The Prediction of Moment of Inertia of Rotating Nuclei*

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Abstract: In this paper the mathematical expression which is given by Bohr for the moment of inertia of even - even nuclei on the basis of the hydrodynamical model is modified. This modification is on the kinetic energy of the surface oscillations including the second and third terms of $R$- expansion as well as the first term which was already carried out by Bohr. Therefore, this work can be considered the continuation and support of the hydrodynamic model of Bohr. This procedure results in a Bohr formula to be multiplied by a factor which depends on the deformation parameter. Bohr (modified) formula is examined by applying it on axially symmetric even–even nuclei with atomic mass ranged between 150 and 190 as well as to some triaxial symmetry nuclei. The modification of Bohr’s formula are discussed including the information on how stable this modification with including second and third terms of $R$- expansion of Bohr’s formula. The results of calculation are compared with the experimental data and the results of Bohr, based earlier. The obtained results are in a good agreement with experimental data by describing almost 0.7 and better than that of the unmodified ones.

Keywords: Moment of inertia, Hydrodynamical model, Irrotational motion

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1 Introduction

The problem of calculating the moment of inertia (MOI) of even-even nuclei has received a considerable attention since 1950’s Refs. [1-3]. The hydrodynamical model, firstly introduced by Bohr [4], considered the nucleus as a droplet of incompressible irrotational fluid. As a consequence, the collective motion of the nucleus was pictured as a quadrupole classical oscillations similar to the oscillations of the liquid droplet that had been discussed in detail by Rayleigh [5]. This description of Bohr leads to simple relationship between moment of inertia $\mathcal{I}$ and deformation parameter $\beta$, as $\mathcal{I} = 3B\beta^2$, where $B$ is the inertial parameter and this relationship is called Bohr's formula. In particular case of small oscillations and hence $R(\theta, \phi)$, the radial coordinate of the surface at polar coordinates $(\theta, \phi)$ can be approximated to $R_0$, the radius of the equilibrium spherical shape of the liquid drop, Rayleigh's calculations showed that $B = \frac{\rho R_0^5}{2}$. The values of $B$ derived by Rayleigh, are five times smaller than from the values of MOI which is obtained experimentally. Furthermore, one cannot explain the MOI of deformed nuclei by considering the extreme case of a rigid deformed shape [6].

An alternative approach to describe the MOI of deformed nuclei was based on the cranking model which was introduced by Inglis [7]. In this model the kinetic energy of rotation is obtained by considering the motion of the nucleons in rotating self–consistent field. In contrast to the hydrodynamical model, the results of the cranking model are found to be 2-3 times larger than that of the experimental ones. Since then, both models: the hydrodynamical model of Bohr and the cranking model of Inglis have been modified by several authors Refs. [2, 8-14]. Recently, a valuable thorough quantitative comparison of predictions for both form factor and moment of inertia of four different models (Hartree-Fock, cranking model, rigid rotator and irrotational fluid flow) for the rare earth nuclei $^{154}\text{Sm}$, $^{156}\text{Gd}$, $^{164}\text{Dy}$, $^{166,168}\text{Er}$ and $^{174}\text{Yb}$ were proposed in reference [9]. The authors of reference [9] extended their work to observed the electromagnetic form

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factors for some odd mass of $Hf$ isotopes [15].

In this paper, we assumed that the approach of Rayleigh within the approximation of $R$ to $R_0$ is the reason for the poor agreement of the Bohr results. Accordingly, we expanded $R$ about $R_0$ and we worked out the second and third terms of this expansion in addition to the first one which was already discussed by Rayleigh. The obtained equation which is called modified form of Bohr’s relationship is checked on axially symmetric nuclei of atomic mass ranged between 150 and 190 as well as on a number of tri-axially nuclei. The results of both modified and original formulas of Bohr are compared with the experimental data in Table 1 and also compared with the numerical values proposed in Ref. [9] as shown in Table 2. It is found, even though, the agreement with experimental data and also with numerical ones is not so good, the enhancement in the results are very large where the ratios of the modified Bohr (current work) to the experimental results are almost 0.6 to sometimes 0.7 instead of 0.2 in the case of original Bohr.

### Table 1
Comparison of calculated results $\mathcal{G}^{Total}$ with the experimental data $\mathcal{G}^{Exp}$ Ref. [22], $\mathcal{G}^{Bohr}_{Hdeformed}$, and also the values of first $\mathcal{G}^{r}$ and second $\mathcal{G}^{s}$ modifications of MOI for even-even axially deformed nuclei

