Geometric Constraints via Page Curves: Insights from Island Rule and Quantum Focusing Conjecture*

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Abstract: Exploring the inverse problem tied to the Page curve phenomenon and island paradigm, we investigate the geometric conditions underpinning black hole evaporation where information is preserved and islands manifest, giving rise to the characteristic Page curve. Focusing on a broad class of static spherical symmetry black hole metrics in asymptotically Minkowski or (anti-)de Sitter spacetimes, we derive a pivotal constraint: the second derivative of blacken factor $f''(r_h) < \frac{6\kappa A'(r_h)}{cG_N}$ for which the island exists and reproduce the Page curve. On the other hand, starting from the quantum focusing conjecture theory, we obtain another constraint on the blacken factor: $f''(r_h) < \frac{6\kappa^2 r_h A'(r_h) e^{2\kappa r_\star(b)}}{cG_N f(b)}$ that the theory can be satisfied. In particular, by studying these two constraints, we find that a common properties. Specifically, we reveal that a universally criterion – manifested in the negativity of the second derivative of $f(r)$, i.e. $f''(r) < 0$, in proximity to the event horizon where $r \sim r_h + O(G_N)$, ensures the emergence of Page curves and follows the quantum focusing conjecture in a manner transcending specific theoretical models. Finally, we argue that the negativity of the second derivative of the blacken factor $f(r)$ near the event horizon strongly indicates negative heat capacity, which implies that black holes with a negative heat capacities must have islands and satisfy the quantum focusing conjecture.

Keywords: black holes, black hole information loss, page curve

DOI: **CSTR:**

I. INTRODUCTION

Black holes are the strongest evidence of general relativity (GR). In modern physics, this surprising and fascinating object has becomes one of the most controversial areas of theoretical physics. When some of the result of quantum mechanics (QM) are inserted into the framework of GR, something amazing occurs. This approach was first proposed by Hawking in 1975 (known as the Hawking radiation) [1]. However, it leads to a very acute dilemma: the information (loss) paradox [2]. QM requires that the evolution of a black hole formed in a pure state must respect the unitary principle, namely, it remains a pure state at the end of evaporation. In contrast, Hawking radiation indicates that radiation in a thermal (mixed) state 1). It was not until the Page curve was proposed that this issue gradually became sharp [3, 4]. Significant breakthroughs have been made in the last 20 years. A key catalyst was the anti-de Sitter/Conformal Field Theory (AdS/CFT), or the holographic duality [5].

The AdS/CFT duality opens a window for us to look at the problem of gravity in AdS from the perspective of CFT though this theory. A milestone work is the RT formula proposed by Ryu and Takayanagi to calculate the holographic entanglement entropy [6]. The RT formula establishes the relation between the entanglement entropy of the subregion and its homologous extremal surface (the RT surface) area. Next, the quantum correction of the RT formula is also followed [7]. In 2015, the modified RT formula with high-order corrections, the quantum extremal surface (QES) prescription was proposed [8].

At now, all the problems of evaluating the entanglement entropy at the boundary translate into finding the minimal extremal surface in bulk spacetime. After the Page time, we have another additional extremal surface, which is located inside the event horzion of the evaporat-

Received 20 December 2024; Accepted 23 December 2024

* The study was partially supported by NSFC, China (Grant No. 12275166 and No. 12311540141)

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1) For analytical simplicity, we omit the consideration of the grey-body factor in this study. Consequently, Hawking radiation is effectively modeled as pure black-body radiation, adhering rigorously to the Planckian spectral distribution.

ing AdS black hole, called the "island" [9−11]. Considering its contribution leads to the unitary Page curve. At this point, the black hole information paradox is declared to be preliminarily solved. Interested readers can refer to a nice pedagogical review [12].

The formula for calculating the fine-grained (entanglement) entropy (or the von Neumann entropy) of Hawking radiation obtained by the QES prescription is summarized as the "island formula" :

$$S_{\text{Rad}} = \text{Min}\left\{\text{Ext}\left[\frac{\text{Area}(\partial I)}{4G_N} + S_{\text{bulk}}(R\cup I)\right]\right\}, \tag{1}$$

where I refer to the island region and its boundary is denoted as ∂I. The entropy of bulk fields consists of two contributions, namely, the island I inside the black hole and the radiation region R outside the black hole. The words "Min" and "Ext" guide us to extremize the generalized entropy first to find saddle points,

$$\frac{\partial S_{\text{gen}}}{\partial x^\mu} \equiv \frac{\partial}{\partial x^\mu}\left(\frac{\text{Area}(\partial I)}{4G_N} + S_{\text{bulk}}(R\cup I)\right) = 0. \tag{2}$$

These saddle points correspond to the candidate "QES". Then we pick the one with the smallest value, which is the final correct result of the fine-grained entropy of Hawking radiation. In addition, the island formula (1) can be derived equivalently by strict gravitational path integral [12, 13]:

$$S_{\text{Rad}} = \lim_{n\to 1}\frac{1}{1-n}\log \text{Tr}(\rho_R^n), \tag{3}$$

in which, the contribution of the connected replica wormhole (saddle) will dominate at late times, and the Page curves can be reproduced naturally[1)].

Recently, studies have demonstrated that the island formula does not depend on the AdS/CFT correspondence and has been applied far beyond the asymptotically AdS black holes. Example include the study of islands in the asymptotically flat or dS spacetime, as well as the combination with some intersecting fields. One can refer to a non-exhaustive list of progress in this field [15−90].

There are two motivations for this article: up to now, most studies have focused on the reproduction of Page curves in *special* spacetime. They all found that islands emerges at late times could curb the growth of entropy and respect the unitarity[16−26, 30, 33, 34, 36, 37, 40, 41, 47−49, 54, 55, 65]. A natural question is what are the constraints on obtaining a unitary Page curve using the island paradigm for *general* spacetime? Or equivalently, we can consider the inverse of this problem: If a Page curve already exists, namely, the unitary is maintained, what constraints does the spacetime geometry need to satisfy? So the first motivation is to find out the constraints on general spacetime when the Page curve exists. On the other hand, the quantum focusing conjecture (QFC) also has a constraint on the generalized entropy at late times. How does this constraint relate to those imposed by the island paradigm? Therefore, the second motivation is based on QFC perspective, we again consider the requirements on spacetime geometry. Incorporating these dual considerations, we discern that the sufficient and necessary conditions for the existence of Page curves is $f''(r_h) < \frac{6\kappa A'(r_h)}{cG_N}$ by the second derivative of the blacken function in the vicinity of the horizon, while and the necessary and sufficient conditions for QFC theorem to be established is $f''(r_h) < \frac{6\kappa^2 r_h A'(r_h) e^{2\kappa r_\star(b)}}{cG_N f(b)}$. In particular, there is a relationship $f''(r_h) < 0$ that satisfies both of these conditions, which implies that black holes with negative heat capacities must have islands and satisfy the QFC theorem. These discoveries culminates in the formulation of overarching geometric principles.

We begin with a general metric that represents a static spherical symmetry black hole. In the static coordinate system under the Schwarzschild gauge, such metrics are written as ($D \geq 3$):

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\omega_{D-2}^2, \tag{4}$$

where $d\omega_{D-2}^2$ is the volume of the unit $(D-2)$-sphere, and the area of the $(D-2)$-sphere with radius r is equal to $r^{D-2}\omega_{D-2}$. Moreover, the angular direction should be removed when we focus on two-dimensional (2D) black holes. To guarantee the existence of a black hole solution, we need to impose some requirements on the blacken function $f(r)$: It must have simple and positive zeros, and then it is also required to have a value for its corresponding radial coordinates that exceeds the horizon and extends to the space-like infinity. Only in this way is the domain of exterior communication is "outside" the black hole.

In some special cases, the blacken functions for radial and time coordinates are not equal. Actually, this corresponds to the configuration with the Einstein-Maxwell-dilation field equation[2)]. For convenience, we ignore these few special examples in the paper and assume that static spherical symmetry black hole solutions can all be written in the form (4). Moreover, when the cosmologic-

1) More precisely, all QES configurations are saddle points in the path integral of the replica geometry. The entanglement entropy is minimized to achieve the minimum partition functions. So the entanglement entropy is approximately the minimum entanglement entropy at the saddle point.

