Spatial covariant gravity with two degrees of freedom in the presence of an auxiliary scalar field: Perturbation analysis*

Zhi-Chao Wang (王志超) Xian Gao (高显)†

School of Physics and Astronomy, Sun Yat-sen University, Zhuhai 519082, China

Abstract: We investigate a class of gravity theories respecting only spatial covariance, termed spatially covariant gravity, in the presence of an auxiliary scalar field. We examine the conditions on the Lagrangian required to eliminate scalar degrees of freedom, allowing only two tensorial degrees of freedom to propagate. Instead of strict constraint analysis, in this paper, we employ the perturbation method and focus on the necessary conditions to evade the scalar mode at the linear order in perturbations around a cosmological background. Beginning with a general action and solving the auxiliary perturbation variables in terms of a would-be dynamical scalar mode, we derive the condition to remove its kinetic term, thus ensuring that no scalar mode propagates. As an application of the general condition, we study a polynomial-type Lagrangian as a concrete example, in which all monomials are spatially covariant scalars containing two derivatives. We find that the auxiliary scalar field is essential, and new terms in the Lagrangian are allowed. Our analysis provides insights into constructing gravity theories with two degrees of freedom in the extended framework of spatially covariant gravity.

Keywords: modified gravity, cosmological perturbation theory, gravitational theory

DOI: 10.1088/1674-1137/ad47a9

I. INTRODUCTION

Recently, interest has been increasing in forming gravitational theories that differ from general relativity (GR) while propagating the same dynamical degrees of freedom of GR. These degrees of freedom occur as tensorial gravitational waves in a homogeneous and isotropic background, which we thus dub the two tensorial degrees of freedom (TTDOFs). However, this seems impossible because Lovelock's theorem [1, 2] claims that GR is the unique metric theory that respects spacetime diffeomorphism and has second-order equations of motion in four dimensions, which guarantee the propagation of only TTDOFs. Nevertheless, several theories that propagate only TTDOFs have been proposed by bypassing the conditions of Lovelock's theorem.

Theories of TTDOFs can be traced back to the so-called Cuscuton theory [3], which was extensively studied [4–16] and was further extended in the framework of higher derivative scalar-tensor theory [17, 18]. Previously, such a type of theory has also been discussed in a subclass of Hořava gravity [19, 20] (see also [21]), whose relation with the Cuscuton theory has been analyzed [22]. Symmetries underlying these theories, including the relation between the TTDOFs (i.e., transverse and traceless graviton), spacetime symmetry [23, 24], and "scalarless" symmetry [25, 26], have also been investigated[1)].

An alternative framework of theories propagating only two degrees of freedom is the minimally modified gravity (MMG) [28]. The so-called type-II MMG, which differs from GR in a vacuum, is of particular importance [29, 30] and includes the original Cuscuton theory, minimal theory of massive gravity [31, 32] (see also [33–36]), and $4D$ Einstein-Gauss-Bonnet gravity as special cases. MMG has been studied extensively [37–52], and has been generalized in the phase space by introducing auxiliary constraints [53, 54].

A general framework of spatially covariant gravity (SCG) theories that propagate TTDOFs with an extra scalar mode was proposed in [55, 56]. It was further generalized by introducing a dynamical lapse function [57, 58] and nonmetricity [59]. SCG can be considered an alternative approach to generalizing scalar-tensor theories [60–63], and it includes Hořava gravity [64–66] and the effective field theory of inflation/dark energy [67–71] as special cases. SCG has been applied to study cosmology

Received 23 March 2024; Accepted 6 May 2024; Published online 7 May 2024

* Supported by the Natural Science Foundation of China (11975020)

† E-mail: gaoxian@mail.sysu.edu.cn

1) Another attempt was proposed in [27] by taking the $D\to 4$ limit of Einstein-Gauss-Bonnet gravity in higher dimensions, which yields an arguable theory propagating only the massless graviton.

and gravitational waves [72–77]. Within the SCG framework, conditions on the Lagrangian such that only the TTDOFs are propagating were explored in [78, 79] and with a dynamical lapse function in [80]. Owing to the complexity of the conditions, a systematic construction of the TTDOF Lagrangians remains to be performed. Nevertheless, a concrete Lagrangian was found in [78], which has been applied in cosmology [10, 81–84].

Instead of solving the complicated variation-differential equations for the TTDOF conditions in [78], an alternative and more straightforward approach is the perturbation method. The concept is to focus on the kinetic term of the would-be dynamical scalar mode. By eliminating its kinetic term order by order in a perturbative expansion around some (typically the cosmological) background, we can fix terms in the Lagrangian order by order. In particular, such a perturbative approach would stop at some finite order owing to the finite number of TTDOF conditions. This perturbation method was used in [17] to extend the Cuscuton theory in the framework of the higher derivative scalar-tensor theory. In the SCG framework, it was successfully applied in [58] to derive a Lagrangian with a single scalar mode in the presence of a dynamical lapse function and in [79] to build a Lagrangian without any scalar mode, i.e., propagating only TTDOFs.

In this study, we apply the perturbation method to SCG in the presence of an auxiliary scalar field and determine the conditions such that the theory propagates only TTDOFs. The concept of SCG with an auxiliary scalar field was proposed in [85], in which the scalar field was assumed to have a spacelike gradient and thus loses its kinetic term in the so-called "spatial gauge." In this study, we neglect this assumption and simply treat it as an auxiliary field. Introducing the auxiliary scalar field can be considered a generalization of the Lagrange multiplier method in modified gravity [86–88]. The existence of the auxiliary field leads to constraints on the dynamical degrees of freedom, resulting in a novel approach to modifying gravity. In the presence of an auxiliary scalar field, we may have much more room to tune the theory to satisfy the TTDOF conditions and may expect to find a novel class of TTDOF theories. This paper is devoted to this problem.

The remainder of this paper is organized as follows. In Sec. II, we describe our model of spatially covariant gravity with an auxiliary scalar field and present the scalar perturbations around a cosmological background. In Sec. III, we derive the quadratic action for the scalar perturbations. By solving the auxiliary perturbation variables and focusing on the kinetic term of the unwanted scalar mode, we derive the degeneracy condition to evade the scalar mode. In Sec. IV, we use the polynomial-type Lagrangian of $d=2$, where d is the total number of derivatives, as a concrete example and derive the explicit Lagrangian that propagates TTDOFs. In Sec. V, we summarize our results.

II. PERTURBATION METHOD

A. Model and cosmological perturbations

Our starting point is the general action of spatial covariant gravity coupled with a non-dynamical scalar field [85]:

$$S=\int \mathrm{d}t\mathrm{d}^3x N\sqrt{h}\mathcal{L}\left(h_{ij},K_{ij},R_{ij},N,\phi,\mathrm{D}_i\right), \tag{1}$$

where h_{ij} is the spatial metric, N is the lapse function, R_{ij} is the spatial Ricci tensor, and K_{ij} is the extrinsic curvature defined by

$$K_{ij}=\frac{1}{2N}\left(\partial_t h_{ij}-\mathrm{D}_iN_j-\mathrm{D}_jN_i\right), \tag{2}$$

where N_i is the shift vector, and D_i is the spatial derivative adapted to h_{ij}. Scalar field ϕ has no kinetic term and thus acts as an auxiliary variable. We emphasize that although this auxiliary scalar field was originally motivated from a scalar field with a spacelike gradient in the so-called spatial gauge, here (and also in the analysis in [85]), we do not make such an assumption and merely treat it as an auxiliary field, i.e., without any time derivative terms. Ref. [85] showed that the theory in (1) generally propagates three degrees of freedom, i.e., two tensorial and one scalar degrees of freedom. The aim of this study is to find the condition on the Lagrangian to eliminate the scalar mode.

The most strict approach to analyzing the number of degrees of freedom is the constraint analysis, either in the Hamiltonian or Lagrangian formalism. In the absence of auxiliary scalar field ϕ, the exact conditions for the TTDOFs were found in [78]. These conditions are variation-differential equations for the Lagrangian, which are difficult (if not impossible) to solve in order to produce the concrete Lagrangian systematically. To address this problem, a perturbation method was employed in [79] (and previously in [58] with a dynamical lapse function). The advantage of the perturbation method is that one can determine the conditions on the Lagrangian order by order in perturbations, which by themselves are much simpler than the fully nonlinear conditions.

In this study, we use this perturbation method to determine the conditions on action (1) such that no scalar mode propagates. To be precise, our task is to study the perturbations of action (1) around a cosmological background and eliminate the scalar modes at the linear order in perturbations. The difference between this study and previous reports [58, 79] is that we first perform the perturbation analysis on the general form of the Lagrangian instead of a specific Lagrangian (e.g., in the polynomial form). Therefore, the resulting conditions are generic and can be applied to more general Lagrangians.

To eliminate the unwanted scalar degree of freedom, we focus solely on the scalar perturbations. We consider perturbations around a Friedmann-Robertson-Walker background. After the residual gauge freedom of spatial diffeomorphism is fixed, the Arnowitt-Deser-Misner (ADM) variables are parameterized as follows:

$$N=\bar{N}\mathrm{e}^{A}, \tag{3}$$

$$N_i=\bar{N}a\partial_i B, \tag{4}$$

$$h_{ij}=a^2 e^{2\zeta}\delta_{ij}, \tag{5}$$

where a is the scale factor. Here, A, B, and ζ are the scalar perturbations of the metric. The auxiliary scalar field is perturbed as usual:

$$\phi=\bar{\phi}+\delta\phi. \tag{6}$$

All perturbation variables are set to zero at the background level, i.e., $A=B=\zeta=\delta\phi=0$. $\bar{N}$ and $\bar{\phi}$ are the background values of the lapse function and scalar field, respectively, which are functions of time. Note that because of the loss of time reparametrization invariance, we cannot fix the background value of lapse function $\bar{N}$ to unity. In this paper, we use the same shorthand as that in [72]:

$$\dot{X}=\frac{1}{\bar{N}}\frac{\partial X}{\partial t}, \tag{7}$$

and

$$f'=\bar{N}\left.\frac{\partial f}{\partial N}\right|_{N=\bar{N}},\quad f''=\bar{N}^2\left.\frac{\partial^2 f}{\partial N^2}\right|_{N=\bar{N}}, \tag{8}$$

etc.

