering a possible resolution to the black hole information paradox. Our findings may lead to a similar conclusion, due to the pre-factor of the logarithmic correction in eq. (37) is positive. However, a subtlety arises because our analysis assumes $M^2\gg\alpha$, which may not hold during the later stages of evaporation. This suggests that these two frameworks might describe different phases of black hole evaporation. Investigating the relationship between these two approaches would be an interesting avenue for future research. Moreover, As discussed in [54], a bounce occurs at the end of collapse. Exploring how to incorporate both the bounce and black hole evaporation into a unified framework remains a challenging but intriguing task.

1) The Greek letters μ, ν are the abstract indices, which take the value $\mu,\nu=0,1,2,3$.

• **Modified dispersion relations approach**

Inspired by quantum gravity, the modified dispersion relations (MDR) approach arises from modifying energy-momentum dispersion relation [104, 105]

$$\vec{p}^2 = E^2 - \mu^2 + \alpha_1 l_p E^3 + \alpha_2 l_p^2 E^4 + \alpha_3 l_p^3 E^5 + \alpha_4 l_p^4 E^6 + O\left(l_p^5 E^7\right), \tag{47}$$

with $E \ll 1/l_p$. The value of the coefficients α_i depend on the specific quantum gravity theory. Applying the modified dispersion relation, the BH entropy is given by

$$\begin{aligned} S_{\rm MDR} \simeq & \frac{\mathscr{A}}{4l_p^2} + \frac{\alpha_1 \pi^{\frac{1}{2}}}{l_p}\mathscr{A}^{\frac{1}{2}} + \pi\left(\frac{3}{2}\alpha_2 - \frac{3}{8}\alpha_1^2\right)\log\frac{\mathscr{A}}{l_p^2} \\ & - 4\pi^{\frac{3}{2}} l_p\left(-\alpha_1\alpha_2 + \frac{1}{4}\alpha_1^3 + 2\alpha_3\right)\mathscr{A}^{\frac{-1}{2}} \\ & - 4\pi^2 l_p^2\left(-\frac{5}{4}\alpha_1\alpha_3 - \frac{5}{8}\alpha_2^2 + \frac{15}{16}\alpha_1^2\alpha_2 - \frac{25}{128}\alpha_1^4\right)\mathscr{A}^{-1} \\ & - \frac{16}{3}\pi^{\frac{5}{2}} l_p^3\left(\frac{9}{8}\alpha_1^2\alpha_3 - \frac{45}{48}\alpha_1^3\alpha_2 + \frac{9}{8}\alpha_1\alpha_2^2 \right. \\ & \left. + \frac{21}{128}\alpha_1^5 - \frac{3}{2}\alpha_2\alpha_3\right)\mathscr{A}^{-\frac{3}{2}}. \end{aligned} \tag{48}$$

As observed in [105], if the odd powers of energy in (47) are set to 0 ($\alpha_1 = \alpha_3 = 0$), then the BH entropy reduces to

$$S_{\rm MDR} \simeq \frac{\mathscr{A}}{4l_p^2} + \frac{3}{2}\pi\alpha_2 \log\frac{\mathscr{A}}{l_p^2} + \frac{5}{2}\pi^2\alpha_2^2\frac{l_p^2}{\mathscr{A}}. \tag{49}$$

This result is consistent with our finding (37). In fact, BH entropy formula imposes constraints on the form of MDR.

• **Generalized uncertainty principle approach**

The general uncertainty principle (GUP) approach is based on the modification of the ordinary eisenbergnuncertainty principle (HUP) [104, 106]

$$\delta x \geq \frac{1}{\delta p} + \beta l_p^2 \delta p + O\left(l_p^3 \delta p^2\right), \tag{50}$$

with β is a small parameter depends on the specific quantum gravity theory. Based on this assumption, the BH entropy is computed as

$$S_{\rm GUP} \simeq \frac{\mathscr{A}}{4l_p^2} - \beta\pi \ln\frac{\mathscr{A}}{l_p^2}. \tag{51}$$

This result is consistent with our finding (37) up to a difference in the sign. Furthermore, by considering higher order effects, Ref. [106] derives the BH entropy with additional corrections and predicts that the existence of a remnant as the BH mass M approaches to the order of Planck mass M_p during evaporation. This prediction aligns with the results of the noncommutative spacetime geometry approach. Incorporating the GUP framework into the quantum OS BH model would be an intriguing direction for future research.

