Traces of quantum fuzziness on the black hole shadow and particle deflection in the multi-fractional theory of gravity

Reggie C. Pantig[†]

Physics Department, Mapúa University, 658 Muralla St., Intramuros, Manila 1002, Philippines

Abstract: In this study, we investigate the properties of black holes within the framework of multi-fractional theories of gravity, focusing on the effects of q-derivatives and weighted derivatives. These modifications, which introduce scale-dependent spacetime geometries, alter black hole solutions in intriguing ways. Within these frameworks, we analyze two key observable phenomena - black hole shadows and particle deflection angle in the weak field limit - using both analytical techniques and observational data from the Event Horizon Telescope (EHT) for M87* and Sgr A*. The study using the q-derivative formalism reveals that the multi-scale length ℓ_* influences the size of the black hole shadow in two ways and modifies the weak deflection angle. Constraints on ℓ_* are derived from the EHT observations, showing significant deviations from standard Schwarzschild black hole predictions, which range from 10^9 to 10^{10} orders of magnitude. Additionally, the weak deflection angle is computed using the non-asymptotic generalization of the Gauss-Bonnet theorem (GBT) to reveal the effects of finite-distance and multi-scale parameters. Using the Sun in the Solar System test, we observe that the constraints for ℓ_* range from 10^8 to 10^9 orders of magnitude. Results from the weighted derivative formalism generate a dS/AdS-like behavior, where smaller deviations are found in the strong field regime than in the weak field regime. The results suggest that, while these effects are subtle, they provide a potential observational signature of quantum gravity effects. The findings presented here contribute to the broader effort of testing alternative theories of gravity through black hole observations, offering a new perspective on the quantum structure of spacetime at cosmological and astrophysical scales.

Keywords: multi-fractional spacetime, black holes in higher dimensions, supermassive black holes, black hole shadow, weak deflection angle

DOI: 10.1088/1674-1137/adb9c6 **CSTR:** 32044.14.ChinesePhysicsC.49065102

I. INTRODUCTION

The exploration of black holes has long captivated the field of theoretical physics, not only for their dramatic properties within general relativity (GR) but also for the possibility they present in testing fundamental theories beyond Einstein's framework. In particular, black holes in higher dimensions provide a compelling arena for investigating the behavior of gravity when the standard four-dimensional (4D) picture is expanded [1, 2]. Higher-dimensional spacetimes, as explored in the context of string theory [3−5] and braneworld models, introduce novel features into black hole physics, such as modified horizons, thermodynamics [6], and new forms of gravitational radiation, which provide clues about the subtle and non-trivial structure of spacetime and create a pathway for probing alternative gravitational theories [7−16].

Black holes have been explored within the framework of multi-fractional theories of gravity [17], specifically focusing on two models: those with q-derivatives and those with weighted derivatives [18]. In these models, spacetime dimensionality changes with the scale being probed, influenced by a fundamental length, ℓ_*, and such modifications to the fabric of spacetime yield intriguing deformations to black hole solutions. In particular, Schwarzschild black holes within these multi-fractional contexts can exhibit altered event horizons and singularities. For instance, in the q-derivatives scenario, an additional ring singularity emerges, and the black hole's Hawking temperature can be higher than that in GR. In the context of higher-dimensional black holes, its importance lies in its demonstration of how the multi-fractional geometry can modify black hole characteristics even at macroscopic scales, such as shifts in the event horizon, thermodynamic properties, and the presence of novel singularities. These effects, although subtle, provide an exciting opportunity to test phenomenological aspects of quantum gravity [19]. Thus, the exploration of black holes within multi-fractional models can contribute to a

Received 9 December 2024; Accepted 25 February 2025; Published online 26 February 2025

[†] E-mail: rcpantig@mapua.edu.ph

deeper understanding of how deviations from standard 4D GR can manifest in observable astrophysical phenomena, providing insights into quantum gravity frameworks.