<table>
<thead>
<tr>
<th>Nuclei</th>
<th>$\beta$</th>
<th>$\mathcal{G}^{Exp}$</th>
<th>$\mathcal{G}^{Bohr}$</th>
<th>$\mathcal{G}^{s}$</th>
<th>$\mathcal{G}^{r}$</th>
<th>$\mathcal{G}^{Total}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{152}Sm$</td>
<td>0.290</td>
<td>0.380</td>
<td>0.069</td>
<td>0.018</td>
<td>0.116</td>
<td>0.203</td>
</tr>
<tr>
<td>$^{154}Sm$</td>
<td>0.336</td>
<td>0.551</td>
<td>0.092</td>
<td>0.028</td>
<td>0.206</td>
<td>0.325</td>
</tr>
<tr>
<td>$^{154}Gd$</td>
<td>0.280</td>
<td>0.373</td>
<td>0.065</td>
<td>0.016</td>
<td>0.101</td>
<td>0.182</td>
</tr>
<tr>
<td>$^{156}Gd$</td>
<td>0.320</td>
<td>0.498</td>
<td>0.083</td>
<td>0.024</td>
<td>0.170</td>
<td>0.277</td>
</tr>
<tr>
<td>$^{160}Gd$</td>
<td>0.346</td>
<td>0.547</td>
<td>0.097</td>
<td>0.030</td>
<td>0.231</td>
<td>0.358</td>
</tr>
<tr>
<td>$^{162}Dy$</td>
<td>0.354</td>
<td>0.561</td>
<td>0.101</td>
<td>0.032</td>
<td>0.252</td>
<td>0.385</td>
</tr>
<tr>
<td>$^{164}Dy$</td>
<td>0.310</td>
<td>0.490</td>
<td>0.074</td>
<td>0.020</td>
<td>0.134</td>
<td>0.228</td>
</tr>
<tr>
<td>$^{166}Dy$</td>
<td>0.320</td>
<td>0.512</td>
<td>0.083</td>
<td>0.024</td>
<td>0.170</td>
<td>0.277</td>
</tr>
<tr>
<td>$^{166}Er$</td>
<td>0.334</td>
<td>0.558</td>
<td>0.090</td>
<td>0.027</td>
<td>0.201</td>
<td>0.319</td>
</tr>
<tr>
<td>$^{168}Er$</td>
<td>0.306</td>
<td>0.456</td>
<td>0.077</td>
<td>0.021</td>
<td>0.143</td>
<td>0.240</td>
</tr>
<tr>
<td>$^{170}Er$</td>
<td>0.323</td>
<td>0.496</td>
<td>0.085</td>
<td>0.025</td>
<td>0.176</td>
<td>0.286</td>
</tr>
<tr>
<td>$^{172}Er$</td>
<td>0.320</td>
<td>0.496</td>
<td>0.083</td>
<td>0.024</td>
<td>0.170</td>
<td>0.277</td>
</tr>
<tr>
<td>$^{176}Er$</td>
<td>0.310</td>
<td>0.484</td>
<td>0.078</td>
<td>0.022</td>
<td>0.150</td>
<td>0.250</td>
</tr>
<tr>
<td>$^{176}Yb$</td>
<td>0.304</td>
<td>0.455</td>
<td>0.076</td>
<td>0.021</td>
<td>0.139</td>
<td>0.235</td>
</tr>
<tr>
<td>$^{178}Yb$</td>
<td>0.311</td>
<td>0.477</td>
<td>0.079</td>
<td>0.022</td>
<td>0.152</td>
<td>0.253</td>
</tr>
<tr>
<td>$^{180}Yb$</td>
<td>0.308</td>
<td>0.475</td>
<td>0.078</td>
<td>0.022</td>
<td>0.146</td>
<td>0.245</td>
</tr>
<tr>
<td>$^{176}Hf$</td>
<td>0.301</td>
<td>0.445</td>
<td>0.074</td>
<td>0.020</td>
<td>0.134</td>
<td>0.228</td>
</tr>
<tr>
<td>$^{178}Hf$</td>
<td>0.300</td>
<td>0.410</td>
<td>0.074</td>
<td>0.020</td>
<td>0.132</td>
<td>0.226</td>
</tr>
<tr>
<td>$^{180}Hf$</td>
<td>0.310</td>
<td>0.380</td>
<td>0.078</td>
<td>0.022</td>
<td>0.150</td>
<td>0.250</td>
</tr>
<tr>
<td>$^{182}W$</td>
<td>0.270</td>
<td>0.380</td>
<td>0.060</td>
<td>0.015</td>
<td>0.087</td>
<td>0.162</td>
</tr>
<tr>
<td>$^{184}W$</td>
<td>0.280</td>
<td>0.340</td>
<td>0.065</td>
<td>0.016</td>
<td>0.101</td>
<td>0.182</td>
</tr>
<tr>
<td>$^{186}W$</td>
<td>0.250</td>
<td>0.310</td>
<td>0.052</td>
<td>0.012</td>
<td>0.065</td>
<td>0.128</td>
</tr>
<tr>
<td>$^{186}Os$</td>
<td>0.259</td>
<td>0.272</td>
<td>0.056</td>
<td>0.013</td>
<td>0.074</td>
<td>0.143</td>
</tr>
<tr>
<td>$^{188}Os$</td>
<td>0.201</td>
<td>0.247</td>
<td>0.034</td>
<td>0.006</td>
<td>0.027</td>
<td>0.068</td>
</tr>
<tr>
<td>$^{190}Os$</td>
<td>0.191</td>
<td>0.214</td>
<td>0.031</td>
<td>0.005</td>
<td>0.022</td>
<td>0.059</td>
</tr>
<tr>
<td>$^{192}Os$</td>
<td>0.180</td>
<td>0.180</td>
<td>0.027</td>
<td>0.004</td>
<td>0.018</td>
<td>0.050</td>
</tr>
<tr>
<td>$^{194}Pt$</td>
<td>0.160</td>
<td>0.160</td>
<td>0.022</td>
<td>0.003</td>
<td>0.011</td>
<td>0.036</td>
</tr>
<tr>
<td>$^{196}Pt$</td>
<td>0.152</td>
<td>0.097</td>
<td>0.020</td>
<td>0.003</td>
<td>0.009</td>
<td>0.032</td>
</tr>
<tr>
<td>$^{198}Pt$</td>
<td>0.122</td>
<td>0.089</td>
<td>0.013</td>
<td>0.001</td>
<td>0.004</td>
<td>0.018</td>
</tr>
<tr>
<td>$^{222}Ra$</td>
<td>0.130</td>
<td>0.076</td>
<td>0.015</td>
<td>0.002</td>
<td>0.005</td>
<td>0.021</td>
</tr>
<tr>
<td>$^{224}Ra$</td>
<td>0.184</td>
<td>0.223</td>
<td>0.029</td>
<td>0.005</td>
<td>0.019</td>
<td>0.053</td>
</tr>
</tbody>
</table>
2 Formalism