2) For instance, for Garfinkle-Horowitz-Strominger black holes, the metric cannot be written in the form of (4) [57]; for Kaluza-Klein black holes, its area is a function of the dilation field ϕ [92].

al constant Λ is non-positive, it is asymptotically associated with flat or AdS black holes. They usually have only one horizon[1)]. However, for a positive cosmological constant, such black holes have a cosmic horizon in addition to their event horizons. For simplicity, we focus mainly on the case of a single horizon. In the case of multiple horizons, the corresponding calculation only requires parameter substitution without affecting the physical meaning. One can refer to [63] for the explicit calculations. Besides, due to the special property of the vanishing temperature of extremal black holes [55], in this paper we only discuss non-extremal black holes.

The rest of the paper is organized as follows. In section II, we calculate the entanglement entropy for Hawking radiation by the island paradigm. We first prove that island is absent at early times. Subsequently, we focus on the behavior of entropy at late times. We derive the constraint condition that the spacetime geometry needs to satisfy when the island appears, and we must obtain a unitary Page curve. In section III, we apply the QFC to test our result and acquire a self-consistent conclusion. Finally, we display the discussions and summary in section IV. The Planck units $\hbar = k_B = c = 1$ is used through the paper.

II. ISLAND PARADIGM FOR BLACK HOLES

In this section, we evaluate the entanglement entropy of Hawking radiation using the island formula (1). We directly assume that there is an island in black hole spacetime due to the fact that islands are necessary and sufficient to reproduce the Page curve based on the island paradigm. We investigate the behavior of the generalized entropy in the early and late stage respectively. Consequently, we indicate that there no islands at early times and leads to information loss. Then, we focus on the behavior at late times. Finally, We obtain a constraint equation for the spacetime geometry to ensure the appearance of Page curves.

A schematic of the Penrose diagram is shown in Figure.1 [2)]. In order to extend the metric (4) to the left and right wedges, a Kruskal transformation is allowed:

$$\begin{aligned}&\text{Right Wedge}: U \equiv -e^{-\kappa u} = -e^{-\kappa(t-r_\star(r))};\ V \equiv +e^{+\kappa v} = +e^{+\kappa(t+r_\star(r))},\\ &\text{Left Wedge}: U \equiv +e^{-\kappa u} = +e^{-\kappa(t-r_\star(r))};\ V \equiv -e^{+\kappa v} = -e^{+\kappa(t+r_\star(r))},\end{aligned} \tag{5}$$

with the surface gravity κ:

$$\kappa \equiv 2\pi T_H = \frac{1}{2} f'(r_h), \tag{6}$$

where T_H is the Hawking temperature and $'$ represents the derivative with respect to the radial coordinate r, and r_h is denoted the radius of event horizons. Here we set $f(r_h) = 0$. The tortoise coordinates is defined by

$$r_\star(r) = \int^r \frac{1}{f(r)} d\tilde{r}. \tag{7}$$

After the Kruskal transformation, the metric (4) can be recast as:

$$ds^2 = -\Omega^2(r) dU dV + r^2 d\omega_{D-2}^2, \tag{8}$$

with the conformal factor[3)]:

$$\Omega_{\text{BH}}(r) = \frac{\sqrt{f(r)}}{\kappa e^{\kappa r_\star(r)}}, \tag{9a}$$

$$\Omega_{\text{bath}}(r) = \frac{1}{\kappa e^{\kappa r}}. \tag{9b}$$

A. No islands at early times

At first, due to the fact that the explicit expression of the entanglement entropy is complicated in higher-dimensional case, we need to resort to the “s-wave” approximation [30] [4)]. That is to say, we neglect the angular part of the wave function. The expression can be well approximated by the theory of two-dimensional CFT at the low energy limit. In this case, we just need to focus on the radial direction of the metric (8). In addition, we assume that the black holes is a pure state in the beginning of evaporation. Moreover, in this paper, we merely focus on the case in which the cut-off surface is distant from the black hole ($b \gg r_h$) to facilitate subsequent calculations. In the case where the cut-off surface is close to a black hole, one can refer to [30, 48].

In the construction of the no-island, only radiation remains. We can only consider the complementary region of radiation based on the complementary of von Neu-

1) Sometimes the black hole has topological horizons or an inner horizon due to charge and angular momentum, but this does not significantly affect our results. We do not consider these cases in this paper.

2) For the bath region, it refer to half-Minkowski spacetime. We usually assume that bath regions have no gravitational effect, or that the gravitational effect can be ignored. Some studies have considered the gravitational bath [50].

3) We assume that the bath region is a Minkowski patch without gravitational effect. So, for the bath region, we have $f(r) = 1, r_\star = r$ and can obtain the expression (9b) from (9a).

4) Although there exists the massive modes in Kaluza-Klein tower of the spherical part, only the s-wave with zero angular momentum has contribution when the distance is much larger than the coherence length of massive modes.

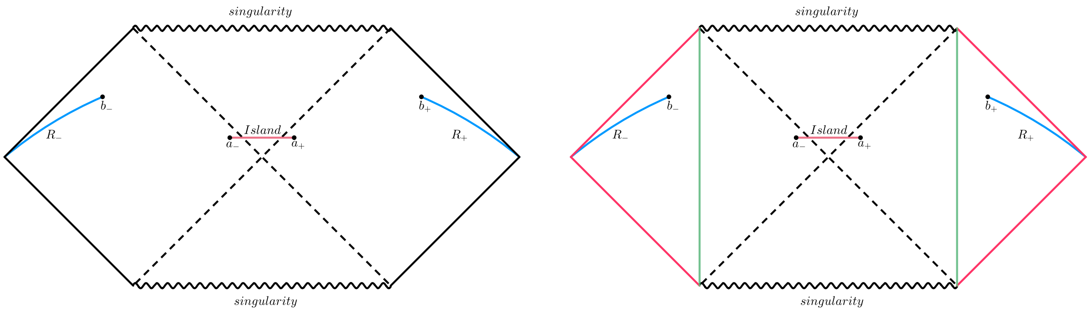

Fig. 1. (color online) The schematic Penrose diagram of black holes (with the single horizon). The radiation regions are denoted by $R\pm$, and their boundaries are the cut-off surfaces. The coordinates of boundaries of the radiation are $b_\pm = (\pm t_b, b)$. The coordinates of the islands boundaries are $a_\pm = (\pm t_a, a)$. On the left, this represents an asymptotically flat black hole. Hawking radiation can naturally diffuse to null infinity. On the right, it represents an asymptotically AdS black hole in thermal equilibrium with the bath (red region). We then impose the transparent boundary condition on the black hole region (black region) [10]. In such way, Hawking radiation can also be collected by observers at space-like infinity.

mann entropy. As a consequence (see Appendix A):

$$S_{\rm Rad} = S(R) = \frac{c}{6}\log\left[d^2(b_-,b_+)\Omega(b_-)\Omega(b_+)\right]$$
$$= \begin{cases} \dfrac{c}{6}\log\left(\dfrac{4f(b)}{\kappa^2}\cosh^2(\kappa t_b)\right), & \text{for asymptotically flat black holes} \\ \dfrac{c}{6}\log\left(\dfrac{4}{\kappa^2}\cosh^2(\kappa t_b)\right), & \text{for asymptotically AdS black holes} \end{cases} \tag{10}$$

where c represents the central charge. In the limit of late times and large distances, we can take the approximation: $\cosh(\kappa t_b) \simeq \frac{1}{2}e^{\kappa t_b}$. Then the above equation is equal to:

$$S_{\rm Rad}(\text{without island}) \simeq \frac{c}{3}\kappa t_b. \tag{11}$$

Apparently, without island construction, the entanglement entropy of radiation grows linearly with time at late times, which leads to the information loss and consistent with Hawking's view. In addition, the result (11) does not depend on the geometry $f(r)$, which implies that the information paradox is always exists.