To obtain the action up to the second order in perturbations, we first expand all the quantities in the following form:

$$Q=Q^{(0)}+\delta Q^{(1)}+\delta Q^{(2)}, \tag{9}$$

where $Q^{(0)}$ represents the background value of any quantity Q, and $\delta Q^{(n)}$ represents the n-th order perturbation of Q. Precisely, we obtain

$$\phi^{(0)}=\bar{\phi},\quad \delta\phi^{(1)}=\delta\phi, \tag{10}$$

$$N^{(0)}=\bar{N},\quad \delta N^{(1)}=\bar{N}A,\quad \delta N^{(2)}=\frac{1}{2}\bar{N}A^2, \tag{11}$$

$$N^{i(0)}=0,\quad \delta N^{i(1)}=\bar{N}a^{-1}\partial^i B, \tag{12}$$

$$h_{ij}^{(0)}=a^2\delta_{ij},\quad \delta h_{ij}^{(1)}=2\zeta a^2\delta_{ij},\quad \delta h_{ij}^{(2)}=2\zeta^2 a^2\delta_{ij}, \tag{13}$$

$$K_{ij}^{(0)}=a\dot{a}\delta_{ij},\quad \delta K_{ij}^{(1)}=a\dot{a}\delta_{ij}(-A+2\zeta)+a^2\delta_{ij}\dot{\zeta}-a\partial_i\partial_j B, \tag{14}$$

$$\begin{aligned}\delta K_{ij}^{(2)}=&\,a\dot{a}\delta_{ij}\left(-2\zeta A+\frac{1}{2}A^2+2\zeta^2\right)+a^2\delta_{ij}\dot{\zeta}(2\zeta-A)\\&+aA\partial_i\partial_j B+a\left(\partial_i\zeta\partial_j B+\partial_j\zeta\partial_i B-\delta_{ij}\partial^l B\partial_l\zeta\right),\end{aligned} \tag{15}$$

$$R_{ij}^{(0)}=0,\quad \delta R_{ij}^{(1)}=-\partial_i\partial_j\zeta-\delta_{ij}\partial^2\zeta, \tag{16}$$

and

$$\delta R_{ij}^{(2)}=\partial_i\zeta\partial_j\zeta-\delta_{ij}\partial_l\zeta\partial^l\zeta. \tag{17}$$

In the above and henceforth, spatial indices are raised and lowered by δ^{ij} and δ_{ij}, respectively. Note that only linear-order terms are considered for the perturbations of auxiliary scalar field ϕ and shift vector N^i. With the above results, we can proceed to calculate the perturbations of the action in the subsequent sections.

B. Background equations of motion

We first derive the background equations of motion, which can be used to simplify calculations at higher orders. Hence, we must obtain linear-order action S_1:

$$S_1=\int \mathrm{d}t\mathrm{d}^3x\left.\frac{\delta S}{\delta\Phi_I}\right|_{\mathrm{bg}}\delta\Phi_I^{(1)}, \tag{18}$$

where

$$\Phi_I\equiv\left\{h_{ij},K_{ij},R_{ij},N,\phi\right\} \tag{19}$$

stands for the basic quantities in our model (1), and "$Q|_{\mathrm{bg}}$" denotes the background value of any quantity Q. We obtain

$$S_1=\int \mathrm{d}t\mathrm{d}^3x\left[\bar{N}a^3\left.\frac{\delta\mathcal{L}}{\delta K_{ij}}\right|_{\mathrm{bg}}\delta K^{(1)}_{ij}+\left(\bar{N}\frac{1}{2}a\delta^{ij}\bar{\mathcal{L}}+\bar{N}a^3\left.\frac{\delta\mathcal{L}}{\delta h_{ij}}\right|_{\mathrm{bg}}\right)\times\delta h^{(1)}_{ij}+\bar{N}a^3\left.\frac{\delta\mathcal{L}}{\delta\phi}\right|_{\mathrm{bg}}\delta\phi+\left(a^3\bar{\mathcal{L}}+\bar{N}a^3\left.\frac{\delta\mathcal{L}}{\delta N}\right|_{\mathrm{bg}}\right)\delta N^{(1)}+\bar{N}a^3\left.\frac{\delta\mathcal{L}}{\delta R_{ij}}\right|_{\mathrm{bg}}\delta R^{(1)}_{ij}\right],\tag{20}$$

where $\bar{\mathcal{L}}$ represents the background Lagrangian density. We emphasize that in (20) (and henceforth), when making the variation, h_{ij}, K_{ij}, R_{ij}, N, and ϕ are treated as independent variables[1)]. Note that at the background level, $N=\bar{N}$, $N_i=0$, $h_{ij}=a^2\delta_{ij}$, and $\phi=\bar{\phi}$. Substituting (10)–(17) into (20), after some simplifications, we obtain

$$S_1[A,\zeta,\delta\phi]=\int \mathrm{d}t\mathrm{d}^3x\bar{N}a^3\left(\mathcal{E}_A A+\mathcal{E}_\zeta\zeta+\mathcal{E}_{\delta\phi}\delta\phi\right),\tag{21}$$

where we define

$$\mathcal{E}_A=\bar{\mathcal{L}}+\bar{N}\left.\frac{\delta\mathcal{L}}{\delta N}\right|_{\mathrm{bg}}-a^2H\delta_{ij}\left.\frac{\delta\mathcal{L}}{\delta K_{ij}}\right|_{\mathrm{bg}},\tag{22}$$

with $H\equiv\dot{a}/a\equiv\partial_t a/(a\bar{N})$,

$$\mathcal{E}_\zeta=3\bar{\mathcal{L}}+a^2\delta_{ij}\left[2\left.\frac{\delta\mathcal{L}}{\delta h_{ij}}\right|_{\mathrm{bg}}-3H\left.\frac{\delta\mathcal{L}}{\delta K_{ij}}\right|_{\mathrm{bg}}-\frac{1}{\bar{N}}\frac{\partial}{\partial t}\left(\left.\frac{\delta\mathcal{L}}{\delta K_{ij}}\right|_{\mathrm{bg}}\right)-\frac{\dot{\bar{N}}}{\bar{N}}\left.\frac{\delta\mathcal{L}}{\delta K_{ij}}\right|_{\mathrm{bg}}\right],\tag{23}$$

and

$$\mathcal{E}_{\delta\phi}=\left.\frac{\delta\mathcal{L}}{\delta\phi}\right|_{\mathrm{bg}}.\tag{24}$$

Requiring the linear variation of the background action to be vanishing, i.e., $S_1[A,\zeta,\delta\phi]=0$, we obtain

$$\mathcal{E}_A=0,\quad \mathcal{E}_\zeta=0,\quad \mathcal{E}_{\delta\phi}=0,\tag{25}$$

which are the background equations of motion. At this point, note that because SCG explicitly breaks the time diffeomorphism, all three of these equations are independent. This differs from generally covariant theories, in which only two of the three equations are independent owing to the time diffeomorphism.

III. QUADRATIC ACTION AND THE DEGENERACY CONDITION

The quadratic action for the perturbations receives contributions from both first-order and second-order quantities. Its general form is given by

$$S_2[A,B,\zeta,\delta\phi]=\int \mathrm{d}t\mathrm{d}^3x\left(\left.\frac{\delta S}{\delta\Phi_I}\right|_{\mathrm{bg}}\delta\Phi^{(2)}_I+\frac{1}{2}\left.\frac{\delta^2 S}{\delta\Phi_I\delta\Phi_J}\right|_{\mathrm{bg}}\delta\Phi^{(1)}_I\delta\Phi^{(1)}_J\right),\tag{26}$$

where (henceforth, we suppress subscript "bg" for simplicity)

$$\int \mathrm{d}t\mathrm{d}^3x\frac{\delta S}{\delta\Phi_I}\delta\Phi^{(2)}_I=\int \mathrm{d}t\mathrm{d}^3x\left[\frac{\delta\left(N\sqrt{h}\mathcal{L}\right)}{\delta K_{ij}}\delta K^{(2)}_{ij}+\frac{\delta\left(N\sqrt{h}\mathcal{L}\right)}{\delta h_{ij}}\delta h^{(2)}_{ij}+\frac{\delta\left(N\sqrt{h}\mathcal{L}\right)}{\delta N}\delta N^{(2)}+\frac{\delta\left(N\sqrt{h}\mathcal{L}\right)}{\delta R_{ij}}\delta R^{(2)}_{ij}\right],\tag{27}$$

and

$$\int \mathrm{d}t\mathrm{d}^3x\frac{\delta^2 S}{\delta\Phi_I\delta\Phi_J}\delta\Phi^{(1)}_I\delta\Phi^{(1)}_J=\int \mathrm{d}t\mathrm{d}^3x\Bigg[2\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta h_{mn}\delta K_{ij}}\delta h^{(1)}_{mn}\delta K^{(1)}_{ij}+\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta h_{mn}\delta h_{ij}}\delta h^{(1)}_{mn}\delta h^{(1)}_{ij}+\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta K_{mn}\delta K_{ij}}\delta K^{(1)}_{mn}\delta K^{(1)}_{ij}$$
$$+2\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta\phi\delta K_{ij}}\delta\phi\delta K^{(1)}_{ij}+2\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta N\delta K_{ij}}\delta N^{(1)}\delta K^{(1)}_{ij}+2\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta\phi\delta h_{ij}}\delta\phi\delta h^{(1)}_{ij}$$

1) For example, although K_{ij}, R_{ij}, etc., contain h_{ij} implicitly, when evaluating $\frac{\delta\mathcal{L}}{\delta h_{ij}}$, only the explicit functional dependence of $\mathcal{L}$ on h_{ij} will be taken into account.

$$+2\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta N\delta h_{ij}}\delta N^{(1)}\delta h^{(1)}_{ij}+2\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta\phi\delta N}\delta\phi\delta N^{(1)}+\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta\phi^2}\delta\phi\delta\phi+\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta N^2}\delta N^{(1)}\delta N^{(1)}+2\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta h_{mn}\delta R_{ij}}\delta h^{(1)}_{mn}\delta R^{(1)}_{ij}$$
$$+2\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta K_{mn}\delta R_{ij}}\delta K^{(1)}_{mn}\delta R^{(1)}_{ij}+2\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta N\delta R_{ij}}\delta N^{(1)}\delta R^{(1)}_{ij}+\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta R_{mn}\delta R_{ij}}\delta R^{(1)}_{mn}\delta R^{(1)}_{ij}+2\frac{\delta^2\left(N\sqrt{h}\mathcal{L}\right)}{\delta\phi\delta R_{ij}}\delta\phi\delta R^{(1)}_{ij}\Big]. \tag{28}$$