The surface of the nucleus that is represented in polar coordinates by \( R(\theta, \phi) \) can be expanded in spherical harmonics as [1]

\[
R(\theta, \phi) = R_0 \left[ 1 + \sum_\mu \alpha_{\mu}^+ Y_{\mu}(\theta, \phi) \right],
\]

where \( R_0 \) is the radius of the spherical nucleus (i.e. when all \( a_{\mu} \) are vanish). Because of the reality of \( R \), then \( a_{\mu}^+ = (-1)^\mu a_{-\mu} \) and since \( R \) is rotationally invariant, then \( a_{\mu} = \sum D_{\mu\nu}^\dagger(\theta_j) a_{\nu} \) where \( D_{\mu\nu}^\dagger(\theta_j) \) is the transformation operator and \( a_{\mu} \) are the deformation parameters in space-fixed coordinates [16] and in body-fixed coordinates respectively, \( \theta_j \) are the Euler angles connecting the space-fixed and the rotated frames (where \( j = 1, 2, 3 \) [17,18]. The functions \( Y_{\mu} \) is the spherical harmonic of order \( \lambda, \mu \).

If we consider only the quadratic deformation of order 2 (i.e. \( \lambda = 2 \)) and the rotating coordinates are chosen to be in coinciding with the principal coordinates, then one can easily verify that \( a_{21} = a_{22} = 0 \), \( a_{20} \neq 0 \) (we will drop the index \( \lambda = 2 \) from the deformation parameters henceforth). So, we are now left with only two parameters \( a_0 \) and \( a_2 \) to describe the shape of the nucleus, and with the three Euler angels to specify the orientation of the principal axes of the nucleus. Sometimes, it is more convenient to use \( \beta \) and \( \gamma \) instead of \( a_0 \) and \( a_2 \) with the definition \( a_0 = \beta \cos(\gamma) \) and \( a_2 = \frac{\beta}{\sqrt{2}} \sin(\gamma) \), referred as Bohr notation [19].

3 Hydrodynamical Model

The hydrodynamical model assumes the irrotational flow \( (\vec{\nabla} \times \vec{v}(r) = 0) \), and the incompressibility \( \vec{\nabla} \cdot \vec{v}(r) = 0 \).
of the nuclear matter [20]. Hence, the velocity of the 
volume element $dt$ can be derived from the scalar 
potential $\chi(r)$ as $v(r) = \hat{\nabla} \chi(r)$. As a result, $\chi(r)$ is the general 
solution of the Laplace equation $\nabla^2 \chi(r) = 0$ [6,17]:

\[
\chi(r) = \sum \xi_{\mu} r^2 Y_{2\mu}(\theta, \phi) = \frac{1}{2} \sum \alpha_{\mu}^2 r^2 Y_{2\mu}(\theta, \phi), \quad \text{where } \xi_{\mu} \text{ is}
\]

a parameter which is related to $\dot{\alpha}_{\mu}^r$ (the time derivative of $\alpha$) by the relation $\xi_{\mu} = \frac{R_0}{2R} \dot{\alpha}_{\mu}^r$. If the oscillations are 
asumes to be small, then $R$ can be approximated to $R_0$ and $\dot{\alpha}_{\mu}^r$ becomes $\frac{1}{2} \dot{\alpha}_{\mu}^r$. Therefore, the kinetic energy of the entire liquid drop with constant density $\rho_0$ is

\[
T = \frac{1}{B} \sum_{\mu \nu} \dot{\alpha}_{\mu}^r \dot{\alpha}_{\nu}^r \int \frac{d\Omega}{4} \sum_{j, l} \langle 2jL^2_{lj} \rangle a_j a_{j'}^\dagger
\]

\[
= \frac{1}{B} \sum_{\mu \nu} \dot{\alpha}_{\mu}^r \dot{\alpha}_{\nu}^r \int \frac{d\Omega}{4} \sum_{j, l} \langle 2jL^2_{lj} \rangle a_j a_{j'}^\dagger
\]

\[
= \frac{1}{B} \sum_{\mu \nu} \dot{\alpha}_{\mu}^r \dot{\alpha}_{\nu}^r \int \frac{d\Omega}{4} \sum_{j, l} \langle 2jL^2_{lj} \rangle a_j a_{j'}^\dagger
\]

\[
T = \frac{1}{2} B \sum_{\mu \nu \sigma \sigma'} \alpha_{\mu}^r \alpha_{\nu}^r \alpha_{\sigma}^r \alpha_{\sigma'}^r \int \frac{d\Omega}{4} \sum_{j, l} \langle 2jL^2_{lj} \rangle a_j a_{j'}^\dagger
\]

\[
+ \frac{1}{2} B \sum_{\mu \nu \sigma \sigma'} \alpha_{\mu}^r \alpha_{\nu}^r \alpha_{\sigma}^r \alpha_{\sigma'}^r \int \frac{d\Omega}{4} \sum_{j, l} \langle 2jL^2_{lj} \rangle a_j a_{j'}^\dagger
\]

\[
+ \frac{1}{2} B \sum_{\mu \nu \sigma \sigma'} \alpha_{\mu}^r \alpha_{\nu}^r \alpha_{\sigma}^r \alpha_{\sigma'}^r \int \frac{d\Omega}{4} \sum_{j, l} \langle 2jL^2_{lj} \rangle a_j a_{j'}^\dagger
\]

\[
T = \frac{1}{B} \sum_{\mu \nu \sigma \sigma'} \alpha_{\mu}^r \alpha_{\nu}^r \alpha_{\sigma}^r \alpha_{\sigma'}^r \int \frac{d\Omega}{4} \sum_{j, l} \langle 2jL^2_{lj} \rangle a_j a_{j'}^\dagger
\]

\[
+ \frac{1}{2} B \sum_{\mu \nu \sigma \sigma'} \alpha_{\mu}^r \alpha_{\nu}^r \alpha_{\sigma}^r \alpha_{\sigma'}^r \int \frac{d\Omega}{4} \sum_{j, l} \langle 2jL^2_{lj} \rangle a_j a_{j'}^\dagger
\]

\[
+ \frac{1}{2} B \sum_{\mu \nu \sigma \sigma'} \alpha_{\mu}^r \alpha_{\nu}^r \alpha_{\sigma}^r \alpha_{\sigma'}^r \int \frac{d\Omega}{4} \sum_{j, l} \langle 2jL^2_{lj} \rangle a_j a_{j'}^\dagger
\]