Next, we turn to the construction with an island to obtain the Page curve. Similarly, referring to the Penrose diagram in Figure 1, we see the entire Cauchy slice is divided into three intervals. For the disconnected union interval $R\cup I$, the expression of the entanglement entropy is converted from (10) (only valid for a single interval) to the following form [93, 94]:

$$S_{\rm bulk}(R\cup I) = \frac{c}{3}\log\left(\frac{d(a_+,a_-)d(b_+,b_-)d(a_+,b_+)d(a_-,b_-)}{d(a_+,b_-)d(a_-,b_+)}\right) = \frac{c}{6}\log\left[16\Omega^2(a)\Omega^2(b)e^{2\kappa(r_\star(a)+r_\star(b))}\cosh^2(\kappa t_a)\cosh^2(\kappa t_b)\right]$$
$$+\frac{c}{3}\log\left[\frac{\cosh[\kappa(r_\star(a)-r_\star(b))]-\cosh[\kappa(t_a-t_b)]}{\cosh[\kappa(r_\star(a)-r_\star(b))]+\cosh[\kappa(t_a+t_b)]}\right], \tag{12}$$

where

$$\Omega^2(a)\Omega^2(b) = \Omega^2_{\rm BH}(a)\Omega^2_{\rm BH}(b) = \frac{f(a)f(b)}{\kappa^4 e^{2\kappa(r_\star(a)+r_\star(b))}}, \quad \text{for asymptotically flat cases} \tag{13a}$$

$$\Omega^2(a)\Omega^2(b) = \Omega^2_{\rm BH}(a)\Omega^2_{\rm bath}(b) = \frac{f(a)}{\kappa^4 e^{2\kappa(r_\star(a)+b)}}. \quad \text{for asymptotically AdS cases} \tag{13b}$$

Accordingly, the generalized entropy read as[1)]:

$$S_{\text{gen}} = \frac{A(a)}{2G_N} + \frac{c}{6}\log\left[\frac{16f(a)f(b)}{\kappa^4}\cosh^2(\kappa t_a)\cosh^2(\kappa t_b)\right] + \frac{c}{3}\log\left[\frac{\cosh[\kappa(r_\star(a)-r_\star(b))]-\cosh[\kappa(t_a-t_b)]}{\cosh[\kappa(r_\star(a)-r_\star(b))]+\cosh[\kappa(t_a+t_b)]}\right], \tag{14}$$

where $A(a)$ is the area of island, which is a positive constant for $D\geq 3$. After here, We default to the dimension $D\geq 3$ to ensure that the area term $A(r)$ is always non-negative and we will discuss the case of 2D black holes specifically later in Appendix B.

At very early times, we assume that $t_a\simeq t_b\simeq 0\ll\kappa b$. Then the generalized entropy becomes:

$$S_{\text{gen}}^{(\text{early})} \simeq \frac{A(a)}{2G_N} + \frac{c}{6}\log\left[\frac{16f(a)f(b)}{\kappa^4}\cosh^2(\kappa t_a)\cosh^2(\kappa t_b)\right]. \tag{15}$$

In order to obtain the QES, we extremize the above expression with respect to a and t_a:

$$\frac{\partial S_{\text{gen}}^{(\text{early})}}{\partial t_a} = \frac{c\kappa}{3}\tanh(\kappa t_a) = 0. \tag{16}$$

The only solution is $t_a = 0$, so the approximation is right. Then the location of QES can be obtained by the following equation:

$$\frac{\partial S_{\text{gen}}^{(\text{early})}}{\partial a} = \frac{A'(a)}{2G_N} + \frac{c}{6}\frac{f'(a)}{f(a)} = 0. \tag{17}$$

We can rewrite this expression to obtain the constraint equation that is satisfied if the island appears at early times:

$$-\frac{3A'(a)f(a)}{f'(a)} = cG_N \sim O(G_N) \ll 1. \tag{18}$$

Here, we assume that the central charge is relatively small: $c\sim O(1)$. Because the area term $A'(a)$ is finite and non-negative. Then there two solutions that satisfy the above equation:

$$f(a)\sim 0, \qquad a\gtrsim r_h, \qquad f'(a)<0, \tag{19a}$$

$$\text{or}\ \ f(a)\sim 0, \qquad a\lesssim r_h, \qquad f'(a)>0. \tag{19b}$$

However, in the region $r\gtrsim r_h$, the expression $f'(a\gtrsim r_h)$ is related to the surface gravity $f'(r_h)=2\kappa>0$ (6). So first solution is not reasonable. While the second solution suggests that the island is located inside the event horizon. In fact, we demonstrate explicitly in Appendix C that the island cannot be inside the event horizon. Therefore, has no physical solution for the constraint equation (18) at early times. We can infer that islands absent at early times, which does not depend on the metric (4).

B. Constraints on the background geometry at late times

By contrast, at large distances and late times, the left wedge and right wedges are significantly separated. To simplify this, we can perform the following approximation [30]:

$$d(a_+,a_-)\simeq d(b_+,b_-)\simeq d(a_+,b_-)\simeq d(a_-,b_+) \gg d(a_+,b_+)\simeq d(a_-,b_-). \tag{20}$$

Then, the entanglement entropy at late times is simplified as:

$$\begin{aligned} S_{\text{gen}}^{(\text{late})} &\simeq \frac{A(a)}{2G_N} + \frac{c}{3}\log[d(a_+,b_+)d(a_-,b_-)] \\ &= \frac{A(a)}{2G_N} + \frac{c}{6}\log\Bigg[\frac{4f(a)f(b)}{\kappa^4} \times\Big(\cosh(\kappa(r_\star(a)-r_\star(b)))-\cosh(\kappa(t_a-t_b))\Big)^2\Bigg]. \end{aligned} \tag{21}$$

In same way, we extremize it with respect to time t_a firstly,

$$\frac{\partial S_{\text{gen}}^{(\text{late})}}{\partial t_a} = -\frac{c}{3}\frac{\kappa\sinh[\kappa(t_a-t_b)]}{\cosh[\kappa(r_\star(a)-r_\star(b))]-\cosh[\kappa(t_a-t_b)]} = 0. \tag{22}$$

The only solution is to set t_a equal to t_b, and then substitute this relation into the original expression and extremize it with respect to a,

$$\begin{aligned} \frac{\partial S_{\text{gen}}^{(\text{late})}}{\partial a} &= \frac{A'(a)}{2G_N} + \frac{c}{6}\left[\frac{f'(a)}{f(a)} + \frac{2\kappa}{f(a)}\coth\left[\frac{\kappa}{2}(r_\star(a)-r_\star(b))\right]\right] = 0 \\ &= \frac{A'(a)}{2G_N} + \frac{cf'(a)}{6f(a)} - \frac{c\kappa}{3f(a)}\left(1+\frac{2}{e^{\kappa x}-1}\right) = 0, \end{aligned} \tag{23}$$

1) Hereafter, we only present the results for asymptotically flat black holes for the sake of simplicity. In order to fit the AdS black holes, one simply set $f(b)=1$ and $r_\star(b)=b$.

where we have used $r'_\star(a)=\dfrac{1}{f(a)}$ and setting $x\equiv r_\star(b)-r_\star(a)$ to simplify the equation. Here we assume that the location of cutoff surface ($r=b$) has the same order with the island ($r=a$), i.e. x is big enough ($e^{\kappa x}-1\gg 0$). Thus the last term in the second line of the above equation does not become a big negative number and cause the equation (23) to have no solution[1]. Following (18), we rewrite this expression as:

$$\frac{3A'(a)f(a)}{4\kappa e^{-\kappa x}+2\kappa-f'(a)}=cG_N\ll 1, \tag{24}$$

Now we make the near horizon limit: $a\simeq r_h$ and obtain:

$$\begin{aligned} f(r) &\simeq f'(r_h)(r-r_h)+O[(r-r_h)^2] \\ &= 2\kappa(r-r_h)+O[(r-r_h)^2], \end{aligned} \tag{25a}$$

$$f'(r)\simeq f'(r_h)+f''(r_h)(r-r_h)+O[(r-r_h)^2] \tag{25b}$$

$$r_\star(r)=\int^r\frac{d\tilde r}{f(r)}\simeq\frac{1}{2\kappa}\log\frac{|r-r_h|}{r_h}. \tag{25c}$$