The quadratic action for the scalar perturbations takes the general form

$$S_2\left[A,B,\zeta,\delta\phi\right]=\int \mathrm{d}t\mathrm{d}^3x\bar{N}a^3\left(A\hat{O}_{AA}A+B\hat{O}_{BB}B+\zeta\hat{O}_{\zeta\zeta}\zeta+\delta\phi\hat{O}_{\phi\phi}\delta\phi+A\hat{O}_{AB}B+A\hat{O}_{A\zeta}\zeta+A\hat{O}_{A\phi}\delta\phi\right.$$
$$\left.+\delta\phi\hat{O}_{\phi\zeta}\zeta+\delta\phi\hat{O}_{\phi B}B+\zeta\hat{O}_{\zeta B}B+\dot{\zeta}\hat{O}_{\dot{\zeta}A}A+\dot{\zeta}\hat{O}_{\dot{\zeta}B}B+\dot{\zeta}\hat{O}_{\dot{\zeta}\zeta}\zeta+\dot{\zeta}\hat{O}_{\dot{\zeta}\phi}\delta\phi+\dot{\zeta}\hat{O}_{\dot{\zeta}\dot{\zeta}}\dot{\zeta}\right), \tag{29}$$

where $\hat{O}_{XY}$ represents the operator involving perturbation variables X and Y (e.g., $\hat{O}_{A\phi}$ represents the operator involving A and $\delta\phi$). Generally, $\hat{O}_{XY}$ is time dependent and may contain spatial derivatives. By substituting (10)−(17) into (27) and (28) and after some tedious manipulations, we obtain

$$\hat{O}_{AA}=\frac{1}{2}\bar{\mathcal{L}}-\frac{1}{2}a\dot{a}\delta_{ij}\frac{\delta\mathcal{L}}{\delta K_{ij}}+\frac{1}{2}a^2\dot{a}^2\delta_{mn}\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta K_{ij}}+\frac{3}{2}\bar{N}\frac{\delta\mathcal{L}}{\delta N}-\bar{N}a\dot{a}\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta N\delta K_{ij}}+\frac{1}{2}\bar{N}^2\frac{\delta^2\mathcal{L}}{\delta N^2}, \tag{30}$$

$$\hat{O}_{BB}=\frac{1}{2}a^2\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta K_{ij}}\partial_m\partial_n\partial_i\partial_j, \tag{31}$$

$$\hat{O}_{\zeta\zeta}=\frac{9}{2}\bar{\mathcal{L}}+8a^2\delta_{ij}\frac{\delta\mathcal{L}}{\delta h_{ij}}+2a^4\delta_{mn}\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta h_{mn}\delta h_{ij}}+8a\dot{a}\delta_{ij}\frac{\delta\mathcal{L}}{\delta K_{ij}}+4a^3\dot{a}\delta_{ij}\delta_{mn}\frac{\delta^2\mathcal{L}}{\delta h_{mn}\delta K_{ij}}+2a^2\dot{a}^2\delta_{mn}\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta K_{ij}}$$
$$+\frac{1}{2}\frac{\delta^2\mathcal{L}}{\delta R_{mn}\delta R_{ij}}\left(\partial_i\partial_j\partial_m\partial_n+2\delta_{ij}\partial_m\partial_n\partial^2+\delta_{ij}\delta_{mn}\partial^4\right)-2a\dot{a}\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta R_{ij}}\delta_{mn}\partial_i\partial_j-2a^2\delta_{mn}\frac{\delta^2\mathcal{L}}{\delta h_{mn}\delta R_{ij}}\left(\partial_i\partial_j+\delta_{ij}\partial^2\right)$$
$$-2\frac{\delta\mathcal{L}}{\delta R_{ij}}\left(2\partial_i\partial_j+\delta_{ij}\partial^2\right), \tag{32}$$

$$\hat{O}_{\phi\phi}=\frac{1}{2}\frac{\delta^2\mathcal{L}}{\delta\phi^2}, \tag{33}$$

$$\hat{O}_{AB}=\left(-\bar{N}a\frac{\delta^2\mathcal{L}}{\delta N\delta K_{ij}}+a^2\dot{a}\delta_{mn}\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta K_{ij}}\right)\partial_i\partial_j, \tag{34}$$

$$\hat{O}_{A\zeta}=3\bar{\mathcal{L}}+2a^2\delta_{ij}\frac{\delta\mathcal{L}}{\delta h_{ij}}+3\bar{N}\frac{\delta\mathcal{L}}{\delta N}+2\bar{N}a^2\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta N\delta h_{ij}}-3a\dot{a}\delta_{ij}\frac{\delta\mathcal{L}}{\delta K_{ij}}-2a^3\dot{a}\delta_{ij}\delta_{mn}\frac{\delta^2\mathcal{L}}{\delta h_{mn}\delta K_{ij}}+2\bar{N}a\dot{a}\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta N\delta K_{ij}}$$
$$-2a^2\dot{a}^2\delta_{mn}\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta K_{ij}}-\left(\frac{\delta\mathcal{L}}{\delta R_{ij}}+\bar{N}\frac{\delta^2\mathcal{L}}{\delta N\delta R_{ij}}\right)\left(\partial_i\partial_j+\delta_{ij}\partial^2\right)+a\dot{a}\delta_{mn}\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta R_{ij}}\partial_i\partial_j, \tag{35}$$

$$\hat{O}_{A\phi}=\frac{\delta\mathcal{L}}{\delta\phi}+\bar{N}\frac{\delta^2\mathcal{L}}{\delta\phi\delta N}-a\dot{a}\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta\phi\delta K_{ij}}, \tag{36}$$

$$\hat{O}_{\phi\zeta}=3\frac{\delta\mathcal{L}}{\delta\phi}+2a^2\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta\phi\delta h_{ij}}+2a\dot{a}\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta\phi\delta K_{ij}}-\frac{\delta^2\mathcal{L}}{\delta\phi\delta R_{ij}}\left(\partial_i\partial_j+\delta_{ij}\partial^2\right), \tag{37}$$

$$\hat{O}_{\phi B}=-a\frac{\delta^2\mathcal{L}}{\delta\phi\delta K_{ij}}\partial_i\partial_j, \tag{38}$$

$$\hat{O}_{\zeta B}=-5a\frac{\delta\mathcal{L}}{\delta K_{ij}}\partial_i\partial_j+a\frac{\delta\mathcal{L}}{\delta K_{ij}}\delta_{ij}\partial^2-2a^3\delta_{mn}\frac{\delta^2\mathcal{L}}{\delta h_{mn}\delta K_{ij}}\partial_i\partial_j-2a^2\dot{a}\delta_{mn}\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta K_{ij}}\partial_i\partial_j+a\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta R_{ij}}\left(\partial_i\partial_j\partial_m\partial_n+\delta_{ij}\partial^2\partial_m\partial_n\right), \tag{39}$$

$$\hat{O}_{\dot{\zeta}A}=-a^3\dot{a}\delta_{mn}\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta K_{ij}}+\bar{N}a^2\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta N\delta K_{ij}}, \tag{40}$$

$$\hat{O}_{\dot{\zeta}B}=-a^3\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta K_{ij}}\delta_{mn}\partial_i\partial_j, \tag{41}$$

$$\hat{O}_{\dot{\zeta}\zeta}=5a^2\delta_{ij}\frac{\delta\mathcal{L}}{\delta K_{ij}}+2a^4\delta_{ij}\delta_{mn}\frac{\delta^2\mathcal{L}}{\delta h_{mn}\delta K_{ij}}+2a^3\dot{a}\delta_{mn}\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta K_{ij}}-a^2\delta_{mn}\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta R_{ij}}\left(\partial_i\partial_j+\delta_{ij}\partial^2\right), \tag{42}$$

$$\hat{O}_{\dot{\zeta}\phi}=a^2\delta_{ij}\frac{\delta^2\mathcal{L}}{\delta\phi\delta K_{ij}}, \tag{43}$$

and

$$\hat{O}_{\dot{\zeta}\dot{\zeta}}=\frac{1}{2}a^4\frac{\delta^2\mathcal{L}}{\delta K_{mn}\delta K_{ij}}\delta_{mn}\delta_{ij}. \tag{44}$$

By deriving the above results, we utilize integrations by parts to simplify the expressions.

By varying the quadratic action (29), one can easily obtain the equations of motion for perturbation variables $\{A,B,\delta\phi,\zeta\}$ as

$$2\hat{O}_{AA}A+\hat{O}_{AB}B+\hat{O}_{A\phi}\delta\phi+\hat{O}_{A\zeta}\zeta+\hat{O}_{\dot{\zeta}A}\dot{\zeta}=0, \tag{45}$$

$$\hat{O}_{AB}A+2\hat{O}_{BB}B+\hat{O}_{\phi B}\delta\phi+\hat{O}_{\zeta B}\zeta+\hat{O}_{\dot{\zeta}B}\dot{\zeta}=0, \tag{46}$$

$$\hat{O}_{A\phi}A+\hat{O}_{\phi B}B+2\hat{O}_{\phi\phi}\delta\phi+\hat{O}_{\phi\zeta}\zeta+\hat{O}_{\dot{\zeta}\phi}\dot{\zeta}=0, \tag{47}$$

$$\begin{aligned}&\frac{1}{\bar{N}}\Big[2\partial_t\left(a^3\hat{O}_{\dot{\zeta}\dot{\zeta}}\dot{\zeta}\right)+\partial_t\left(a^3\hat{O}_{\dot{\zeta}\phi}\delta\phi\right)\\&+\partial_t\left(a^3\hat{O}_{\dot{\zeta}B}B\right)+\partial_t\left(a^3\hat{O}_{\dot{\zeta}A}A\right)+\partial_t\left(a^3\hat{O}_{\dot{\zeta}\zeta}\zeta\right)\Big]\\&-a^3\left(2\hat{O}_{\zeta\zeta}\zeta+\hat{O}_{A\zeta}A+\hat{O}_{\phi\zeta}\delta\phi+\hat{O}_{\zeta B}B\right)=0.\end{aligned} \tag{48}$$