3.1 The original work of Rayleigh and Bohr

Since oscillations are assumed to be small, then 
$R(\theta, \phi)$, the upper limit of the integral in Eq. (3) can be 
approximated to $R_0$ and the integral results in $R_0^5$, the 
first term in the expansion of Eq. (3). The integration in 
Eq. (2) along with only the first term of Eq. (3) was 
carried out by Rayleigh. He obtained

\[
T = \frac{1}{B} \sum_{\mu \nu} \dot{\alpha}_{\mu}^r \dot{\alpha}_{\nu}^r
\]

\[
= \frac{1}{B} \sum_{\mu \nu} \dot{\alpha}_{\mu}^r \dot{\alpha}_{\nu}^r
\]

\[
= \frac{1}{B} \sum_{\mu \nu} \dot{\alpha}_{\mu}^r \dot{\alpha}_{\nu}^r
\]

where $B$ is the inertial parameter and it is equal to

$B = \frac{\rho_0 R_0^5}{2}$

Eq. (4) describes the kinetic energy of the surface oscil-
lations of a classical liquid drop in a space-fixed co-
ordinates. According to Bohr, for deformed nuclei, the 
collective motions should be of vibrational and rotational 
 modes. In order to distinguish between these two modes 
Bohr rewrite Eq. (4) in terms of $\alpha_0, \alpha_2$ and the three Euler 
angles. For this purpose we use $\dot{\alpha}_{\mu}^r = \Sigma \alpha_{\mu}^r \dot{\alpha}_{\mu}^r + D_{\mu \nu}^r \dot{\alpha}_{\nu}^r$, therefore

\[
+ \frac{1}{2} B \sum_{\mu \nu \sigma \sigma'} \alpha_{\mu}^r \alpha_{\nu}^r \alpha_{\sigma}^r \alpha_{\sigma'}^r \int \frac{d\Omega}{4} \sum_{j, l} \langle 2jL^2_{lj} \rangle a_j a_{j'}^\dagger
\]

(7)

By using the unitary property of the transformation 
matrix [23] $D_{\mu \nu}^{\mu \nu}$ (i.e. \(D_{\mu \mu}^{\mu \mu} = \delta_{\mu, \mu'}\)), Eq. (7) can be 
rewrite as

\[
T = \frac{1}{2} B \sum_{\nu \nu'} \sum_k (2\nu L_k^2 2\nu' a_j a_{\nu} a_{\nu'} \omega_k^2 = \frac{1}{2} \sum_k \omega_k^2
\]

(8)

where $k = k'$ since $\nu$ and $\nu'$ should be even. Further, this 
leads to the MOI along the axis $k$:
\[ \mathcal{I}_{\text{Bohr}}^{k} = B \sum_{vv'} (2v'L_{2v}^2) a_v a_{v'} \quad (9) \]

we will call this relationship as the original form Bohr's formula.

Till now, we have briefly presented the well known Bohr results with the first term of radial distribution as \( R_0 \) in Eq. (3). The full algebraic details can be followed from the books of Preston [16], Eisenberg [17] and Pal [21]. Further, \( \mathcal{I}_{\text{Bohr}}^{1}, \mathcal{I}_{\text{Bohr}}^{2} \) and \( \mathcal{I}_{\text{Bohr}}^{3} \) (that is the MOI along each of the three body-fixed axes) can be simplified by using the identities of Ladder operators [24, 25] as:

\[ \mathcal{I}_{\text{Bohr}}^{1} = B \sum_{vv'} (2v'L_{2v}^2) a_v a_{v'} = \frac{1}{4} B \sum_{vv'} (2v'L_{2v}^2 + L_{2v}^2 + 2L_{2v}L_{2v}) a_v a_{v'} \]
\[ = B(2 \sqrt{6}a_0 a_2 + 2a_2^2 + 3a_0^2) \quad (10) \]

Similarly,

\[ \mathcal{I}_{\text{Bohr}}^{2} = B(2 \sqrt{6}a_0 a_2 + 2a_2^2 + 3a_0^2) \quad (11) \]

\[ \mathcal{I}_{\text{Bohr}}^{3} = B \sum_{v} (2v'L_{2v}^2) a_v a_{v'} = 8Ba_2^2 \quad (12) \]

In general,

\[ \mathcal{I}_{\text{Bohr}}^{k} = 4B\beta^2 \sin^2 \left( \gamma - \frac{2\pi k}{3} \right) \quad (13) \]

For the special case of axially symmetric nuclei (\( \gamma = 0 \)) and \( \mathcal{I}_{\text{Bohr}}^{2} = 3B\beta^2, \mathcal{I}_{\text{Bohr}}^{3} = 0 \) \quad (14)

Unfortunately, the values of the moment of inertia calculated using Eq. (13) are nearly five times smaller than that measured by the empirical fitting of the first few low-lying levels. We think that this poor agreement is due to the assumption that the oscillations are small to an extent that \( R \) has been approximated to \( R_0 \) at several places in the work of Rayleigh. In this work, this assumption is verified by considering the second and third terms in Eq. (3). The second term with the integration in Eq. (2) will be discussed in subsection 3.1 under the heading "first order modification of MOI". In subsection 3.2 below, the third term will be discussed under the heading "second order modification of MOI".