Substituting these equations into (24), yields the following constraint equation:

$$0<y(a)=\frac{6\kappa A'(a)(a-r_h)}{4\kappa e^{-\kappa r_\star(b)}\sqrt{\frac{(a-r_h)}{r_h}}-f''(r_h)(a-r_h)}=cG_N\ll 1. \tag{26}$$

Firstly, we know that the necessary and sufficient condition for the existence of islands is that the above equation (26) must have a solution. Based on the non-negativity of the area term $A'(a)$ (for $D\geq 3$), we obtain the constraint equation as follows:

$$a>r_h,\qquad f''(r_h)<\frac{4\kappa e^{-\kappa r_\star(b)}}{\sqrt{(a-r_h)r_h}}, \tag{27a}$$

$$a<r_h,\qquad f''(r_h)<\frac{-4\kappa e^{-\kappa r_\star(b)}}{\sqrt{(r_h-a)r_h}}. \tag{27b}$$

These two solutions corresponding to the island located outside or inside the event horizon. However, we show in Appendix B that islands can not be inside the event horizon, so the second solution (27b) should be discarded. Next, when the above condition is satisfied, there must be an island located outside the event horizon:

$$a\simeq r_h+\frac{4c^2G_N^2e^{-2\kappa r_\star(b)}}{9r_h(A'(r_h))^2}+O[(cG_N)^3]. \tag{28}$$

Substituting this location to the constraint equation (27a), we obtain the necessary and sufficient condition for the existence of the island:

$$f''(r_h)<\frac{6\kappa A'(r_h)}{cG_N}\equiv\tilde\alpha. \tag{29}$$

At first sight, one might naively assume that the result (29) is trivial. Since in the semiclassical frame, the Newton constant G_N is sufficient small, then the constraint (29) is easily satisfied. However, on the one hand, we now pay attention to the behavior of entropy at late times (21). At this time, the black hole is at the end of evaporation. The quantum effect dominates and should not be ignored. On the other hand, even at the early stage, the results (29) is trivial only for non-extremal black holes with high temperature ($T_H\sim\kappa\gg 1$). But for near extremal black hole with almost vanishing temperature ($T_H\sim 0$), this constrain need to be treated with great caution. Therefore, the constrain equation (29) is a significant conclusion. Finally, according to the location (28), we obtain the entanglement entropy of radiation at late times:

$$\begin{aligned} S_{\text{Rad}}(\text{with island}) &\simeq \frac{A(r_h)}{2G_N}+O(G_N) \\ &\simeq 2S_{\text{BH}}. \end{aligned} \tag{30}$$

It is what we expect. Recall the result without island (11), the Page time is determined by

$$t_{\text{Page}}=\frac{6S_{\text{BH}}}{c\kappa}=\frac{3S_{\text{BH}}}{c\pi T_H}. \tag{31}$$

Besides, we can also calculate the scrambling time as a by-product. Drawing from the insights of the Hayden-Preskill thought experiment [95], it is posited that an external observer, situated asymptotically relative to the black hole, must patiently await the elapsed duration known as the “scrambling time” before information initially engulfed by the black hole can be retrieved through analyzing the emitted Hawking radiation. In the language of the entanglement wedge reconstruction, the scrambling time corresponds to the time when the information reaches the boundary of island ($r=a$) from the cut-off surface ($r=b$) [11]:

1) Even for cases that the cutoff surface is very close to the island: $b\gtrsim a$, the last term in equation (23) is large than 1 due to the exponential dependence. Therefore the equation (23) always has a solution under reasonable approximation.

$$\begin{aligned} t_{\rm scr} &\equiv {\rm Min}[v(t_b,b)-v(t_a,a)] = r_\star(b)-r_\star(a) \\ &\simeq r_\star(b)-\frac{1}{2\kappa}\log\frac{a-r_h}{r_h} \simeq \frac{1}{2\kappa}\log\frac{A'(r_h)r_h}{cG_N} \\ &\simeq \frac{1}{2\pi T_H}\log S_{\rm BH}, \end{aligned} \tag{32}$$

where t_a, t_b is the time of sending and receiving information, respectively. In the penultimate line, we employed the approximation delineated in equation (25c) to facilitate our calculations and assume that b has the same order of the event horizon $b\sim r_h$. Concludingly, we adopted the established outcome for the four-dimensional scenario, aligning seamlessly with the findings reported in the seminal Hayden-Preskill thought experiment [96, 97], thus ensuring theoretical consistency.

Above all, we protect the unitary by the island formula. In particular, we obtain a sufficient and necessary condition for deriving the Page curve (29). In addition, since the area term $A'(r_h)$ is non-negative, the critical value $\tilde{\alpha}$ is always positive. Therefore, we can further infer that there must exist a Page curve when the following constraint is satisfied. Namely, a sufficient and *unnecessary* for Page curves:

$$f''(r_h+O(G_N))<0. \tag{33}$$

Specifically, the radial coordinate r is confined to a region situated just outside the event horizon, adhering to the condition $r\gtrsim r_h$, reflecting our focus on the immediate vicinity of the horizon through implementation of the near-horizon approximation. The impact of the condition on the result will be discussed in detail in the following section.

III. ISLAND AND QUANTUM FOCUSING CONJECTURE

Up to the present, we calculate the Page curve using the island formula (1). Combing the results of the previous section, we obtain the behavior of entanglement entropy in the entire process of black hole evaporation is

$$S_{\rm Rad} = {\rm Min}\left[\frac{2\pi c}{3}T_H t, 2S_{\rm BH}\right]. \tag{34}$$

In particular, we find that if the constraint condition (29) is satisfied, the Page curve must be reproduced, and there must exist an island outside the event horizon (28). This conclusion is universal and not depend on the explicit form of the metric (4). In this sense, we provide the constraint conditions of spacetime when Page curve is established.

Now in this section, we further study the constraint of Page curves on space-time from the perspective of QFC, and compare the results with (33) that given by the island paradigm. The classical focusing theorem asserts that the expansion θ of the congruence of null geodesic never increases:

$$\frac{d\theta}{d\lambda}\le 0, \tag{35}$$

where λ is the affine parameter. An important application of this theorem is to prove the second law of black holes. For a black hole with area A and entropy $S_{\rm BH}(=\frac{A}{4G_N})$, the expansion θ is defined by:

$$\theta = \frac{1}{A}\frac{dA}{d\lambda}. \tag{36}$$

Then one can infer to the second law: $dS_{\rm BH}\ge 0$.

However, once quantum effects are considered[1)], i.e. the black hole emits Hawking radiation. The second law is violated. For the sake of rationality, this law should be upgraded to the generalized second law. Accordingly, the black hole entropy should be replaced by the generalized entropy: $dS_{\rm gen}\ge 0$. Therefore, the classical focusing theorem is also being extended to the QFC [98, 99], in which the quantum expansion is given by replacing the area in the classical expansion with the generalized entropy (21):

$$\frac{d\Theta}{d\lambda}\le 0, \tag{37}$$

where Θ is the quantum expansion, which can be expressed in terms of the generalized entropy:

$$\Theta = \frac{1}{A}\frac{d}{d\lambda}S_{\rm gen}. \tag{38}$$

Now, we investigate the QFC for the construction with an island. For the entanglement entropy at late times (21), the quantum expansion is written as:

$$\Theta = \frac{1}{A}\frac{d}{d\lambda}S_{\rm gen} = \frac{1}{A}\frac{dv_b}{d\lambda}\left[\frac{\partial S_{\rm gen}}{\partial v_b}+\frac{dv_a}{dv_b}\frac{\partial S_{\rm gen}}{\partial v_a}+\frac{du_a}{dv_b}\frac{\partial S_{\rm gen}}{\partial u_a}\right]. \tag{39}$$

Here we introduce the affine parameter [99]:

1) Even though our metric (4) looks static. However, it is actually in dynamic equilibrium with the external bath. More specifically, the outgoing Hawking radiation is perfectly balanced by the replenished energy flow from the bath (see Penrose diagram Figure.1). Therefore, the area of a black hole is actually change with time, the classical focusing theorem (35) can be violated, and we need to consider the quantum correction (37).

$$d\lambda \equiv -\frac{\partial r(u,v)}{\partial u}dv, \tag{40}$$

for simplicity. Due to the fact that QES makes the entanglement entropy to extremized, which means that:

$$\frac{\partial S_{\text{gen}}}{\partial u_a} = \frac{\partial S_{\text{gen}}}{\partial v_a} = 0. \tag{41}$$