Equations (45)−(48) show that A, B, and $\delta\phi$ are all auxiliary variables, as the equations of motion do not contain their first-order time derivative terms. In the case of a non-degenerate coefficient matrix for perturbation variables A, B, and $\delta\phi$, i.e.,

$$\Omega\equiv\det\begin{pmatrix}2\hat{O}_{AA}&\hat{O}_{AB}&\hat{O}_{A\phi}\\\hat{O}_{AB}&2\hat{O}_{BB}&\hat{O}_{\phi B}\\\hat{O}_{A\phi}&\hat{O}_{\phi B}&2\hat{O}_{\phi\phi}\end{pmatrix}\neq0, \tag{49}$$

they can be formally solved from their equations of motion (45)−(47) as

$$\begin{aligned}A=&-\frac{1}{2\Omega}\Big[\left(\hat{O}_{A\zeta}\hat{O}^2_{\phi B}-\hat{O}_{AB}\hat{O}_{\phi B}\hat{O}_{\phi\zeta}-\hat{O}_{A\phi}\hat{O}_{\phi B}\hat{O}_{\zeta B}+2\hat{O}_{A\phi}\hat{O}_{BB}\hat{O}_{\phi\zeta}+2\hat{O}_{AB}\hat{O}_{\phi\phi}\hat{O}_{\zeta B}-4\hat{O}_{A\zeta}\hat{O}_{BB}\hat{O}_{\phi\phi}\right)\zeta\\&+\left(\hat{O}_{\dot{\zeta}A}\hat{O}^2_{\phi B}-\hat{O}_{AB}\hat{O}_{\phi B}\hat{O}_{\dot{\zeta}\phi}-\hat{O}_{A\phi}\hat{O}_{\phi B}\hat{O}_{\dot{\zeta}B}+2\hat{O}_{A\phi}\hat{O}_{BB}\hat{O}_{\dot{\zeta}\phi}+2\hat{O}_{AB}\hat{O}_{\phi\phi}\hat{O}_{\dot{\zeta}B}-4\hat{O}_{\dot{\zeta}A}\hat{O}_{BB}\hat{O}_{\phi\phi}\right)\dot{\zeta}\Big],\end{aligned} \tag{50}$$

$$B=-\frac{1}{2\varOmega}\Big[\Big(\hat{O}_{\zeta B}\hat{O}^2_{A\phi}-\hat{O}_{AB}\hat{O}_{A\phi}\hat{O}_{\phi\zeta}-\hat{O}_{A\phi}\hat{O}_{\phi B}\hat{O}_{A\zeta}+2\hat{O}_{\phi B}\hat{O}_{AA}\hat{O}_{\phi\zeta}+2\hat{O}_{AB}\hat{O}_{\phi\phi}\hat{O}_{A\zeta}-4\hat{O}_{\zeta B}\hat{O}_{AA}\hat{O}_{\phi\phi}\Big)\zeta$$
$$+\Big(\hat{O}_{\dot{\zeta}B}\hat{O}^2_{A\phi}-\hat{O}_{AB}\hat{O}_{A\phi}\hat{O}_{\dot{\zeta}\phi}-\hat{O}_{A\phi}\hat{O}_{\phi B}\hat{O}_{\dot{\zeta}A}+2\hat{O}_{\phi B}\hat{O}_{AA}\hat{O}_{\dot{\zeta}\phi}+2\hat{O}_{AB}\hat{O}_{\phi\phi}\hat{O}_{\dot{\zeta}A}-4\hat{O}_{\dot{\zeta}B}\hat{O}_{AA}\hat{O}_{\phi\phi}\Big)\dot{\zeta}\Big], \tag{51}$$

$$\delta\phi=-\frac{1}{2\varOmega}\Big[\Big(\hat{O}_{\phi\zeta}\hat{O}^2_{AB}-\hat{O}_{AB}\hat{O}_{A\zeta}\hat{O}_{\phi B}-\hat{O}_{AB}\hat{O}_{A\phi}\hat{O}_{\zeta B}+2\hat{O}_{\phi B}\hat{O}_{AA}\hat{O}_{\zeta B}+2\hat{O}_{BB}\hat{O}_{A\phi}\hat{O}_{A\zeta}-4\hat{O}_{BB}\hat{O}_{AA}\hat{O}_{\phi\zeta}\Big)\zeta$$
$$+\Big(\hat{O}_{\dot{\zeta}\phi}\hat{O}^2_{AB}-\hat{O}_{AB}\hat{O}_{\dot{\zeta}A}\hat{O}_{\phi B}-\hat{O}_{AB}\hat{O}_{A\phi}\hat{O}_{\dot{\zeta}B}+2\hat{O}_{\phi B}\hat{O}_{AA}\hat{O}_{\dot{\zeta}B}+2\hat{O}_{BB}\hat{O}_{A\phi}\hat{O}_{\dot{\zeta}A}-4\hat{O}_{BB}\hat{O}_{AA}\hat{O}_{\dot{\zeta}\phi}\Big)\dot{\zeta}\Big], \tag{52}$$

with

$$\varOmega=\hat{O}^2_{A\phi}\hat{O}_{BB}+\hat{O}_{AA}\hat{O}^2_{\phi B}+\hat{O}^2_{AB}\hat{O}_{\phi\phi}-\hat{O}_{AB}\hat{O}_{A\phi}\hat{O}_{\phi B}-4\hat{O}_{AA}\hat{O}_{BB}\hat{O}_{\phi\phi}. \tag{53}$$

Although we assume that $\varOmega\neq 0$ to ensure that auxiliary variables A, B, and $\delta\phi$ are solvable (at least formally), note that even if $\varOmega\neq 0$ is not satisfied, we can still obtain conditions to eliminate the unwanted scalar degree of freedom. We will show this explicitly through a detailed analysis of a specific example in the next section.

Finally, by substituting solutions (50)−(52) into the equation of motion for ζ (48), we obtain an equation of motion for single variable ζ. If no further conditions are assumed, the effective equation of motion for ζ will contain its second-order time derivative term, $\ddot{\zeta}$, indicating a propagating degree of freedom carried by ζ. For our purpose, to eliminate the scalar degree of freedom, we must ensure that ζ is not propagating, at least at linear order in a cosmological background. To this end, we must make the coefficient of $\ddot{\zeta}$ vanish. After some manipulations, the coefficient of $\ddot{\zeta}$ is found to be

$$\varDelta=4\hat{O}_{\dot{\zeta}\dot{\zeta}}\Big(\hat{O}^2_{A\phi}\hat{O}_{BB}+\hat{O}_{AA}\hat{O}^2_{\phi B}+\hat{O}^2_{AB}\hat{O}_{\phi\phi}-\hat{O}_{AB}\hat{O}_{A\phi}\hat{O}_{\phi B}-4\hat{O}_{AA}\hat{O}_{BB}\hat{O}_{\phi\phi}\Big)-\hat{O}^2_{\phi B}\hat{O}^2_{\dot{\zeta}A}-\hat{O}^2_{A\phi}\hat{O}^2_{\dot{\zeta}B}-\hat{O}^2_{AB}\hat{O}^2_{\dot{\zeta}\phi}+2\hat{O}_{A\phi}\hat{O}_{\phi B}\hat{O}_{\dot{\zeta}A}\hat{O}_{\dot{\zeta}B}$$
$$+2\hat{O}_{AB}\hat{O}_{\phi B}\hat{O}_{\dot{\zeta}A}\hat{O}_{\dot{\zeta}\phi}+2\hat{O}_{AB}\hat{O}_{A\phi}\hat{O}_{\dot{\zeta}B}\hat{O}_{\dot{\zeta}\phi}+4\hat{O}_{BB}\hat{O}_{\phi\phi}\hat{O}^2_{\dot{\zeta}A}+4\hat{O}_{AA}\hat{O}_{BB}\hat{O}^2_{\dot{\zeta}\phi}+4\hat{O}_{AA}\hat{O}_{\phi\phi}\hat{O}^2_{\dot{\zeta}B}$$
$$-4\hat{O}_{AA}\hat{O}_{\phi B}\hat{O}_{\dot{\zeta}B}\hat{O}_{\dot{\zeta}\phi}-4\hat{O}_{AB}\hat{O}_{\phi\phi}\hat{O}_{\dot{\zeta}A}\hat{O}_{\dot{\zeta}B}-4\hat{O}_{A\phi}\hat{O}_{BB}\hat{O}_{\dot{\zeta}A}\hat{O}_{\dot{\zeta}\phi}. \tag{54}$$

Therefore, we require that

$$\varDelta=0, \tag{55}$$

which is a necessary condition to eliminate the scalar degree of freedom (i.e., the TTDOF conditions), at least at the linear order in perturbations around a cosmological background.

IV. THEORY OF $d=2$

The analysis and conditions obtained in the above section are general but quite formal. In this section, we apply condition (54) to a concrete model as an example. According to the classification of SCG monomials [60, 61], we consider a polynomial-type Lagrangian built of monomials of $d=2$, where d is the total number of derivatives. Precisely, the action is given by

$$S_2=\int \mathrm{d}t\mathrm{d}^3x N\sqrt{h}(\mathcal{L}-\Lambda), \tag{56}$$

with

$$\mathcal{L}=c_1K_{ij}K^{ij}+c_2K^2+c_3R+c_4a_ia^i+d_1\mathrm{D}_i\phi\mathrm{D}^i\phi+d_2a_i\mathrm{D}^i\phi, \tag{57}$$

where c_i and d_i are the general functions of N and ϕ, and Λ is the cosmological constant. Additionally, acceleration a_i is defined as $a_i=\partial_i\ln N$. We make the assumption that both ϕ and N are homogeneous and isotropic at the background, i.e., $\bar{\phi}=\bar{\phi}(t)$, $\bar{N}=\bar{N}(t)$. To simplify our calculation, we denote

$$\mathcal{L}_c=\mathcal{L}-\Lambda. \tag{58}$$

We will observe that non-vanishing cosmological constant Λ, which may be time dependent, is necessary to have a cosmological background solution.