### 3.2 First order modification of MOI

Let us denote the integration in Eq. (2) with the second term in Eq. (3) as \( T' \), then

\[ T' = \frac{1}{8} \rho_0 R_0^5 \sum_{\mu \mu'} \alpha_{\mu}^* \alpha_{\mu} \sum_{\sigma} \alpha_{\sigma}^* \int d\Omega \]
\[ \times \left\{ (4-\mu \mu') Y_{2\mu} Y_{2\mu'} - L_{2\mu} Y_{2\mu'} \right\} Y_{\sigma} \quad (15) \]

In Eq. (15) using the property \( \alpha_\mu^* = (-1)^\mu \alpha_{-\mu}^* \) of the deformation parameter and since \( \alpha_{\mu} \) and \( \alpha_{\mu'} \) represent two arbitrary components for the velocity of a given point on the surface of the nucleus then the expression

\[ \sum_{\mu \mu'} \alpha_{\mu}^* \alpha_{\mu} = \sum_{\mu \mu'} \alpha_{\mu}^* (-1)^\mu \alpha_{-\mu} \] should be zero unless \( \mu = -\mu' \) (orthogonal property of \( a \) ) this result can be written mathematically as

\[ \sum_{\mu \mu'} \alpha_{\mu}^* \alpha_{\mu} = \sum_{\mu \mu'} \alpha_{\mu}^* (-1)^\mu \alpha_{-\mu} \delta_{\mu, -\mu'} \quad (16) \]

Putting this result into Eq. (15) we get

\[ T' = \frac{1}{8} \rho_0 R_0^5 \sum_{\mu} |\alpha_{\mu}|^2 \sum_{\sigma} \alpha_{\sigma}^* \]
\[ \times \int d\Omega \left\{ (4+\mu^2) Y_{2\mu} (-1)^\mu Y_{2\mu} \right\} Y_{\sigma} \]
\[ - (2-\mu) \sum_{\mu'} Y_{2\mu+1} (-1)^\mu Y_{2\mu+1} \right\} Y_{\sigma} \]. \quad (17)

Since the subscript index \( \mu + 1 \) in the second part in Eq. (18) is dummy variable, it can be replaced by \( \mu \) without any change in the value of the integration. This leads to

\[ T' = \frac{1}{8} \rho_0 R_0^5 \sum_{\mu} |\alpha_{\mu}|^2 \sum_{\sigma} \alpha_{\sigma}^* (10+\mu) \int d\Omega Y_{2\mu} Y_{2\mu} \]
\[ = CB \sum_{\mu} |\alpha_{\mu}|^2 \alpha_\mu^* (220\mu|2\mu), \quad (19) \]

where, \( C = \frac{5}{2} \frac{\sqrt{5}}{4\pi} \) \( (2200|20) \), \( B = \frac{1}{2} \rho_0 R_0^5 \). In getting the second line of Eq. (19) form the first we use the identity

\[ \int d\Omega Y_{\lambda, \mu_1}, Y_{\lambda, \mu_2}, Y_{\lambda, \mu_3} = \frac{(2\lambda_1 + 1)(2\lambda_2 + 1)}{4\pi(2\lambda_3 + 1)} (\lambda_1 \lambda_2 \mu_1 \mu_2 | \lambda_3 \lambda_3 \lambda_3)(\lambda_1 \lambda_1 \lambda_0 | \lambda_3 \lambda_3 \lambda_3) \]. \quad (20)

one can get

\[ T' = CB \sum_{\mu} |\alpha_{\mu}|^2 \alpha_\mu^* (220\mu|2\mu), \quad (21) \]

where \( C = \frac{5}{2} \frac{\sqrt{5}}{4\pi} \) \( (2200|20) \) and \( B = \frac{1}{2} \rho_0 R_0^5 \). In body
fixed coordinate the rotational kinetic energy part of Eq. (19) can be written as
\[ T'_\text{Rot} = CB \sum_{\mu} \sum_{\nu \sigma \nu'} D_{\mu \nu}^2 D_{\sigma \nu'}^2 \frac{\partial \varphi}{\partial \varphi} \cdot a_{\nu} a_{\nu'} a_{\sigma} a_{\sigma'} (22\nu | 2\mu) \]
\[ = CB \sum_{\nu \sigma \nu'} \sum_{m \nu \sigma} \sum_{\mu} D_{\mu \nu m}^2 D_{\nu' \sigma m'}^2 \frac{\partial \varphi}{\partial \varphi} (22\nu | 2\mu), \]
\[ \times (2m|L_0 | 2\nu)^* (2m'|L_0 | 2\nu') a_{\sigma} a_{\sigma'} a_{\nu} a_{\nu'}, \] (22)
where \( k' \) and \( k \) should be equal each other since \( \nu \) and \( \nu' \)
can only be even. Using this condition together with the
identity \( \sum_{\mu} D_{\mu \nu}^2 D_{\nu' \sigma}^2 \langle A_1 A_2 A_3 | 2 \rangle = \langle A_1 A_2 A_3 | 2 \rangle \) \( k \rightarrow k' \) \( \mu \rightarrow \mu' \) \( \nu \rightarrow \nu' \) \( \sigma \rightarrow \sigma' \),
[24,26] we can get
\[ T_{\text{Rot}'} = 2CB \sum_{\nu \sigma \nu'} \sum_{m} (22m|2m|2\nu | 2\nu') a_{\nu} a_{\nu'} a_{\sigma} a_{\sigma'} a_{0} a_{0}. \]
\[ = 2CB \sum_{\nu \sigma \nu'} \sum_{m} (22m|2m|2\nu | 2\nu') a_{\nu} a_{\nu'} a_{\sigma} a_{\sigma'} a_{0} a_{0}. \] (23)