Then we have

$$\begin{aligned}\Theta &= \frac{1}{A}\frac{dv_b}{d\lambda}\frac{\partial S_{\text{gen}}}{\partial v_b} = \frac{1}{A(b)f(b)}\left[\frac{\partial S_{\text{gen}}}{\partial t_b} + f(b)\frac{\partial S_{\text{gen}}}{\partial b}\right] \\ &= \frac{A'(b)}{2A(b)G_N} - \frac{c\kappa}{3A(b)f(b)}\coth \\ &\quad\times\left[\frac{\kappa}{2}((t_a - t_b) + (r_\star(a) - r_\star(b))\right] + \frac{c}{6A(b)f(b)}f'(b) \\ &= \frac{A'(b)}{2A(b)G_N} + \frac{c}{6A(b)f(b)}f'(b) \\ &\quad+ \frac{c\kappa}{3A(b)f(b)}\left(1 + \frac{1}{e^{\kappa(r_\star(b)-r_\star(a))} - 1}\right) > 0.\end{aligned} \tag{42}$$

Here, we have used the relation $t_a = t_b$. Sine the cut-off surface is far from the event horizon, $f(b > r_h \simeq a) > 0$. Therefore, the entanglement entropy is always increasing with the null time v_b and the quantum expansion is positive. Moreover, following the QFC, we obtain the derivative of the quantum expansion as:

$$\begin{aligned}\frac{d\Theta}{d\lambda} &= \frac{d}{d\lambda}\left(\frac{1}{A}\frac{dS_{\text{gen}}}{d\lambda}\right) = \frac{1}{d\lambda}\left(\frac{1}{A}\frac{dv_b}{d\lambda}\frac{\partial S_{\text{gen}}}{\partial v_b}\right) = \frac{1}{f(b)}\frac{d}{dv_b}X \\ &= -\frac{[A'(b)]^2 - A(b)A''(b)}{2G_N A^2(b)} - \frac{c}{6A^2(b)f^2(b)}(Y+Z),\end{aligned} \tag{43}$$

where

$$\begin{aligned}X &= \frac{A'(b)}{A(b)G_N} + \frac{c\kappa}{3A(b)f(b)}\coth\left(\frac{\kappa}{2}\left(r_\star(b) - r_\star(a)\right)\right) \\ &\quad+ \frac{c}{6A(b)}\frac{f'(b)}{f(b)},\end{aligned} \tag{44a}$$

$$Y = f(b)A'(b)\left[2\kappa\coth\left(\frac{\kappa}{2}\left(r_\star(b) - r_\star(a)\right)\right) + f'(b)\right] > 0, \tag{44b}$$

$$\begin{aligned}Z &= A(b)\left[4\kappa^2\frac{e^{\kappa(r_\star(a)-r_\star(b))}}{\left(e^{\kappa(r_\star(a)-r_\star(b))} - 1\right)^2}\right. \\ &\quad\left.+ \frac{2\kappa f'(b)\left(e^{\kappa r_\star(b)} + e^{\kappa r_\star(a)}\right)}{e^{\kappa r_\star(b)} - e^{\kappa r_\star(a)}} + (f'(b))^2 - f(b)f''(b)\right].\end{aligned} \tag{44c}$$

In above calculations, the QES condition (41) is used to be simplified in the second line of the expression (43) to eliminate terms related to u_A and v_a. The first term of equation (43) is related to the area, which is always positive for spherically symmetric black holes because the area term is a linear function of the radius r. Therefore, the only requirement that the QFC theorem must be satisfied is non-vanishing Z. Further, because the location of the cutoff surface $(r = b)$ is artificially selected, if we assume that it has the same order of the horizon: $b \sim r_h \simeq a$. Then the expression Z can be reduced to the following form:

$$\begin{aligned}Z &\sim A(r_h)\left(\frac{4\kappa^2}{e^{\kappa(r_\star(a)-r_\star(b))}} + 2\kappa f'(r_h) + (f'(r_h))^2 - f(b)f''(r_h)\right) \\ &= A(r_h)\left(4\kappa^2\left(\frac{1}{e^{\kappa(r_\star(a)-r_\star(b))}} + 2\right) - f(b)f''(r_h)\right) \\ &\simeq A(r_h)\left(\frac{4\kappa^2 e^{\kappa r_\star(b)}}{\sqrt{\frac{a-r_h}{r_h}}} - f(b)f''(r_h)\right) > 0.\end{aligned} \tag{45}$$

In the first line, we have used $e^{\kappa r_\star(b)} \gg e^{\kappa r_\star(a)} \gg 1$ and $b \sim r_h$. In the last line. we do not expand $f(b)$, because although the cut-off surface has the same order as the horizon $(b \sim r_h)$, its gravitational effect is so small (the asymptotic region of the observer) that it can not be included in the near-horizon region $(r \simeq r_h)$. Therefore, we can acquire that the sufficient and necessary conditions for QFC to be valid by the location of island (28)

$$f''(r_h) < \frac{6\kappa^2 r_h A'(r_h)e^{2\kappa r_\star(b)}}{cG_N f(b)} \equiv \tilde{\beta} = \tilde{\alpha}\cdot\frac{\kappa r_h e^{2\kappa r_\star(b)}}{f(b)}. \tag{46}$$

The explanation of the physical significance here is consistent with the one below equation (29), and this constraint is also a non-trivial result. In particular, based on the non-negativity of $\tilde{\beta}$, we also obtain a sufficient and *unnecessary* for QFC to be hold:

$$f''(r_h) < 0. \tag{47}$$

Compare this constraint and the result from the island paradigm (33), we find that the derived from QFC result (47) contain (33). Namely, the applicability of QFC is wider. Secondly, both are a sufficient and *unnecessary* condition for the Page curve and QFC to be established. Therefore, we can conclude that, a sufficient and *unnecessary* condition for a Page curve for general spacetime (4) to exist and satisfy QFC is the second derivative of the blacken function is negative in the near horizon region. We stress that this conclusion is only valid at the semi-classical level, where the whole spacetime is can be regards as static.

Now we display some physical meaning of the result. As we know, the first derivative of the blacken function $f(r)$ at the event horizon r_h is the Hawking temperature $T_H = \frac{f'(r_h)}{4\pi}$ of black hole. Therefore, its second derivative is related to the heat capacity:

$$C_H \equiv T_H\left(\frac{\partial S_{BH}}{\partial T_H}\right) = T_H\left(\frac{\partial S_{BH}/\partial r_h}{\partial T_H/\partial r_h}\right) \sim f'(r_h)\cdot\frac{A'(r_h)}{f''(r_h)}. \tag{48}$$

Then the positive or negative heat capacity is consistent with $f''(r_h)$. Namely, when the condition (33) is satisfied, the capacity is always negative. Therefore, $f''(r_h)<0$ is *necessary and sufficient* for the heat capacity of a black hole to be negative. Then, we can further summarize secondary conclusions: a black hole with a negative heat capacity must have islands at late times. Moreover, this is also supported by the QFC. We present the results of calculations for some typical black holes in the following Table 1.

IV. DISCUSSION AND CONCLUSION

In summary, we calculate the Page curve from the general static spherical symmetry metric (4) and obtain the entanglement entropy of radiation behaves as (34). We find that island is always outside the event horizon (28). Moreover, we also obtain a *sufficient and necessary condition* (29) for the emergence of islands. This methodology sets a benchmark for employing the island paradigm in Page curve computations. In particular, we use the Liouville black hole [22] as an example to support our conclusion. In addition, we emphasize again that this conclusion is valid only at the *semi-classical* approximation, namely, the metric is static and the approximation is valid (25). When the size of the black hole evaporates to the final stage is small enough, the quantum effect can not be ignored, and the near-horizon approximation (25) may also fail due to the effect of high curvature at the event horizon of black holes at final stage. On the other hand, we follow the perspective of QFC to prove our results. Explicit calculations indicates that QFC is always satisfied when condition (46) is present, which also ensures the validity of behavior of the entanglement entropy at late times (21). In particular, we find that a *common constraint equation* that satisfies conditions (29) and (46) is the condition (33) and (47). They are both self-consistent, which implies the rationality of our calculation.