One of the most observable aspects of black holes lies in their shadows — dark regions formed by the gravitational bending of light near the event horizon, primarily governed by the photon region in rotating black holes and the photonsphere in spherically symmetric cases [20, 21]. The study of black hole shadows has evolved from theoretical predictions [22–24] to direct observational tests, such as those conducted by the Event Horizon Telescope (EHT), which captured the shadow of the supermassive black holes M87* and Sgr. A* [25–29]. Notably, in black hole imaging, the term shadow often refers to the inner shadow, whereas the bright ring observed in EHT images corresponds to lensed photon trajectories rather than the actual shadow. Recent studies, including those in Refs. [30, 31], have further clarified this distinction, emphasizing the importance of understanding the photon region and the role of multiple light trajectories in black hole imaging. Shadows are not only dependent on the mass and spin of the black hole but also serve as probes for the underlying gravitational theory [32, 33]. Because of such importance, many authors have analyzed black hole shadows in various alternative gravity frameworks. In particular, recent advancements have explored the influence of strong magnetic fields [34–37], studied deviations from GR [38–48], and proposed novel observational techniques [49], which are crucial to contextualizing the present study within the broader landscape of black hole physics. In the context of multi-dimensional black holes, the reader may refer to published references [50–65]. Some recently published references on the topic are also available [66, 67].

Another important black hole phenomenon that we can consider is the deflection angle, which has two regimes: weak and strong fields. The former measures the bending of light by black holes at larger impact parameters and is particularly sensitive to deviations from GR. That is, observing relativistic image lensing formations can provide limitations on the compactness of massive dark entities, which is independent of their mass and distance [68–74]. Using tools such as the Gauss-Bonnet theorem (GBT) [75], which links the geometry of the spacetime to its topology, we can compute deflection angles that may differ in higher-dimensional or modified gravity settings. Recently, the GBT was extended to include the finite distance of the source (of photons) and the receiver [76]. Furthermore, its generalization, in which the integration domains were modified to include the photon sphere, was artfully considered [77]. It has been used to explore the deflection angle, which is also generalized to include massive time-like particles [78–83].

This study aims to probe whether the multi-fractional theory leaves measurable traces on these observable phenomena. Specifically, we investigate the effect of a parameter emerging from the multi-fractional framework on the black hole shadow and the weak deflection angle when viewed in the context of higher-dimensional black holes. Such an approach not only serves as a test of the multi-fractional theory itself but also enhances our understanding of the broader implications of non-standard dimensionality in gravitational systems. By examining these effects through established techniques such as the non-asymptotic generalization of the GBT, we aim to bridge the gap between theoretical predictions and potential observational signatures, enhancing our understanding of the deep structure of spacetime. At the time of writing this paper, no study has explored the multi-fractional nature of a black hole spacetime.

The remainder of this paper is organized as follows. In Sec. II, we briefly review the application of multi-fractional theory to black hole solutions. Subsequently, in Sec. III, we study the shadow and find constraints to the multi-scale length ℓ_* using the EHT results for M87* and Sgr. A*. In Sec. IV, we test the weak field regime for the parameter ℓ_*. Finally, in Sec. V, we state final remarks and possible research prospects. Unless specified differently, we use the metric signature $(-,+,+,+)$ and geometrized units by setting $G = c = 1$.

II. BRIEF REVIEW OF A BLACK HOLE IN MULTI-FRACTIONAL THEORY

In this section, we first review the black hole solution in multi-fractional theory with q-derivatives. In a seminal work [18], the standard GR equations were modified by replacing ordinary derivatives with q-derivatives, which reflect a scale-dependent structure of spacetime. The first modification appears in the Riemann tensor, where the usual derivative ∂_μ is replaced by the q-derivative:

$$
{}^qR^\rho_{\mu\sigma\nu} = \frac{1}{v_\sigma}\partial_\sigma {}^q\Gamma^\rho_{\mu\nu} - \frac{1}{v_\nu}\partial_\nu {}^q\Gamma^\rho_{\mu\sigma} + {}^q\Gamma^\tau_{\mu\nu}\,{}^q\Gamma^\rho_{\tau\sigma} - {}^q\Gamma^\tau_{\mu\sigma}\,{}^q\Gamma^\rho_{\nu\tau}, \tag{1}
$$

where $v_\mu = \partial_\mu q_\mu(x_\mu)$ is a function that accounts for the measure dependence. Subsequently, the Christoffel symbols are modified as

$$
{}^q\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\frac{1}{v_\mu}\partial_\mu g_{\nu\sigma} + \frac{1}{v_\nu}\partial_\nu g_{\mu\sigma} - \frac{1}{v_\sigma}\partial_\sigma g_{\mu\nu}\right), \tag{2}
$$

and directly impact the geometry of spacetime in the multi-fractional framework. The Einstein-Hilbert action is generalized by introducing the measure $v(x)$, which is a product of the q-derivative terms:

$$
{}^qS = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\, v(x)\sqrt{-g}\,({}^qR - 2\Lambda) + S_m, \tag{3}
$$

where $v(x)=\prod_\mu v_\mu(x_\mu)$, and ${}^qR$ is the q-version of the Ricci scalar. This modification introduces a background scale dependence into the action.