Where \( S'_1 \) in the last line of Eq. (23) is called first order
differentiation in the value of the moment of inertia
along the body fixed axes \( k \), and it can be defined as
\[ S'_1 = 2CB \sum_{\nu \sigma \nu'} \sum_{m} (22m|2m|2\nu | 2\nu') a_{\nu} a_{\nu'} a_{\sigma} a_{\sigma'} a_{0} a_{0}. \] (24)

In order to extract the \( S'_1 \), \( S'_2 \) and \( S'_3 \) from Eq. (24),
it should be noted first that the allowed choices of \( \nu, \nu', m \)
are \( \nu = 0, 2, -2, \nu' = \nu, \nu \pm 2 \) and \( m = \nu, \nu \pm 1 \) respectively.
The first correction in \( \text{MOI} \) corresponding to \( k \) can be obtained as
\[ S'_1 = 2CB \sum_{\nu \sigma \nu'} \sum_{m} (22m|2m|2\nu | 2\nu') a_{\nu} a_{\nu'} a_{\sigma} a_{\sigma'} a_{0} a_{0} \]
\[ = \frac{1}{4} \times 2CB \sum_{\nu \sigma \nu'} \sum_{m} (22m|2m|2\nu | 2\nu') a_{\nu} a_{\nu'} a_{\sigma} a_{\sigma'} a_{0} a_{0} \]
\[ \times (2m|L_0 | 2\nu)^* (2m'|L_0 | 2\nu') a_{\sigma} a_{\sigma'} a_{0} a_{0} \]
\[ = 2CB \left[ \frac{2}{7} \left( 2a_{2}^{2} a_{0} - 3a_{0}^{3} - 6a_{2} a_{2} a_{0} \right) \right]. \] (25)

Similarly,
\[ S'_2 = 2CB \sum_{\nu \sigma \nu'} \sum_{m} (22m|2m|2\nu | 2\nu') a_{\nu} a_{\nu'} a_{\sigma} a_{\sigma'} a_{0} a_{0} \]
\[ = \frac{1}{4} \times 2CB \sum_{\nu \sigma \nu'} \sum_{m} (22m|2m|2\nu | 2\nu') a_{\nu} a_{\nu'} a_{\sigma} a_{\sigma'} a_{0} a_{0} \]
\[ \times (2m|L_0 | 2\nu)^* (2m'|L_0 | 2\nu') a_{\sigma} a_{\sigma'} a_{0} a_{0} \]
\[ = 2CB \left[ \frac{3}{7} \left( 2a_{2}^{2} a_{0} - 3a_{0}^{3} - 6a_{2} a_{2} a_{0} \right) \right]. \] (26)

In the case of axially symmetric we have \( \gamma = 0 \),
\( a_{2} = 0 \), and \( a_{0} = \beta \), the values of \( S'_1 \), \( S'_2 \) and \( S'_3 \) are given as
\[ S'_1 = S'_2 = -6 \sqrt{\frac{7}{2}} CB \beta^{3}, \quad S'_3 = 0. \] (28)

3.3 Second order modification of \( \text{MOI} \)

Similarly, Eq. (2) with the third term \( 2R_{0}^{2} \sum_{\sigma \nu} a_{\sigma} a_{\nu} Y_{2\sigma} Y_{2\nu} \) of radial integration which we are
denoted by \( T'' \) it can be written as
\[ T'' = \frac{1}{4} \rho R_{0}^{5} \sum_{\mu \sigma} a_{\mu} a_{\sigma} \sum_{\nu} a_{\nu} a_{\nu} \]
\[ \times \int d \Omega \left[ (4 - \mu') Y_{2\nu} Y_{2\mu'} - L_{x} Y_{2\mu} L_{x} Y_{2\nu} \right] Y_{2\nu} Y_{2\mu} \]
\[ \times \left[ (10 - \mu) \right] d \Omega \left[ (10 - \mu Y_{2\nu} Y_{2\nu} -(10 - \mu) Y_{2\nu} Y_{2\mu} \right], \] (29)
where \( B = \frac{1}{4} \rho R_{0}^{5} \) is the inertial parameter which was
mentioned in section 3. Each pair \( Y_{2\nu} Y_{2\nu} \) and \( Y_{2\nu} Y_{2\mu} \) in
second line of Eq. (27) can be coupled using the identity
\[ Y_{j m} Y_{j m} = \frac{\sqrt{2j + 1}}{4\pi} \sqrt{\frac{2j + 1}{4\pi}} \]
\[ \times \sum_{j} \frac{4\pi}{j} Y_{m j} (j_{1} j_{2} m_{1} m_{2} | j m) (j_{1} j_{2} 00 | j m). \] The integration in Eq. (29) becomes
\[ \int d \Omega \left[ (10 - \mu) Y_{2\nu} Y_{2\nu} -(10 - \mu) Y_{2\nu} Y_{2\mu} \right] = \frac{25}{4\pi} \frac{(10 - \mu)^{2}}{4\pi} \sum_{j} \frac{1}{4\pi} \frac{1}{j + 1} (220 | 2j + 1\rangle \langle 220 \rangle (-1)^{j + 1}(10 - \mu)). \] (30)

If we use the identity
\[ \sum_{j m} \langle j_{1} j_{2} m_{1} m_{2} | j_{1} j_{2} m_{1}' m_{2}' \rangle = \delta_{m_{1}, m_{1}'} \delta_{m_{2}, m_{2}'} \]