A. Degeneracy condition

As both ϕ and N depend only on time at the background level, the background values of each quantity in the Lagrangian are given by

$$\bar{K}_{ij}=a\dot{a}\delta_{ij},\quad \bar{K}^{ij}=Ha^{-2}\delta^{ij},\quad \bar{K}=3H, \tag{59}$$

and

$$\bar{R}=0, \quad \bar{a}_i=0, \quad \partial_i\bar{\phi}=0. \tag{60}$$

The background value of the Lagrangian is

$$\bar{\mathcal{L}}_c=3H^2b-\Lambda, \tag{61}$$

where we define

$$b=c_1+3c_2 \tag{62}$$

for short. Thus, the non-vanishing derivatives of Lagrangian $\mathcal{L}_c$ with respect to various quantities are

$$\frac{\delta\mathcal{L}_c}{\delta N}=\frac{3}{\bar{N}}H^2b', \tag{63}$$

$$\frac{\delta\mathcal{L}_c}{\delta K_{ij}}=2Ha^{-2}\delta^{ij}b, \tag{64}$$

$$\frac{\delta\mathcal{L}_c}{\delta\phi}=3H^2\frac{\partial b}{\partial\phi}, \tag{65}$$

$$\frac{\delta^2\mathcal{L}_c}{\delta N^2}=\frac{3}{\bar{N}^2}H^2b''-\frac{2}{\bar{N}^2}c_4\partial^2, \tag{66}$$

$$\frac{\delta^2\mathcal{L}_c}{\delta K_{mn}\delta K_{ij}}=2a^{-4}\left(\frac{1}{2}c_1\left(\delta^{im}\delta^{jn}+\delta^{in}\delta^{jm}\right)+c_2\delta^{ij}\delta^{mn}\right), \tag{67}$$

$$\frac{\delta^2\mathcal{L}_c}{\delta\phi^2}=3H^2\frac{\partial^2b}{\partial\phi^2}-2d_1\partial^2, \tag{68}$$

$$\frac{\delta^2\mathcal{L}_c}{\delta N\delta K_{ij}}=\frac{2}{\bar{N}}Ha^{-2}\delta^{ij}b', \tag{69}$$

$$\frac{\delta^2\mathcal{L}_c}{\delta\phi\delta N}=3H^2\frac{\partial^2b}{\partial\phi\partial N}-\frac{d_2}{\bar{N}}\partial^2, \tag{70}$$

and

$$\frac{\delta^2\mathcal{L}_c}{\delta\phi\delta K_{ij}}=2Ha^{-2}\delta^{ij}\frac{\partial b}{\partial\phi}. \tag{71}$$

We do not present derivatives with respect to R_{ij}, as they are irrelevant to the degeneracy analysis. According to (61)−(71), the background equations of motion (25) are explicitly given by

$$\mathcal{E}_A=3H^2(-b+b')-\Lambda=0, \tag{72}$$

$$\mathcal{E}_\zeta=-9H^2b-6\dot{H}b-\frac{6H}{\bar{N}}\dot{\bar{N}}b-3\Lambda=0, \tag{73}$$

$$\mathcal{E}_{\delta\phi}=3H^2\frac{\partial b}{\partial\phi}=0, \tag{74}$$

From (74), to have a homogeneous and isotropic background, b defined in (62) must be a function of lapse function N only. Background equations of motion (72)−(74) can significantly simplify the calculation of the quadratic action for the perturbations in the following.

By substituting (61)−(71) into (30)−(44), we can evaluate operators $\hat{O}_{XY}$ that are relevant to eliminating the scalar mode, which are given by

$$\hat{O}_{AA}=\frac{3}{2}H^2(2b-2b'+b'')-c_4\partial^2, \tag{75}$$

$$\hat{O}_{BB}=a^{-2}(c_1+c_2)\partial^4, \tag{76}$$

$$\hat{O}_{\phi\phi}=\frac{3}{2}H^2\frac{\partial^2b}{\partial\phi^2}-d_1\partial^2, \tag{77}$$

$$\hat{O}_{AB}=2Ha^{-1}(-b'+b)\partial^2, \tag{78}$$

$$\hat{O}_{A\phi}=3H^2\frac{\partial(b'-b)}{\partial\phi}-d_2\partial^2, \tag{79}$$

$$\hat{O}_{\phi B}=-2Ha^{-1}\frac{\partial b}{\partial\phi}\partial^2, \tag{80}$$

$$\hat{O}_{\dot{\zeta}A}=6H(-b+b'), \tag{81}$$

$$\hat{O}_{\dot{\zeta}B}=-2a^{-1}b\partial^2, \tag{82}$$

$$\hat{O}_{\dot{\zeta}\phi}=6H\frac{\partial b}{\partial\phi}, \tag{83}$$

and

$$\hat{O}_{\dot{\zeta}\dot{\zeta}}=3b. \tag{84}$$

Subsequently, by substituting (75)−(84) into the determinant of coefficient matrix (53), and recalling that $\frac{\partial b}{\partial\phi}=0$ from background equation of motion (74), we find

$$\varOmega = a^{-2}(c_1+c_2)\left(d_2^2-4d_1c_4\right)\partial^8 + 2H^2a^{-2}\left[3(2b-2b'+b'')(c_1+c_2)-2(-b'+b)^2\right]d_1\partial^6. \tag{85}$$

Degeneracy condition (55) is expressed as

$$0 \equiv \varDelta = 8a^{-2}bc_1\left(d_2^2-4c_4d_1\right)\partial^8 + 48H^2a^{-2}d_1c_1\left(2bb'-2b'^2+bb''\right)\partial^6. \tag{86}$$

In deriving (85) and (86), we use background equations of motion (72)−(74) to simplify the expressions.

In the following, we discuss two cases based on whether $\varOmega$ is vanishing or not.

B. Case $\varOmega \neq 0$

When $\varOmega \neq 0$, all auxiliary variables A, B, $\delta\phi$ are solvable; thus, the degeneracy condition $\varDelta = 0$ holds. According to (86), to eliminate the unwanted scalar degree of freedom, we require the coefficients of operators ∂^6 and ∂^8 to be vanishing simultaneously, i.e.,

$$\left(2bb'-2b'^2+bb''\right)c_1d_1 = 0, \tag{87}$$

and

$$bc_1\left(d_2^2-4c_4d_1\right) = 0, \tag{88}$$

which are two constraints among c_1, c_4, d_1, d_2, and b. Before solving these two equations, let us conduct a brief analysis.

First, (87) and (88) are trivially satisfied if $b \equiv c_1+3c_2 = 0$. However, this conflicts with background equation of motion (72), which leads to $\Lambda = 0$. As mentioned above, a non-vanishing (and actually positive) cosmological constant is required to obtain a cosmological background solution. Therefore, we must require $b \neq 0$. Second, (87) and (88) are also satisfied if $c_1 = 0$. However, although we focus only on the scalar perturbations in this study, further calculation reveals that only $c_1K_{ij}K^{ij}$ will contribute to the kinetic term for the tensor perturbations (i.e., the gravitational waves). As a result, vanishing c_1 would also eliminate the gravitational waves, which is unacceptable based on the observation of gravitational waves. Thus, we must also require $c_1 \neq 0$. After considering that $b, c_1 \neq 0$, (88) holds only if $d_2^2-4c_4d_1 = 0$. Finally, if $d_1 = 0$, it follows that $d_2 = 0$. From (85), this will lead to $\varOmega = 0$, which conflicts with our requirement $\varOmega \neq 0$. Thus, we must have $d_1 \neq 0$.

Based on the above arguments, equations (87)−(88) hold only if

$$d_2^2-4c_4d_1 = 0, \tag{89}$$

$$2bb'-2b'^2+bb'' = 0, \tag{90}$$

are satisfied. The general solutions for c_4 and b to (89)−(90) are

$$c_4 = \frac{d_2^2}{4d_1}, \tag{91}$$

$$b = \frac{C_2N}{1+C_1N}, \tag{92}$$

where C_1 and C_2 are constants. Finally, the Lagrangian containing no dynamical scalar degree of freedom at linear order in a cosmological background is

$$\mathcal{L} = c_1\hat{K}_{ij}\hat{K}^{ij} + \frac{1}{3}\frac{C_2N}{1+C_1N}K^2 + c_3R + \frac{d_2^2}{4d_1}a_ia^i + d_1\mathrm{D}_i\phi\mathrm{D}^i\phi + d_2a_i\mathrm{D}^i\phi, \tag{93}$$

where $\hat{K}_{ij} = K_{ij} - \frac{1}{3}Kh_{ij}$ is the traceless part of K_{ij}, and coefficients c_1, c_3, d_1, and d_2 are the general functions of N and ϕ.

It is interesting to compare (93) with the result obtained in [79]. In [79] (see Eq. (61)), only the first three terms in (93) are present. In particular, acceleration term a_ia^i is not allowed. Here, owing to the presence of auxiliary field ϕ, the acceleration term can be introduced, together with two other terms involving $\mathrm{D}_i\phi$. By applying appropriate limit $d_1, d_2 \to 0$, we can return to the result in [79].

C. Case $\boldsymbol{\varOmega = 0}$

If $\varOmega = 0$, not all of auxiliary variables A, B, $\delta\phi$ are solvable. Nevertheless, we can still determine the conditions on the coefficients to eliminate the scalar degree of freedom. First, according to (85), $\varOmega = 0$ implies two equations:

$$(c_1+c_2)\left(d_2^2-4d_1c_4\right) = 0, \tag{94}$$

and

$$\left[3(2b-2b'+b'')(c_1+c_2)-2(-b'+b)^2\right]d_1 = 0. \tag{95}$$

Several branches of solutions to (94) and (95) exist, which we discuss below.

1. *Case 1*

The simplest case is to assume $d_1 = d_2 = 0$. Here, Lagrangian (57) reduces to

$$\mathcal{L} = c_1 K_{ij}K^{ij} + c_2 K^2 + c_3 R + c_4 a_i a^i, \tag{96}$$

which is simply the case that has already been discussed in [79]. The Lagrangian that propagates only two tensor degrees of freedom (up to the linear order in perturbations) is given by

$$\mathcal{L} = c_1 \hat{K}_{ij}\hat{K}^{ij} + \frac{1}{3}\frac{C_2 N}{1+C_1 N}K^2 + c_3 R, \tag{97}$$

where c_1 and c_3 are general functions of N and ϕ.