Therefore, we consider the inverse problem of calculating Page curves and conclude that: When the constraint equation (29) is satisfied, one can always obtain the unitary Page curve from the generic metric (4). While for the view of QFC, the QFC is always be hold under the constraint (46). Our study significantly contributes to the comprehension of black hole evaporation dynamics and the resolution of the information paradox, leveraging the insights from the island paradigm and QFC. Finally, the common constraint of spacetime (47) affirms the universality of Page curves, transcending model-specific restrictions and reinforcing the compatibility of information conservation within the semi-classical gravity framework. This is also suggests that spherically symmetric static black holes with a negative heat capacity must have the island and satisfy QFC theorem.

Our calculation have very broad applicability beyond specific model dependencies. As long as the metric satisfies (4), one can use our calculation to obtain the corresponding island (28), Page curves (29) and the condition for the QFC theorem (46). Therefore, our calculation can be used as a standard procedure to obtain the Page curve (34) and the QFC theorem (37).

In the future, we would like to consider the following points:

Firstly, Our metric (4) is only fit the eternal black hole. Although the information paradox for eternal black holes is more straightforward, we except to acquire universal results from more realistic models of evaporating black holes. When the dynamical black hole is taken account, the back-reaction should be considered seriously [100, 101]. The constraint equations (29) and (46) may be modified. Besides, although the calculations about QFC in this paper are based on a static black hole background, QFC is a more general theorem. The QFC theorem is rarely studied in the dynamic black holes background [99, 102]. Subsequent studies can extend our results ((27a) and (29)) to the evaporating version.

In addition, we only focus on the contribution of "s-wave" and the other modes with angular momentum are omitted. Nevertheless, we still need to be caution when using this reduction. In particular, when the observer is close enough to black holes, this approximation is not valid. Some calculations beyond the s-wave approximation are discussed in [99]. On the other side, In the case of non-spherical symmetry, there is a lack of well-defined conformal transformations, such as the Kruskal coordinate transformation (8). The metric (4) can not be maximally extended to the two-sided geometry form (8). The calculation method presented in this paper is difficult to perform under non-spherical symmetric configuration. Another interesting and beneficial aspect is to contemplate the scenario where the cut-off surface is close to the black hole ($b-r_h \ll 0$). Although the outcomes of the island in this case have been examined in [30, 48], it is worthwhile to investigate whether the QFC theorem holds in this situation.

Finally, although we can infer that black holes with negative heat capacities must exists islands and follows the QFC. We should use this conclusion with caution. For

Table 1. The related results for several black holes. Here we assume that the the cosmological constant Λ, the AdS length ℓ, and the horizon r_h, r_-, r_U have the same order. The Newton constant is small enough: $cG_N \sim O(G_N) \ll 1$. So $\tilde{\alpha} \sim \tilde{\beta} \gg f''(r_h)$ in most cases. In particular, for the Liouville black holes [22], there is no island because of its blacken factor unsatisfied the relation $f''(r_h) < \tilde{\alpha}$. We discuss this special 2D black hole in detail in Appendix B.