One can easily verify that the summation over \( j \) in
Eq. (30) is unity. Putting this result in Eq. (29) we get
\[ T'' = \frac{1}{2} \frac{25}{4\pi} B \sum_{j} \langle j_{\mu} \rangle^{2} \sum_{j} \langle j_{\sigma} \rangle^{2} \langle 4(10 - \mu) \rangle. \] Since \( \mu \) runs from -2 to 2 the summation over \( \mu \)
should be zero. Finally, \( T'' \) can be simplified as
\[ T'' = \frac{1}{2} B \sum_{j} \langle j_{\mu} \rangle^{2} \sum_{j} \langle j_{\sigma} \rangle^{2} \frac{250}{4\pi}. \] (31)

In body fixed coordinates, the rotational part of the
quantity \( \sum_{\mu} \langle j_{\mu} \rangle^{2} \) was treated in detail in subsection 2.1 of
this section and the result is \( \sum \langle 2' | L_d | 2' \rangle \alpha_\alpha \alpha_\alpha \omega_\alpha^2 \) where as it can be easily verified that \( \sum |\alpha_\alpha \alpha_\alpha |^2 = \beta^2 \), it follows

\[
T_{rot}'' = \frac{250}{4\pi} \beta^2 \frac{1}{2} \sum \langle 2' | L_d | 2' \rangle \alpha_\alpha \alpha_\alpha \omega_\alpha^2
\]

\[
= \frac{1}{2} \sum_k 3 k'' \omega_k^2.
\]  

(32)

where the quantities \( k'' \) are denoted as the second modification to \( MOI \) and it can be obtained as:

\[
3_k'' = \frac{125}{2\pi} \beta^2 B \sum_{\nu \nu \nu} \langle 2' | L_d | 2' \rangle \alpha_\alpha \alpha_\alpha \omega_\alpha^2.
\]  

(33)

or,

\[
3_k'' = \frac{125}{2\pi} \beta^2 3_k^{Bohr}.
\]  

(34)

The \( 3_k^{Bohr} \) values are already known and the second order modification of \( MOI \) can be found by multiplying the \( 3_k^{Bohr} \) by a factor of \( \frac{125}{2\pi} \beta^2 \). For the special case of axially symmetric:

\[
3_1'' = 3_2'' = \frac{750}{4\pi} \beta^4, \ 3_3'' = 0.
\]  

(35)

4 Results and Discussion

The total moment of inertia (\( MOI \)) including first and second corrections can be obtained for the special case of axially deformed nuclei, as follows:

\[
3_k^{Total} = 3_k^{Bohr} + 3_k' + 3_k''
\]

\[
= 3B \beta^2 - 6CB \sqrt{\frac{2}{7}} \beta^3 + 750 B \beta^4
\]

\[
= 3k^{Bohr} \left( 1 - 2C \sqrt{\frac{2}{7}} \beta + \frac{250}{4\pi} \beta^2 \right),
\]  

(36)

which is simply \( 3_k^{Bohr} \) multiplied by the modification factor \( 1 - 2C \sqrt{\frac{2}{7}} \beta + \frac{250}{4\pi} \beta^2 \).

The comparison of calculated results and experimental data of \( MOI \) for the even-even axially deformed nuclei is shown in Table 1. The first column of the table denotes the nucleus while the second column represents the \( \beta \) deformation coefficients, which are extracted from the associated electric quadrupole transitions Ref. [22]. The third column presents the experimental values of \( MOI \), which are deduced from the experimental energy spacing of ground state rotational bands Ref. [22]. The fourth, fifth and sixth columns represent the calculated values by using Eqs. (14), (21) and (30) to find Bohr (\( 3_k^{Bohr} \)), first (\( 3' \)) and second (\( 3'' \)) order modification to \( MOI \), respectively. The seventh column presents our total \( MOI \) value from Eq. (36), as \( 3_k^{Total} = 3_k^{Bohr} + 3_k' + 3_k'' \). All values of moment of inertia \( 3_k \) in Table 1, are shown in units of \( L^{\frac{3}{2}} A^{1/2} \) MeV, and \( 3_k^{rig} \) is the \( MOI \) for a rigid body having the same volume, shape and density of the nucleus. One can see from the Table 1 that there is a large enhancement in the calculated values of \( 3_k^{Total} \). It becomes 0.6 of the experimental ones instead of 0.2 in the case of unmodified ones. It is remarkable to note that the second modification (\( 3'' \)) to \( MOI \) contribute \( \frac{250}{4\pi} \beta^2 \) times more than the Bohr value (\( 3_k^{Bohr} \)). On the other hand, first modification (\( 3' \)) is quite negligible. These phenomena are shown clearly in Fig. 1a) and b). Where Fig 1a) shows only the contribution of the first modification.
Bohr+second modifications

In order to illustrate the effects of the first and second corrections separately on the results of Bohr we draw a curve only for the first correction plus the values of Bohr and another curve for Bohr plus only the second correction (the orange curves in Fig. 1 part a) and part b) as a function of the deformation parameter $\beta$.

The graph includes a curves for the original formula of Bohr and the experimental results as a comparison. It is shown clearly from this figure that, for all $\beta < 0.15$, the effects of both of them are small enough to be neglected. As $\beta > 0.15$ the contribution of the second correction increases more rapidly than that of the first one and it becomes greater than the zeroth order when $\beta = 0.3$. This means that for nuclei with large deformation parameter the assumption of small oscillations suggested by Rayleigh is not adequate. That means the first term is not enough to represent situation of the nucleus.