2. *Case 2*

If $d_1 \neq 0$, from (95) we must have

$$3(2b-2b'+b'')(b-2c_2) - 2(-b'+b)^2 = 0. \tag{98}$$

For (94), we assume $c_1 + c_2 \neq 0$; thus, it follows that

$$d_2^2 - 4d_1 c_4 = 0. \tag{99}$$

Because background equation of motion (72) must hold, we require $-b+b' \neq 0$ to guarantee non-vanishing cosmological constant Λ and thus a cosmological background solution. Thus, (98) indicates

$$b - 2c_2 \neq 0, \quad 2b-2b'+b'' \neq 0. \tag{100}$$

Subsequently, (98) implies that

$$c_2 = \frac{1}{2}b - \frac{(b-b')^2}{3(2b-2b'+b'')}, \tag{101}$$

and

$$c_1 = -\frac{1}{2}b + \frac{(b-b')^2}{2b-2b'+b''}, \tag{102}$$

where b is a general function of N only. The Lagrangian now becomes

$$\begin{aligned}\mathcal{L} = &\left[-\frac{1}{2}b + \frac{(b-b')^2}{2b-2b'+b''}\right]K_{ij}K^{ij} \\ &+ \left[\frac{1}{2}b - \frac{(b-b')^2}{3(2b-2b'+b'')}\right]K^2 \\ &+ c_3 R + c_4 a_i a^i + d_1 \mathrm{D}_i\phi \mathrm{D}^i\phi + d_2 a_i \mathrm{D}^i\phi,\end{aligned} \tag{103}$$

where c_3, c_4, d_1, and d_2 are general functions of N and ϕ.

We can easily check that operators $\hat{O}_{XY}$ is the same as those in (75)−(84). The equations of motion for the perturbation variables are

$$2\hat{O}_{AA}A + \hat{O}_{AB}B + \hat{O}_{A\phi}\delta\phi + \hat{O}_{A\zeta}\zeta + \hat{O}_{\dot{\zeta}A}\dot{\zeta} = 0, \tag{104}$$

$$\hat{O}_{AB}A + 2\hat{O}_{BB}B + \hat{O}_{\zeta B}\zeta + \hat{O}_{\dot{\zeta}B}\dot{\zeta} = 0, \tag{105}$$

$$2\hat{O}_{\phi\phi}\delta\phi + \hat{O}_{A\phi}A = 0, \tag{106}$$

$$\begin{aligned}&\frac{1}{N}\Big[2\partial_t\left(a^3\hat{O}_{\dot{\zeta}\dot{\zeta}}\dot{\zeta}\right) \\ &+ \partial_t\left(a^3\hat{O}_{\dot{\zeta}B}B\right) + \partial_t\left(a^3\hat{O}_{\dot{\zeta}A}A\right) + \partial_t\left(a^3\hat{O}_{\zeta\zeta}\zeta\right)\Big] \\ &- a^3\left(2\hat{O}_{\zeta\zeta}\zeta + \hat{O}_{A\zeta}A + \hat{O}_{\zeta B}B\right) = 0.\end{aligned} \tag{107}$$

We rewrite the first three equations in the matrix form:

$$\boldsymbol{D}\boldsymbol{\alpha} = \boldsymbol{\beta}, \tag{108}$$

where

$$\boldsymbol{D} = \begin{bmatrix} 2\hat{O}_{AA} & \hat{O}_{AB} & \hat{O}_{A\phi} \\ \hat{O}_{AB} & 2\hat{O}_{BB} & 0 \\ \hat{O}_{A\phi} & 0 & 2\hat{O}_{\phi\phi} \end{bmatrix}, \tag{109}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} A \\ B \\ \delta\phi \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} -\hat{O}_{A\zeta}\zeta - \hat{O}_{\dot{\zeta}A}\dot{\zeta} \\ -\hat{O}_{\zeta B}\zeta - \hat{O}_{\dot{\zeta}B}\dot{\zeta} \\ 0 \end{bmatrix}. \tag{110}$$

Determinant of coefficient matrix $\boldsymbol{D}$ of auxiliary perturbation variables A, B, and $\delta\phi$ is

$$\begin{aligned}\varOmega &= \det\boldsymbol{D} \\ &= \hat{O}_{A\phi}^2\hat{O}_{BB} + \hat{O}_{AB}^2\hat{O}_{\phi\phi} - 4\hat{O}_{AA}\hat{O}_{BB}\hat{O}_{\phi\phi} = 0.\end{aligned} \tag{111}$$

By multiplying the third row of $\boldsymbol{D}$ by $\hat{O}_{A\phi}\hat{O}_{BB}$, the second row by $\hat{O}_{AB}\hat{O}_{\phi\phi}$, and the first row by $-2\hat{O}_{BB}\hat{O}_{\phi\phi}$ and by adding the third and second rows to the first row, we derive the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ \hat{O}_{AB} & 2\hat{O}_{BB} & 0 \\ \hat{O}_{A\phi} & 0 & 2\hat{O}_{\phi\phi} \end{pmatrix}, \tag{112}$$

which shows that $\mathrm{rank}\boldsymbol{D}=2<3$. The augmented matrix of the set of equations is defined as

$$\boldsymbol{E}=\begin{pmatrix} 2\hat{O}_{AA} & \hat{O}_{AB} & \hat{O}_{A\phi} & -\hat{O}_{A\zeta}\zeta-\hat{O}_{\dot{\zeta}A}\dot{\zeta} \\ \hat{O}_{AB} & 2\hat{O}_{BB} & 0 & -\hat{O}_{\zeta B}\zeta-\hat{O}_{\dot{\zeta}B}\dot{\zeta} \\ \hat{O}_{A\phi} & 0 & 2\hat{O}_{\phi\phi} & 0 \end{pmatrix}. \tag{113}$$

By performing the same operations to $\boldsymbol{D}$, the augmented matrix becomes

$$\begin{pmatrix} 0 & 0 & 0 & \left(2\hat{O}_{BB}\hat{O}_{\phi\phi}\hat{O}_{A\zeta}-\hat{O}_{\zeta B}\hat{O}_{AB}\hat{O}_{\phi\phi}\right)\zeta+\left(2\hat{O}_{BB}\hat{O}_{\phi\phi}\hat{O}_{\dot{\zeta}A}-\hat{O}_{\dot{\zeta}B}\hat{O}_{AB}^{2}\hat{O}_{\phi\phi}\right)\dot{\zeta} \\ \hat{O}_{AB} & 2\hat{O}_{BB} & 0 & -\hat{O}_{\zeta B}\zeta-\hat{O}_{\dot{\zeta}B}\dot{\zeta} \\ \hat{O}_{A\phi} & 0 & 2\hat{O}_{\phi\phi} & 0 \end{pmatrix}, \tag{114}$$

which shows that the crucial term relevant to our analysis is

$$\Xi\equiv\left(2\hat{O}_{BB}\hat{O}_{\phi\phi}\hat{O}_{A\zeta}-\hat{O}_{\zeta B}\hat{O}_{AB}\hat{O}_{\phi\phi}\right)\zeta+\left(2\hat{O}_{BB}\hat{O}_{\phi\phi}\hat{O}_{\dot{\zeta}A}-\hat{O}_{\dot{\zeta}B}\hat{O}_{AB}^{2}\hat{O}_{\phi\phi}\right)\dot{\zeta}. \tag{115}$$

Fortunately, the explicit expression for Ξ is not required in the following analysis.

Based on whether Ξ is identically vanishing or not, we have two cases. If

$$\Xi\neq0, \tag{116}$$

we obtain $\mathrm{rank}\boldsymbol{E}=3>\mathrm{rank}\boldsymbol{D}=2$, which implies that the set of equations for the auxiliary variables has no solution.

In contrast, if

$$\Xi=0, \tag{117}$$

we obtain $\mathrm{rank}\boldsymbol{E}=\mathrm{rank}\boldsymbol{D}=2<3$, which means that the set of equations is solvable. In this case, equations (104)−(106) become

$$2\hat{O}_{BB}B+\hat{O}_{AB}A+\hat{O}_{\zeta B}\zeta+\hat{O}_{\dot{\zeta}B}\dot{\zeta}=0, \tag{118}$$

$$2\hat{O}_{\phi\phi}\delta\phi+\hat{O}_{A\phi}A=0, \tag{119}$$

from which we can solve B as

$$B=-\frac{\hat{O}_{AB}A+\hat{O}_{\zeta B}\zeta+\hat{O}_{\dot{\zeta}B}\dot{\zeta}}{2\hat{O}_{BB}}. \tag{120}$$

Substituting solution (120) into the last equation of motion (107) yields the second order time derivative term of ζ,

$$\Delta_1\ddot{\zeta}, \tag{121}$$

where

$$\Delta_1=\frac{4\hat{O}_{\dot{\zeta}\dot{\zeta}}\hat{O}_{BB}-\hat{O}_{\dot{\zeta}B}^{2}}{2\hat{O}_{BB}}. \tag{122}$$

To eliminate the scalar mode, we must require

$$4\hat{O}_{\dot{\zeta}\dot{\zeta}}\hat{O}_{BB}-\hat{O}_{\dot{\zeta}B}^{2}=0, \tag{123}$$

which corresponds to

$$4a^{-2}b\left[b-3\left(c_1+c_2\right)\right]\partial^4=0. \tag{124}$$

Recall that we have assumed $c_1+c_2\neq0$ from the beginning; thus, the only feasible solution to (124) is $c_1=0$. Again, this will lead to the disappearance of $\hat{K}_{ij}\hat{K}^{ij}$; thus, the kinetic term for the gravitational wave, which is unacceptable.

In conclusion, no viable Lagrangian is obtained in Case 2.

3. *Case 3*

In this case, we assume

$$c_1+c_2=0,\quad -b'+b=0. \tag{125}$$

As mentioned above, this case conflicts with the background equation of motion (72), and thus is unphysical.

4. *Case 4*

In this case, we assume

$$c_1+c_2=0,\quad d_1=0. \tag{126}$$

The Lagrangian reduces to be

$$\mathcal{L}=-\frac{1}{2}bK_{ij}K^{ij}+\frac{1}{2}bK^2+c_3R+c_4a_ia^i+d_2a_i\mathrm{D}^i\phi. \tag{127}$$

Operators $\hat{O}_{XY}$ are given by

$$\hat{O}_{AA}=\frac{3}{2}H^2(2b-2b'+b'')-c_4\partial^2, \tag{128}$$

$$\hat{O}_{AB}=2Ha^{-1}(-b'+b)\partial^2, \tag{129}$$

$$\hat{O}_{A\phi}=-d_2\partial^2, \tag{130}$$

$$\hat{O}_{\dot\zeta A}=6H(-b+b'), \tag{131}$$

$$\hat{O}_{\dot\zeta B}=-2a^{-1}b\partial^2, \tag{132}$$

$$\hat{O}_{\dot\zeta\dot\zeta}=3b, \tag{133}$$

and $\hat{O}_{BB}=\hat{O}_{\phi\phi}=\hat{O}_{\phi B}=\hat{O}_{\dot\zeta\phi}=\hat{O}_{\phi\zeta}=0$.