Black Hole	$f(r)$	$f''(r_h)$	$\tilde{\alpha}$	$\tilde{\beta}$	Location of Islands/QFC
Callan-Giddings-Harvey-Strominger[18, 19]	$1-e^{-2\lambda(r-r_h)}$	$-4\lambda^2$ $\sim -\mathcal{O}\left(\frac{1}{r_h^2}\right)$	$\frac{12\lambda^2 e^{2\lambda r_h}}{cG_N}$ $\sim \mathcal{O}\left(\frac{1}{r_h^2 G_N}\right)$	$\frac{12\lambda^3 r_h e^{\lambda(2r_h+r_\star(b))}}{cG_N(1-e^{-2\lambda(b-r_h)})}$ $\sim \mathcal{O}\left(\frac{1}{r_h^2 G_N}\right)$	$r_h + \frac{c^2 G_N^2}{12\lambda e^{2\lambda r_h}} e^{\lambda r_\star(b)}$ QFC is satisisfied
Jackiw-Teitelboim[24]	$\frac{r^2-r_h^2}{\ell^2}$	$\frac{2}{\ell^2}$ $\sim \mathcal{O}\left(\frac{1}{r_h^2}\right)$	$\frac{6r_h}{cG_N\ell^3}$ $\sim \mathcal{O}\left(\frac{1}{r_h^2 G_N}\right)$	$\frac{6r_h^3 e^{\frac{r_h r_\star(b)}{\ell^2}}}{cG_N(b^2-r_h^2)\ell^3}$ $\sim \mathcal{O}\left(\frac{1}{r_h^2 G_N}\right)$	$r_h + \frac{c^2 G_N^2 \ell}{6} e^{\frac{r_h}{\ell^2} r_\star(b)}$ QFC is satisisfied
Bañados-Teiteboim-Zanelli[79]	$\frac{r^2-r_h^2}{\ell^2}$	$\frac{2}{\ell^2}$ $\sim \mathcal{O}\left(\frac{1}{r_h^2}\right)$	$\frac{12\pi r_h}{cG_N\ell^2}$ $\sim \mathcal{O}\left(\frac{1}{r_h G_N}\right)$	$\frac{12\pi r_h^3 e^{-\frac{r_h r_\star(b)}{\ell^3}}}{cG_N(b^2-r_h^2)\ell^2}$ $\sim \mathcal{O}\left(\frac{1}{r_h G_N}\right)$	$r_h + \frac{c^2 G_N^2}{12\pi} e^{\frac{r_h}{\ell^2} r_\star(b)}$ QFC is satisfied
Rotating Bañados-Teiteboim-Zanelli[47]	$\frac{(r^2-r_h^2)(r^2-r_-^2)}{r^2\ell^2}$	$\frac{2(3r_-^2+r_h^2)}{r_h^2\ell^2}$ $\sim \mathcal{O}\left(\frac{1}{r_h^2}\right)$	$\frac{12\pi(r_h^2-r_-^2)}{cG_N r_h \ell^2}$ $\sim \mathcal{O}\left(\frac{1}{r_h G_N}\right)$	$\frac{12\pi(r_h^2-r_-^2)^2 b^2 e^{\frac{(r_h^2-r_-^2)r_\star(b)}{r_h\ell^2}}}{cG_N r_h \ell^2(b^2-r_h^2)(b^2-r_-^2)}$ $\sim \mathcal{O}\left(\frac{1}{r_h G_N}\right)$	$r_h + \frac{c^2 G_N^2}{18\pi^2 r_h} e^{2\left(\frac{r_h^2-r_-^2}{r_h\ell^2}\right) r_\star(b)}$ QFC is satisfied
Schwarzschild [30, 48]	$1-\frac{r_h}{r}$	$-\frac{2}{r_h^2}$ $\sim -\mathcal{O}\left(\frac{1}{r_h^2}\right)$	$\frac{24\pi}{cG_N}$ $\sim \mathcal{O}\left(\frac{1}{G_N}\right)$	$\frac{12\pi b e^{\frac{r_\star(b)}{2r_h}}}{cG_N(b-r_h)}$ $\sim \mathcal{O}\left(\frac{1}{G_N}\right)$	$r_h + \frac{c^2 G_N^2}{48\pi r_h} e^{\frac{r_\star(b)}{2r_h}}$ QFC is satisfied bn
Schwarzschild-AdS[28]	$1-\frac{r_0}{r}+\frac{r^2}{\ell^2}$	$\frac{2}{\ell^2}-\frac{2r_0}{r_h^2}$ $\sim \mathcal{O}\left(\frac{1}{r_h^2}\right)$	$\frac{24\pi r_h\left(\frac{2r_h}{\ell^2}+\frac{r_0}{r_h^2}\right)}{cG_N}$ $\sim \mathcal{O}\left(\frac{1}{G_N}\right)$	$\frac{12\pi r_h^2\left(\frac{2r_h}{\ell^2}+\frac{r_0}{r_h^2}\right)^2}{cG_N\left(1+\frac{b^2}{\ell^2}-\frac{r_0}{b}\right)}\times$ $\exp\left(\left(\frac{r_h}{\ell^2}+\frac{r_0}{2r_h^2}\right)r_\star(b)\right)$ $\sim \mathcal{O}\left(\frac{1}{G_N}\right)$	$r_h + \frac{c^2 G_N^2}{48\pi r_h} e^{\left(\frac{r_h}{\ell^2}+\frac{r_0}{2r_h^2}\right) r_\star(b)}$ QFC is satisisfied
Schwarzschild-dS[77]	$\frac{(r_U-r)(r-r_h)(r+r_h+r_U)}{\ell^2 r}$	$-\frac{2(r_h^2+r_h r_U+r_U^2)}{\ell^2 r_h^2}$ $\sim -\mathcal{O}\left(\frac{1}{r_h^2}\right)$	$\frac{24\pi(r_U-r_h)(2r_h+r_U)}{cG_N\ell^2}$ $\sim \mathcal{O}\left(\frac{1}{G_N}\right)$	$\frac{12\pi b(r_h-r_U)^2(2r_h+r_U)^2}{cG_N(b-r_h)(r_U-b)(b+r_h+r_U)\ell^2}\times$ $\exp\left(\frac{(r_U-r_h)(2r_h+r_U)r_\star(b)}{2r_h\ell^2}\right)$ $\sim \mathcal{O}\left(\frac{1}{G_N}\right)$	$r_h + \frac{c^2 G_N^2 e^{\frac{(r_c-r_h)(r_h-r_u)}{6r_h\ell^2}}}{48\pi r_h e^{-r_\star(b)}}$ QFC is satisisfied
Reissner-Nordström[54]	$\left(1-\frac{r_h}{r}\right)\left(1-\frac{r_-}{r}\right)$	$\frac{4r_- 2r_h}{r_h^3}$ $\sim \mathcal{O}\left(\frac{1}{r_h^2}\right)$	$\frac{24\pi\left(1-\frac{r_-}{r_h}\right)}{cG_N}$ $\sim \mathcal{O}\left(\frac{1}{G_N}\right)$	$\frac{12\pi b^2(r_h-r_-)^2}{cG_N(b-r_-)(b-r_h)r_h^2}\times$ $\exp\left(\frac{(r_h-r_-)r_\star(b)}{2r_h^2}\right)$ $\sim \mathcal{O}\left(\frac{1}{G_N}\right)$	$r_h + \frac{c^2 G_N^2}{48\pi r_h} e^{\left(\frac{r_h-r_-}{2r_h^2}\right) r_\star(b)}$ QFC is satisisfied
Reissner-Nordström-AdS[85, 90]	$\frac{(r-r_-)(r-r_h)}{\ell^2 r}\times$ $\left(\ell^2+r^2+r_-^2+r_h^2+r_h r_-\right)$	$\frac{2(r_-^3+r_-^2 r_h+2r_h^3)}{r_h^2\ell^2}+$ $\frac{2r_-\ell}{r_h^2\ell^2}$ $\sim \mathcal{O}\left(\frac{1}{r_h}\right)$	$\frac{24\pi(r_h-r_-)}{cG_N\ell^2}\times$ $\left(r_-^2+r_- r_h+2r_h^2+\ell^2\right)$ $\sim \mathcal{O}\left(\frac{r_h}{G_N}\right)$	$\frac{12\pi b((r_-^2+r_- r_h+2r_h^2+\ell^2)^2}{cG_N(b-r_-)(b-r_h)\ell^2}\times$ $\frac{r_h-r_-)^2}{(b^2+r_-^2+r_h r_-+r_h^2+\ell^2)}\times$ $\exp\left(\frac{(r_h-r_-)r_\star(b)}{2r_h\ell^2}\right)$ $\sim \mathcal{O}\left(\frac{r_h}{G_N}\right)$	$r_h + \frac{c^2 G_N^2}{144\pi r_h^3} e^{r_\star(b)}\times$ $e^{\left(\frac{(3r_h^2+2r_h r_-+r_-^2)(r_h-r_-)}{r_h\ell^2}\right)}$ QFC is satisfied
Higher-dimensional Schwarzschild[30]	$1-\left(\frac{r_h}{r}\right)^{d-3}$	$-\frac{6-5d+d^2}{r_h^2}$ $\sim -\mathcal{O}\left(\frac{1}{r_h^2}\right)$	$\frac{3(d-3)(d-2)r_h^{d-4}\omega_{d-2}}{cG_N}$ $\sim \mathcal{O}\left(\frac{r_h^{d-4}}{G_N}\right)$	$\frac{3(d-3)^2(d-2)r_h^{d-4}\omega_{d-2}}{2cG_N\left(1-\left(\frac{r_h}{b}\right)^{d-3}\right)}\times$ $\exp\left(\frac{(d-3)r_\star(b)}{2r_h}\right)$ $\sim \mathcal{O}\left(\frac{r_h^{d-4}}{G_N}\right)$	$r_h + \frac{cG_N e^{\frac{d-3}{2r_h}} r_\star(b)}{12\omega_{d-2} r_h^{d-3}}$ QFC is satisfied
Liouville[22]	$1-e^{-2\sqrt{\lvert C\rvert}r}$	$-4\lvert C\rvert$ $\sim -\mathcal{O}\left(\frac{1}{r_h^2}\right)$	$-\frac{96\lvert C\rvert}{cG_N}$ $\sim -\mathcal{O}\left(\frac{1}{r_h^2 G_N}\right)$	0	**No Island** QFC is satisfied

the more general case, such as axially symmetric black holes or black holes with topological phase transitions, we still need to treat constraint equations (29) and (46) strictly. There may be better physical explanations in the future, and our superficial discussion here may provide some possible references. We end our discussion here.

ACKNOWLEDGMENTS

We would thank to Shuyi Lin, Yuqi-Lei, Ruidong Zhu and Xiaokai He for discussions related to island rule.

APPENDIX A: ENTANGLEMENT ENTROPY IN CURVED SPACETIME

In this appendix, we briefly give the expression of entanglement entropy in curved black hole background and discuss what should be pay attention to when using them.

Initially, different from the 2D simple case, the expression of entanglement entropy in the 4D scenario is complicated and has an area-like divergent. Namely, the entropy for matter fields has following expression:

$$S_{\text{bulk}}(R\cup I)=\frac{\text{Area}(\partial I)}{\epsilon^2}+S_{\text{bulk}}^{\text{finite}}(R\cup I), \tag{A1}$$

where ϵ is the cutoff, which is dominates the area-like divergent term. Then we can absorb this term by renormalizing the Newton constant:

$$\frac{1}{4G_N^{(r)}}\equiv\frac{1}{4G_N}+\frac{1}{\epsilon^2}. \tag{A2}$$

As consequent, we can replace the corresponding part of island formula (1) with $G_N^{(r)}$ and $S_{\text{bulk}}^{\text{finite}}(R\cup I)$, respectively, to yield a finite contribution of the entanglement entropy. Thus, the entanglement entropy in 4D spacetime is

$$S_{\text{Rad}}=\text{Min}\left\{\text{Ext}\left[\frac{\text{Area}(\partial I)}{4G_N^{(r)}}+S_{\text{bulk}}^{(\text{finite})}(R\cup I)\right]\right\}. \tag{A3}$$

Secondly, due to the s-wave approximation, the renormalized von Neumann entropy in vacuum CFT_2 in *flat* spacetime $ds^2=-dx^+dx^-$ (with the light cone coordinate $x^\pm=t\pm r$) is [93, 94]

$$S_{\text{bulk}}(A\cup B)=\frac{c}{3}\log(d_{AB}), \tag{A4}$$

with

$$d_{AB}\equiv\sqrt{[x^+(A)-x^+(B)][x^-(B)-x^-(A)]}, \tag{A5}$$

in the geodesic distance between points A and B in flat metric. In order to apply the formula (51) to the *curved* spacetime, we need to perform the Wely transformation into curved 2D metric $ds^2_{2D}=-\Omega^2(x^+,x^-)dx^+dx^-$ [9]. After the Weyl transformation, we finally obtain the entanglement entropy in general 2D spacetime is [19]:

$$S_{\text{bulk}}(A\cup B)=\frac{c}{6}\log\left[d^2(A,B)\Omega(A)\Omega(B)\right]\Big|_{t_\pm=0}. \tag{A6}$$

For the higher-dimensional case, we can still calculate the entanglement entropy by this formula in a similar way as in (8). For the 3D case, we just replace the area term to the length of the system, and for the 2D case, we replace the area term in terms of the dilaton.