• **Polymeric quantization approach**

The polymeric quantization approach is a method for quantizing gravity, inspired by techniques from LQG and quantum mechanics in the polymer representation[107, 108]. It is based on the premise that, in a quantum theory of gravity, spacetime is discrete and there exists a minimum measurable length scale in the order of the Planck length. Within this framework, the classical phase space is modified to incorporate discrete structures, which is different from the standard continuous treatment. Ref. [109] computes the BH entropy in this framework. We begin by introducing the polymeric area of the black hole in terms of the horizon area, expressed as:

$$\mathscr{A}^{(\rm poly)} = \mathscr{A}\left(1 - \frac{\left(1+\mu^2 M_P/8\right)}{8\pi}\left(\frac{M_P}{M}\right)^2\right)^2. \tag{52}$$

Here, $\mu = \mu_0/\hbar$, with μ_0 is the discreteness parameter. The continuum limit is given by $\mu \to 0$. In the high-temperature limit, the black hole entropy is given by

$$\begin{aligned} S_{\rm poly} = & \frac{\mathscr{A}^{(\rm poly)}}{4l_P^2} - \frac{1}{2}\ln\left[\frac{\mathscr{A}^{(\rm poly)}}{4l_P^2}\right] + M\left(1 - \frac{\mathscr{A}^{(\rm poly)}}{\mathscr{A}}\right) \\ & + O\left[\mathscr{A}^{(\rm poly)^{-1}}\right]. \end{aligned} \tag{53}$$

In the limit $\frac{M_p}{M} \to 0$, the polymeric area $\mathscr{A}^{(\rm poly)}$ reduces to the original horizon area $\mathscr{A}$. The entropy reduces to the well-known form

$$S_{\rm poly} = \frac{\mathscr{A}}{4l_P^2} - \frac{1}{2}\ln\left[\frac{\mathscr{A}}{4l_P^2}\right] + O\left[\mathscr{A}^{-1}\right]. \tag{54}$$

This result is also consistent with our finding (37), up to a minus sign.

IV. CONCLUSION AND OUTLOOK

In this work, we apply the the PW approach to the quantum OS BH and evaluate the emission rate of the outgoing massless scalar particles. Compared to the original results in [68, 69], our findings include quantum correction terms, arising from LQG effects. Following the scheme in [1, 2], we establish the entropy of the OS BH. This entropy formula includes a logarithmic correction, which is consistent with well-known result in the context of quantum gravity [77−79].

So far, we have only considered the massless scalar particles as the emitted particles. It would be interesting to extend the study to the tunneling process of massive particles. This topic has been explored within the framework of classical GR in [1, 2, 76], where BH thermodynamics plays a crucial role. Investigating the tunneling process of massive particles in the OS model potentially provides us deeper insight into the BH thermodynamics in the context of LQG.

Recently, the island scheme is proposed as a resolution to the BH information paradox [110−114]. This approach utilizes the concepts of the minimal quantum extremal surface [115−117] to evaluate the BH entropy, successfully recovering the Page curve [118, 119] and resolving the BH information paradox. In our future work, we plan to extend the island scheme to the OS BH and compare the results with those presented in this paper.

APPENDIX A: A BRIEF REVIEW OF PAINLEVÉ-GULLSTRAND COORDINATES

In this appendix, we briefly review the scheme of obtaining the Painlevé-Gullstrand coordinate transformation of a general static spacetime. Firstly, we introduce the line element as

$$ds^2 = -(1-g(r))\,dt^2 + \frac{1}{1-g(r)}dr^2 + d\Omega^2, \tag{A1}$$

with $g(r)$ is a function of r. We then introduce the coordinate transformation

$$t = \tilde{t} + F(r). \tag{A2}$$

So we have

$$dt = d\tilde{t} + F'(r)dr. \tag{A3}$$

In Painlevé-Gullstrand coordinates, the time slice is required to be Euclidean, which necessitates the coefficient of dr^2 is 1. consequently, we obtain the following equations for $F(r)$

$$\frac{1}{1-g(r)} - [1-g(r)]\left[F'(r)\right]^2 = 1. \tag{A4}$$

Then the new line element reads

$$ds^2 = -[1-g(r)]dt^2 \pm 2\sqrt{g(r)}dt\,dr + dr^2 + r^2\,d\Omega^2. \tag{A5}$$

In our case,

$$g(r) = \frac{2M}{r} - \frac{\alpha M^2}{r^4}. \tag{A6}$$

Then $F(r)$ is given by

$$F(r) = \int_0^r \sqrt{\frac{Mr'^4(2r'^3 - M\alpha)}{(-2Mr'^3 + r'^4 + M^2\alpha)^2}}dr'. \tag{A7}$$

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