Consequently, the quadratic action for the perturbation variables is

$$\begin{aligned}S_2[A,B,\zeta,\delta\phi]=\int \mathrm{d}t\mathrm{d}^3x\bar{N}a^3\Big(&A\hat{O}_{AA}A+\zeta\hat{O}_{\zeta\zeta}\zeta+A\hat{O}_{AB}B\\&+A\hat{O}_{A\zeta}\zeta+A\hat{O}_{A\phi}\delta\phi+\zeta\hat{O}_{\zeta B}B+\dot\zeta\hat{O}_{\dot\zeta A}A\\&+\dot\zeta\hat{O}_{\dot\zeta B}B+\dot\zeta\hat{O}_{\dot\zeta\zeta}\zeta+\dot\zeta\hat{O}_{\dot\zeta\dot\zeta}\dot\zeta\Big).\end{aligned} \tag{134}$$

The equations of motion for the perturbation variables are

$$2\hat{O}_{AA}A+\hat{O}_{AB}B+\hat{O}_{A\zeta}\zeta+\hat{O}_{A\phi}\delta\phi+\hat{O}_{\dot\zeta A}\dot\zeta=0, \tag{135}$$

$$\hat{O}_{AB}A+\hat{O}_{\zeta B}\zeta+\hat{O}_{\dot\zeta B}\dot\zeta=0, \tag{136}$$

$$\hat{O}_{A\phi}A=0, \tag{137}$$

$$\begin{aligned}&\frac{1}{\bar{N}}\left[2\partial_t\left(a^3\hat{O}_{\dot\zeta\dot\zeta}\dot\zeta\right)+\partial_t\left(a^3\hat{O}_{\dot\zeta B}B\right)\right.\\&\left.+\partial_t\left(a^3\hat{O}_{\dot\zeta A}A\right)+\partial_t\left(a^3\hat{O}_{\dot\zeta\zeta}\zeta\right)\right]\\&-a^3\left(2\hat{O}_{\zeta\zeta}\zeta+\hat{O}_{A\zeta}A+\hat{O}_{\phi\zeta}\delta\phi+\hat{O}_{\zeta B}B\right)=0.\end{aligned} \tag{138}$$

To obtain a nontrivial solution for A, from (137), we require that

$$\hat{O}_{A\phi}=0, \tag{139}$$

which implies

$$d_2=0. \tag{140}$$

Thus, we can solve A and B from (135)−(136) to obtain

$$A=-\frac{\hat{O}_{\zeta B}\zeta+\hat{O}_{\dot\zeta B}\dot\zeta}{\hat{O}_{AB}}, \tag{141}$$

and

$$B=\frac{\left(2\hat{O}_{AA}\hat{O}_{\zeta B}-\hat{O}_{AB}\hat{O}_{A\zeta}\right)\zeta+\left(2\hat{O}_{AA}\hat{O}_{\dot\zeta B}-\hat{O}_{AB}\hat{O}_{\dot\zeta A}\right)\dot\zeta}{\hat{O}_{AB}^2}, \tag{142}$$

where $\hat{O}_{AB}$ given in (129) is not vanishing.

Finally, by substituting solutions (141) and (142) into (138), we obtain an effective equation of motion for ζ (with undetermined $\delta\phi$), in which the second-order time derivative term can be expressed as

$$\Delta_2\ddot\zeta, \tag{143}$$

with

$$\Delta_2=2\frac{\hat{O}_{\dot\zeta\dot\zeta}\hat{O}_{AB}^2+\hat{O}_{AA}\hat{O}_{\dot\zeta B}^2-\hat{O}_{AB}\hat{O}_{\dot\zeta A}\hat{O}_{\dot\zeta B}}{\hat{O}_{AB}^2}. \tag{144}$$

Note that although $\delta\phi$ is left undetermined, it will not contribute to the $\ddot\zeta$ term in its equation of motion. To avoid the unwanted scalar mode, we must impose condition $\Delta_2=0$, which is equivalent to

$$\hat{O}_{\dot\zeta\dot\zeta}\hat{O}_{AB}^2+\hat{O}_{AA}\hat{O}_{\dot\zeta B}^2-\hat{O}_{AB}\hat{O}_{\dot\zeta A}\hat{O}_{\dot\zeta B}=0. \tag{145}$$

After substituting (128)−(133) into (145), we obtain

$$6H^2a^{-2}b\left(bb''+2bb'-2b'^2\right)\partial^4-4a^{-2}b^2c_4\partial^6=0, \tag{146}$$

which implies two equations:

$$b\left(bb''+2bb'-2b'^2\right)=0, \tag{147}$$

$$b^2c_4=0. \tag{148}$$

Because the background equations of motion must be satisfied, $b\neq 0$. Thus, the only solutions to $\Delta_2=0$ are

$$c_4=0,\quad b=\frac{C_2N}{1+C_1N}. \tag{149}$$

As a result, in this case, the Lagrangian that does not propagate any scalar degree of freedom at the linear order is

$$\mathcal{L}=-\frac{1}{2}\frac{C_2N}{1+C_1N}\hat{K}_{ij}\hat{K}^{ij}+\frac{1}{3}\frac{C_2N}{1+C_1N}K^2+c_3R, \tag{150}$$

where c_3 is a general function of N and ϕ.

Here, if we select $c_3=1$, let $|C_2|\to\infty$, $|C_1|\to\infty$, and maintain $\dfrac{C_2}{C_1}=-2$, Lagrangian (150) reduces to

$$\mathcal{L}=\hat{K}_{ij}\hat{K}^{ij}-\frac{2}{3}K^2+R, \tag{151}$$

which is nothing but GR.

V. CONCLUSION

Spatially covariant gravity provides us with a broad playground for studying modifications of GR. According to the discussion in [85], the spatial covariant gravity theory with non-dynamical auxiliary fields propagates three dynamical degrees of freedom, i.e., two tensorial and one scalar degrees of freedom. In light of the detection of gravitational waves, which correspond to the two tensorial degrees of freedom (TTDOFs), it is interesting to examine under which conditions the theory propagates only TTDOFs.

In this work, instead of deriving and solving the fully nonlinear TTDOF conditions as in [78], we performed a perturbative analysis similar to [58, 79] and focussed on the necessary conditions such that no scalar mode propagates at the linear order in perturbations around a cosmological background. In Sec. II, starting from general action (1), background equation of motion (25) was easily obtained from linear order action (21). The equation of motion is useful for simplifying the calculations at the quadratic order. Next, in Sec. III, we derived the quadratic action for scalar perturbations (29) and found that A, B, and $\delta\phi$ are auxiliary variables. By solving these auxiliary variables and deriving the effective equation of motion for would-be dynamical perturbation variable ζ, we eliminated this unwanted scalar mode as long as condition (55) was satisfied.

As an illustration, in Sec. IV, we considered polynomial-type Lagrangian (56), in which all the monomials are of $d=2$, with d being the total number of derivatives. Generally, if condition (53) holds, i.e., when all auxiliary perturbation variables A, B, $\delta\phi$ are solvable, TTDOF condition (55) leads to Lagrangian (93), which propagates only TTDOFs at the linear order around a cosmological background. Compared with the result in [79], terms involving the acceleration are allowed in the Lagrangian owing to the presence of the auxiliary scalar field. An interesting observation was that even when condition (53) does not hold, which implies that not all of the auxiliary variables can be solved, we could still derive the degeneracy condition to eliminate the unwanted scalar mode. In this case, we obtained two viable Lagrangians (97) and (150). Both cases included GR as a special scenario as long as appropriate coefficients were selected.

We have several comments on the results presented in this paper. First, the degeneracy condition derived herein is necessary in the sense that the unwanted scalar mode is removed only at the linear order in a cosmological background, which may reappear at nonlinear orders and/or around a general background. Therefore, we may perform an analysis similar to those in [58, 79] to derive further constraints on the Lagrangian such that the scalar mode is fully removed. Second, the re-arising of the scalar mode at nonlinear orders signals the possible strong coupling problem. Nevertheless, the strong coupling problem can be solved using the effective field theory approach [89, 90]. In other words, the strong couple problem may not necessarily appear for Larangians (93), (97), and (150) as long as the strong coupling scale is comparable to the cutoff scale of the relevant theory. Third, in the example of $d=2$, the resulting Lagrangian for $\Omega\neq 0$ generalizes the one derived in [79]. Auxiliary scalar field ϕ not only affects the coefficients in the Lagrangian but also allows the acceleration term. Therefore, more general cases (e.g., with $d\geq 3$) where the auxiliary scalar field may have an essential role would be interesting to explore to enable a novel class of TTDOF theories with such auxiliary fields to be constructed.

References

[1] D. Lovelock, J. Math. Phys. **12**, 498 (1971)

[2] D. Lovelock, J. Math. Phys. **13**, 874 (1972)

[3] N. Afshordi, D. J. H. Chung, and G. Geshnizjani, Phys. Rev. D **75**, 083513 (2007), arXiv:hep-th/0609150