APPENDIX B: THE CASE OF 2D BLACK HOLES

In this appendix, we display the details of the result for 2D black holes in Table 1. The bulk action for the 2D gravity can be written in the following form [18, 19, 22]:

$$I_{\text{bulk}}=\frac{1}{16\pi G_N}\int d^2x\sqrt{-g}\left[\Phi\big(R+K(\Phi)(\nabla\Phi)^2-2V(\Phi)\big)\right], \tag{B1}$$

where $K(\Phi)$ and $V(\Phi)$ are

$$K=0,\qquad V=-\lambda^2,\qquad \text{for JT gravity} \tag{B2a}$$

$$K=\frac{1}{\Phi^2},\qquad V=-2\lambda^2,\qquad \text{for CGHS model} \tag{B2b}$$

$$K=0,\qquad V=-2\lambda^2e^{B\Phi}\qquad \text{for Liouville model} \tag{B2c}$$

where λ determines the length of the cosmological constant, $B>0$ is a constant. We can obtain the vacuum black hole solutions by solving the equations of motions from the action (55). In the Schwarzschild gauge, the vacuum black hole metric are:

$$ds^2=-f(r)dt^2+f(r)^{-1}dr^2, \tag{B3}$$

where

$$f(r)=\frac{r^2-r_h^2}{\ell^2},\qquad \text{for JT gravity} \tag{B4a}$$

$$f(r) = 1 - e^{-2\lambda(r - r_h)}, \qquad \text{for CGHS model} \tag{B4b}$$

$$f(r) = 1 - e^{-2\sqrt{|C|}\cdot r}. \qquad \text{for Liouville model} \tag{B4c}$$

Here ℓ sets the AdS length, $C<0$ is a constant. For the case of JT and CGHS model, we can easily calculate and find that their blacken factors $f(r)$ (58a) (58b) satisfy the constraint equation (29), and then obtain the correct results in Table 1. However, in the case of the Liouville black hole, there is no island due to its special properties. Now we study this situation in detail.

A key property of a Liouville black hole is that its area term $A(r)$ is *negative*. For the Liouville solution (58c), it can be prove that the time t has a periodicity along imaginary axis. We introduce a new coordinate by $r = \frac{\sqrt{|C|}}{2}R^2$. Then the metric (58c) in the Euclidean time $t = i\tau$ near the event horizon $r = 0$ has the following form:

$$ds^2 = R^2 d(\sqrt{|C|}\tau)^2 + dR^2. \tag{B5}$$

Therefore, the Euclidean time has a periodicity of $\frac{2\pi}{\sqrt{|C|}}$. Then we obtain the Hawking temperature: $T_H = \frac{\sqrt{|C|}}{2\pi}$. The expression of the dilaton ϕ in (t,r) coordinates are given by

$$\phi = -\frac{2}{B}\sqrt{|C|}r - \frac{1}{B}\log\frac{\lambda^2 B}{C}. \tag{B6}$$

The mass of black hole is

$$M = \frac{2\sqrt{|C|}}{B\pi}. \tag{B7}$$

It is obviously that B must be positive. Therefore, the full restrictions for the parameters are

$$C<0, \qquad B>0, \qquad \lambda^2<0. \tag{B8}$$

Combine the Hawking temperature, we can find that: $T = \frac{B}{4}M$. Finally, based on the first law of thermodynamics, we obtain the Bekenstein-Hawking entropy as:

$$S_{\rm BH} = \int\frac{dM}{T} = \frac{4}{B}\log M - \frac{2}{B}\log\left(\frac{-4\lambda^2}{B\pi^2}\right). \tag{B9}$$

Therefore, the black hole entropy is related to the dilaton at the event horizon:

$$S_{\rm BH} = 2\phi_H = 2\phi(r=0). \tag{B10}$$

The area and its derivative of Liouville black hole are given by:

$$\begin{aligned} A(r) &= -\frac{16}{B}\sqrt{|C|}r - \frac{8}{B}\log\frac{\lambda^2 B}{C}, \\ A'(r) &= -\frac{16}{B}\sqrt{|C|}. \end{aligned} \tag{B11}$$

Because of the negative value of $A'(r_h)$, the blacken factor (58c) does not satisfy the constraint equation (29), so the Liouville black hole does not have an island. Then, we prove the validity of our results in Table 1. For more information about Liouville black holes see [22].

APPENDIX C: NO ISLAND INSIDE THE EVENT HORIZON

In this appendix, we prove islands cannot exist inside the event horizon. In the Section II, We obtain the location of the island by extremizing the generalized entropy (18), (26). In addition to the solution where the island is outside the event horizon, there is also a solution where the island is inside the event horizon (19b), (27b). The crux of the matter is that our results are based on the Penrose diagram Figure 1, where the island is was already assumed to be outside the event horizon, so this solution should be discarded. Now, we give the corresponding explicit calculation. In this case, the correct Penrose diagram as follows:

In this construction, the island is located on the top wedge of Figure 2. So the corresponding Kruskal coordinate is different from (5). We redefine the Kruskal coordinate as follows:

$$\begin{aligned} &\text{Top Wedge}: \ U \equiv +e^{\kappa u} = +e^{\kappa(t - r_\star(r))}, \\ &V \equiv +e^{\kappa v} = +e^{\kappa(t + r_\star(r))}. \end{aligned} \tag{C1}$$

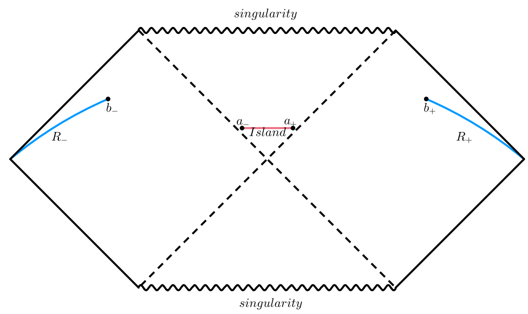

Fig. 2. (color online) The Penrose diagram in which the island is assumed to be inside the event horizon.

The generalized entropy at late times is given by substituting the coordinate of island $a_{\pm} = (\pm t_a, a)$, which is

$$S_{\text{gen}} \simeq \frac{A(a)}{2G_N} + \frac{c}{3}\log\left[d(a_+,b_+)d(a_-,b_-)\right] = \frac{A(a)}{2G_N} + \frac{c}{6}\log\left[\frac{f(a)f(b)}{\kappa^4}\left(1+e^{\kappa(r_\star(a)+r_\star(b)-t_a-t_b)}\right)\left(-1+e^{\kappa(r_\star(a)+r_\star(b)+t_a+t_b)}\right)\times\right.$$
$$\left. -e^{-2\kappa(2r_\star(a)+r_\star(b))}\left(e^{\kappa(r_\star(a)-t_a)}+e^{\kappa(r_\star(b)-t_b)}\right)\left(e^{\kappa(r_\star(a)+t_a)}+e^{\kappa(r_\star(b)-t_b)}\right)\right]. \tag{C2}$$

Extrimizing this equation with respect to a:

$$\frac{\partial S_{\text{gen}}}{\partial a} = \frac{1}{6}\left(\frac{3A'(a)}{G_N}+\right.$$
$$\left.\frac{c\left(\kappa\operatorname{csch}[\kappa(r_\star(a)+r_\star(b))]\operatorname{sech}[\frac{1}{2}\kappa(r_\star(a)+r_\star(b))]\left(\cosh[\frac{1}{2}\kappa(3r_\star(a)+r_\star(b))]+3\left(\sinh[\frac{1}{2}\kappa(3r_\star(a)+r_\star(b))]\right)+f'(a)\right)}{f(a)}\right)=0. \tag{C3}$$

This equation has no solution, namely, there is no island for this construction. Therefore, we can prove that islands cannot exist inside the event horizon, and the solutions (19b) and (27b) is not physical and should be rejected.

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