A comparison of the results of the modified form of Bohr’s relationship (the current work $S^{\text{corr.}}$) Eq. (34) and the results of original Bohr’s relationship Eq. (14) with numerical calculations ($S^{\text{Num.}}$) presented by Berdichevsky at el. [9] and also experimental data for a few nuclei are listed in Table 2. The numerical values were calculated on the basis of cranking model with the Sk-3 field.

Two features can be noted from this Table 2:

i) in general the results of the modified form of Bohr’s relationship are much closer to the numerical results than the original one for all nuclei in question,

ii) the results of this work for the nuclei $^{154}\text{Sm}$ and $^{158}\text{Gd}$ is close to numerical ones (the ratio of the difference are nearly 8% for $^{154}\text{Sm}$ and 16% for $^{158}\text{Gd}$ while they become large for the remaining nuclei). However there are significant enhancement due to the modification factor.

The ratio $S/S_{\text{rig}}$ and $\beta$ deformation for axially even-even deformed nuclei is illustrated in Fig. 2, respectively.

The filled squares denote the experimental data while red line represent $S^{\text{Bohr}}$ values, which lie quite far from the experimental data [22]. Our results with the Bohr+first modification and Bohr+second modification are shown in blue line in figures, which shift towards the experimental data drastically. The shifting increases rapidly with an increase in $\beta$, mainly due to the second modification to $MOI$.

For an average value of $\beta = 0.27$ in the listed axially deformed nuclei, Bohr estimates are only 18% of the experimental value, while the Bohr estimates plus first modifications gets improved by 4-5% reaching to 22-23% of the experimental value. As soon as we include the second modification to this, the $MOI$ values become more than 53% of the experimental value. In mathematics, it happens due to the symmetry properties of spherical harmonics. One can notice even number of involved $Y_{2\mu}$ operators into Bohr estimates and second modification, while odd number of $Y_{2\mu}$ operators in the first modification. Physically, it supports the large amplitude of vibrations at nuclear surface for deformed rotating even-even nuclei. Further improvements can be foreseen while doing the similar exercise for next higher order terms, particularly with next to next term having even number of involved $Y_2$ operators, which is going to be very complicated in nature and are in progress. Furthermore, we can obtain the total $MOI$ in three coordinates for triaxial nuclei, as following

$$S_1^{\text{total}} = S_1^{\text{Bohr}} + S_1' + S_1''$$
$$= 4B\beta^2 \sin^2\left(\frac{\gamma - 2\pi}{3}\right) \left(1 + 2C\sqrt{\frac{2}{7}}\beta \cos\gamma + \frac{250}{4\pi\beta^2}\right)$$
$$- 2CB\sqrt{\frac{2}{7}}\beta^3 \cos\gamma \left(3\sqrt{3}\beta \sin\gamma + 6\cos\gamma\right),$$

$$S_2^{\text{total}} = S_2^{\text{Bohr}} + S_2' + S_2''$$
$$= 4B\beta^2 \sin^2\left(\frac{\gamma - 4\pi}{3}\right) \left(1 + 2C\sqrt{\frac{2}{7}}\beta \cos\gamma + \frac{250}{4\pi\beta^2}\right)$$
$$- 2CB\sqrt{\frac{2}{7}}\beta^3 \cos\gamma \left(3\sqrt{3}\beta \sin\gamma + 6\cos\gamma\right),$$

$$S_3^{\text{total}} = S_3^{\text{Bohr}} + S_3' + S_3''$$
$$= 4B\beta^2 \sin^2\gamma \left(1 + 2C\sqrt{\frac{2}{7}}\beta \cos\gamma + \frac{250}{4\pi\beta^2}\right),$$

where we have used Eqs. (10), (11) and (12) to calculate
the Bohr values of MOI, shown as $\mathcal{I}^{\text{Bohr}}_1, \mathcal{I}^{\text{Bohr}}_2, \mathcal{I}^{\text{Bohr}}_3$, respectively. The first modification to the moment of inertia MOI are obtained by Eqs. (25), (26) and (27), respectively, and Eq. (34) is used for the second order modification to MOI.

The total of moment of inertia $\mathcal{I}^{\text{Total}}_k$ for triaxial nuclei have been calculated by using Eqs. (37), (38) and (39), respectively. The comparison of calculated results with the experimental data and Bohr estimation ($\mathcal{I}^{\text{Bohr}}$) is given in Table 3. All experimental values for the moment of inertia $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ along 1−, 2− and 3− body axes respectively and also the deformation parameters ($\beta, \gamma$) are taken

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from Allmond and Wood Ref. [28]. Our calculated results are clearly in better agreement to the experimental data in comparison to the original Bohr estimation.

5 Conclusion

A detailed theoretical extension to hydrodynamical model has been presented in view of the contributions arising from higher order terms of radial distribution. Such calculated MOI values are found to be in better agreement than the original model for both axially deformed and triaxial nuclei. This highlights the crucial approximation involved in the irrotational picture of liquid droplet in terms of small amplitude vibrations and further supports the large amplitude vibrations at the nuclear surface. Such investigations strengths the irrotational and collective picture of even-even deformed nuclei. Further improvements to this extension are in progress.

It should be mentioned that the real values of $\xi_{2\mu}$ which is given in the section 3 is $\xi_{2\mu} = \frac{R_0}{2R} \alpha_{2\mu}^\ast$, this quantity was approximated by Raleigh to $\frac{1}{2} \alpha_{2\mu}^\ast$ by making $R = R_0$. One can make another modification here by making expansion to $R$ and then treat the higher terms.

In the future we are planning to predict the values of the parameters of inertia and rigidity within the hydrodynamic model by using simple harmonic potential.

The obtained results compared with the experimental data and with the results of Bohr, shown a good agreement with experimental data by describing almost 0.6 to sometimes 0.7.

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