[4] N. Afshordi, D. J. H. Chung, M. Doran *et al*., Phys. Rev. D **75**, 123509 (2007), arXiv:astro-ph/0702002
[5] M. Mylova and N. Afshordi, *Effective Cuscuton Theory*, arXiv: 2312.06066
[6] S. S. Boruah, H. J. Kim, and G. Geshnizjani, JCAP **07**, 022 (2017), arXiv:1704.01131
[7] S. S. Boruah, H. J. Kim, M. Rouben *et al*., JCAP **08**, 031 (2018), arXiv:1802.06818
[8] J. Bhattacharyya, A. Coates, M. Colombo *et al*., Phys. Rev. D **97**, 064020 (2018), arXiv:1612.01824
[9] J. Quintin and D. Yoshida, JCAP **02**, 016 (2020), arXiv:1911.06040
[10] N. Bartolo, A. Ganz, and S. Matarrese, JCAP **05**, 008 (2022), arXiv:2111.06794
[11] S. A. Hosseini Mansoori and Z. Molaee, JCAP **01**, 022 (2023), arXiv:2207.06720
[12] P. Channuie, K. Karwan and J. Sangtawee, Eur. Phys. J. C **83**, 421 (2023), arXiv:2301.07019
[13] K.-i. Maeda and S. Panpanich, Phys. Rev. D **105**, 104022 (2022), arXiv:2202.04908
[14] K. Kohri and K.-i. Maeda, PTEP **2022**, 091E01 (2022), arXiv:2206.11257
[15] S. Panpanich and K.-i. Maeda, Eur. Phys. J. C **83**, 240 (2023), arXiv:2109.12288
[16] F. C. E. Lima and C. A. S. Almeida, Eur. Phys. J. C **83**, 831 (2023), arXiv:2301.01397
[17] A. Iyonaga, K. Takahashi, and T. Kobayashi, JCAP **12**, 002 (2018), arXiv:1809.10935
[18] A. Iyonaga, K. Takahashi, and T. Kobayashi, JCAP **07**, 004 (2020), arXiv:2003.01934
[19] T. Zhu, Q. Wu, A. Wang *et al*., Phys. Rev. D **84**, 101502 (2011), arXiv:1108.1237
[20] T. Zhu, F. W. Shu, Q. Wu *et al*., Phys. Rev. **D85**, 044053 (2012), arXiv:1110.5106
[21] J. Chagoya and G. Tasinato, *A new scalar-tensor realization of Hořava-Lifshitz gravity*, arXiv: 1805.12010
[22] N. Afshordi, Phys. Rev. D **80**, 081502 (2009), arXiv:0907.5201
[23] J. Khoury, G. E. Miller, and A. J. Tolley, Phys. Rev. D **85**, 084002 (2012), arXiv:1108.1397
[24] J. Khoury, G. E. J. Miller, and A. J. Tolley, Int. J. Mod. Phys. D **23**, 1442012 (2014), arXiv:1405.5219
[25] J. Chagoya and G. Tasinato, JHEP **02**, 113 (2017), arXiv:1610.07980
[26] G. Tasinato, Phys. Rev. D **102**, 084009 (2020), arXiv:2009.02157
[27] D. Glavan and C. Lin, Phys. Rev. Lett. **124**, 081301 (2020), arXiv:1905.03601
[28] C. Lin and S. Mukohyama, JCAP **1710**, 033 (2017), arXiv:1708.03757
[29] K. Aoki, A. De Felice, C. Lin *et al*., JCAP **1901**, 017 (2019), arXiv:1810.01047
[30] K. Aoki, C. Lin and S. Mukohyama, Phys. Rev. D **98**, 044022 (2018), arXiv:1804.03902
[31] A. De Felice and S. Mukohyama, Phys. Lett. B **752**, 302 (2016), arXiv:1506.01594
[32] A. De Felice and S. Mukohyama, *Phenomenology in minimal theory of massive gravity*, arXiv: 1512.04008
[33] N. Bolis, A. De Felice, and S. Mukohyama, Phys. Rev. D **98**, 024010 (2018), arXiv:1804.01790
[34] A. De Felice, F. Larrouturou, S. Mukohyama *et al*., Phys. Rev. D **98**, 104031 (2018), arXiv:1808.01403
[35] A. De Felice, S. Mukohyama, and M. C. Pookkillath, JCAP **12**, 011 (2021), arXiv:2110.01237
[36] A. De Felice, S. Kumar, S. Mukohyama *et al*., *Observational bounds on extended minimal theories of massive gravity: New limits on the graviton mass*, arXiv: 2311.10530
[37] S. Mukohyama and K. Noui, JCAP **1907**, 049 (2019), arXiv:1905.02000
[38] A. De Felice, A. Doll, and S. Mukohyama, JCAP **09**, 034 (2020), arXiv:2004.12549
[39] A. De Felice, S. Mukohyama, and M. C. Pookkillath, Phys. Lett. B **816**, 136201 (2021), arXiv:2009.08718
[40] K. Aoki, A. De Felice, S. Mukohyama *et al*., Eur. Phys. J. C **80**, 708 (2020), arXiv:2005.13972
[41] M. C. Pookkillath, Astron. Rep. **65**, 1021 (2021)
[42] A. De Felice, F. Larrouturou, S. Mukohyama *et al*., JCAP **04**, 015 (2021), arXiv:2012.01073
[43] K. Aoki, F. Di Filippo, and S. Mukohyama, JCAP **05**, 071 (2021), arXiv:2103.15044
[44] A. De Felice, A. Doll, F. Larrouturou *et al*., JCAP **03**, 004 (2021), arXiv:2010.13067
[45] A. De Felice and S. Mukohyama, JCAP **04**, 018 (2021), arXiv:2011.04188
[46] A. De Felice, S. Mukohyama, and M.C. Pookkillath, Phys. Rev. D **105**, 104013 (2022), arXiv:2110.14496
[47] A. De Felice, K.-i. Maeda, S. Mukohyama *et al*., Phys. Rev. D **106**, 024028 (2022), arXiv:2204.08294
[48] A. F. Jalali, P. Martens, and S. Mukohyama, Phys. Rev. D **109**, 044053 (2024), arXiv:2306.10672
[49] R. Carballo-Rubio, F. Di Filippo, and S. Liberati, JCAP **1806**, 026 (2018), arXiv:1802.02537
[50] J. Sangtawee and K. Karwan, *Inflationary model in minimally modified gravity theories*, arXiv: 2103.11463
[51] A. Ganz, JCAP **08**, 074 (2022), arXiv:2203.12358
[52] O. Akarsu, A. De Felice, E. Di Valentino *et al*., Λ_s *CDM cosmology from a type-II minimally modified gravity*, arXiv: 2402.07716
[53] Z. B. Yao, M. Oliosi, X. Gao *et al*., Phys. Rev. D **103**, 024032 (2021), arXiv:2011.00805
[54] Z. B. Yao, M. Oliosi, X. Gao *et al*., Phys. Rev. D **107**, 104052 (2023), arXiv:2302.02090
[55] X. Gao, Phys. Rev. D **90**, 081501 (2014), arXiv:1406.0822
[56] X. Gao, Phys. Rev. D **90**, 104033 (2014), arXiv:1409.6708
[57] X. Gao and Z.-B. Yao, JCAP **1905**, 024 (2019), arXiv:1806.02811
[58] X. Gao, C. Kang, and Z. B. Yao, Phys. Rev. D **99**, 104015 (2019), arXiv:1902.07702
[59] Y. Yu, Z. Chen, and X. Gao, *Spatially covariant gravity with nonmetricity*, arXiv: 2402.02565
[60] X. Gao, Sci. China Phys. Mech. Astron. **64**, 210012 (2021), arXiv:2003.11978
[61] X. Gao and Y. M. Hu, Phys. Rev. D **102**, 084006 (2020), arXiv:2004.07752
[62] X. Gao, JCAP **11**, 004 (2020), arXiv:2006.15633
[63] Y. M. Hu and X. Gao, Phys. Rev. D **105**, 044023 (2022), arXiv:2111.08652
[64] P. Horava, JHEP **0903**, 020 (2009), arXiv:0812.4287
[65] P. Horava, Phys. Rev. D **79**, 084008 (2009), arXiv:0901.3775
[66] D. Blas, O. Pujolas, and S. Sibiryakov, JHEP **0910**, 029 (2009), arXiv:0906.3046
[67] P. Creminelli, M. A. Luty, A. Nicolis *et al*., JHEP **0612**, 080 (2006), arXiv:hep-th/0606090
[68] C. Cheung, P. Creminelli, A. L. Fitzpatrick *et al*., JHEP

0803, 014 (2008), arXiv:0709.0293

[69] P. Creminelli, G. D'Amico, J. Norena *et al.*, JCAP **0902**, 018 (2009), arXiv:0811.0827

[70] G. Gubitosi, F. Piazza, and F. Vernizzi, JCAP **1302**, 032 (2013), arXiv:1210.0201

[71] J. K. Bloomfield, E. E. Flanagan, M. Park *et al.*, JCAP **1308**, 010 (2013), arXiv:1211.7054

[72] T. Fujita, X. Gao, and J. Yokoyama, JCAP **1602**, 014 (2016), arXiv:1511.04324

[73] X. Gao and X. Y. Hong, Phys. Rev. D **101**, 064057 (2020), arXiv:1906.07131

[74] C. Gong, T. Zhu, R. Niu *et al.*, Phys. Rev. D **105**, 044034 (2022), arXiv:2112.06446

[75] T. Zhu, W. Zhao, and A. Wang, Phys. Rev. D **107**, 024031 (2023), arXiv:2210.05259

[76] T. Zhu, W. Zhao, and A. Wang, Phys. Rev. D **107**, 044051 (2023), arXiv:2211.04711

[77] T. Zhu, W. Zhao, J. M. Yan *et al.*, *Tests of modified gravitational wave propagations with gravitational waves*, arXiv: 2304.09025

[78] X. Gao and Z. B. Yao, Phys. Rev. D **101**, 064018 (2020), arXiv:1910.13995

[79] Y. M. Hu and X. Gao, Phys. Rev. D **104**, 104007 (2021), arXiv:2104.07615

[80] J. Lin, Y. Gong, Y. Lu *et al.*, Phys. Rev. D **103**, 064020 (2021), arXiv:2011.05739

[81] A. Iyonaga and T. Kobayashi, Phys. Rev. D **104**, 124020 (2021), arXiv:2109.10615

[82] T. Hiramatsu and T. Kobayashi, JCAP **07**, 040 (2022), arXiv:2205.04688

[83] J. Saito and T. Kobayashi, Phys. Rev. D **108**, 104063 (2023), arXiv:2308.00267

[84] S. Chakraborty, K. Karwan, and J. Sangtawee, *Observational predictions of inflationary model in spatially covariant gravity with two tensorial degrees of freedom for gravity*, arXiv: 2308.09508

[85] X. Gao, M. Yamaguchi, and D. Yoshida, JCAP **1903**, 006 (2019), arXiv:1810.07434

[86] V. F. Mukhanov and R. H. Brandenberger, Phys. Rev. Lett. **68**, 1969 (1992)

[87] Y. F. Cai and E. N. Saridakis, Class. Quant. Grav. **28**, 035010 (2011), arXiv:1007.3204

[88] L. Sebastiani, S. Vagnozzi, and R. Myrzakulov, Adv. High Energy Phys. **2017**, 3156915 (2017), arXiv:1612.08661

[89] Y. M. Hu, Y. Zhao, X. Ren *et al.*, JCAP **07**, 060 (2023), arXiv:2302.03545

[90] Y. M. Hu, Y. Yu, Y. F. Cai *et al.*, *The effective field theory approach to the strong coupling issue in $f(T)$ gravity with a non-minimally coupled scalar field*, arXiv: 2311.12645