Thus, the solutions to Einstein's equations are observed to appear the same as in GR when expressed in terms of q-coordinates q_μ [18]. However, non-linear modifications appear when re-expressed in physical coordinates x_μ. Note that this is not simply a coordinate transformation (which it may be confused for) but reflects the intrinsic multi-scale geometry of the theory. Thus, the Schwarzschild solution in the multi-fractional theory is

$$ {}^q\mathrm{d}s^2 = -\left(1-\frac{2M}{q}\right)\mathrm{d}t^2+\left(1-\frac{2M}{q}\right)^{-1}\mathrm{d}q^2 + q^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2), \tag{4}$$

where q is the modified radial coordinate, and the non-trivial aspect of multi-fractal theory is hidden:

$$ q = r \pm \frac{\ell_*}{\alpha}\left(\frac{r}{\ell_*}\right)^{\alpha}. \tag{5}$$

We observe that inherently, q is a function of the radial coordinate r. Furthermore, the length scale parameter ℓ_* marks the transition between the UV and IR behavior of the geometry. We should also note that the standard Schwarzschild metric is recovered in the limit where the multi-fractional effects vanish, *i.e.*, when $q(r)\to r$. If we consider the case of no log oscillations, the transformation simplifies to

$$ q = r \pm \frac{\ell_*^{1-\alpha}}{\alpha} r^{\alpha}. \tag{6}$$

When ℓ_* (the multi-fractional characteristic scale) is small or goes to zero, the correction term disappears, leading to $q(r)\approx r$, which restores the standard Schwarzschild metric:

$$ \mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2+\left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2). \tag{7}$$

Thus, the Schwarzschild solution is recovered in the classical (large-scale) limit where multi-fractional effects are negligible.

In Eq. (5), α plays a fundamental role in determining the scaling of the coordinate transformation and the behavior of gravitational corrections, where α should be within a specific range to ensure a physically viable model [18]. A notable choice is $\alpha=1/2$, which is highlighted as having a "special role in the theory." More generally, the theory suggests that α must be constrained between 0 and 1 to maintain consistency with the underlying fractal-like spacetime modifications. Specifically,

- If $\alpha\approx 1$, the corrections become very small, making the multi-fractional effects negligible.

- If $\alpha\approx 0$, strong oscillatory behavior emerges, requiring logarithmic oscillations to smooth out the solution.

- $\alpha=1/2$ is selected in multiple cases as a representative value that balances multi-fractional effects.

Thus, the practical range of α appears to be $0<\alpha<1$, with special emphasis on $\alpha=1/2$. When $\alpha=1/2$, $\ell_*=l_{\mathrm{Pl}}^2/s$, where s is called the observation scale length [17, 84]. Indeed, the parameter ℓ_* is the boundary at which the difference in physical laws occurs. Ref. [18] further explains that q represents the stochastic fluctuation as can be inferred from the upper and lower signs of the equation. The former simply represents the deformation in the radius, whereas the latter represents the quantum stochastic feature, where the radius suffers fuzziness owing to the multi-fractional effects of spacetime.

The horizon radius is also modified by the multi-fractional effects of spacetime [18]. The key equation that determines the fractional event horizon is $q(r_h)=2M$. Without log oscillations, this equation simplifies to

$$ r_h \pm \frac{\ell_*^{1-\alpha}}{\alpha} r_h^{\alpha} = 2M, \tag{8}$$

where the implications are (1) the horizon radius is smaller than the standard Schwarzschild radius $2M$ - which is the initial-point presentation, and (2) the horizon radius is larger than $2M$ - the final-point presentation. Explicitly solving for r_h when $\alpha=1/2$ yields

$$ r_h^{ip} = 2\ell_* + 2M - 2\sqrt{\ell_*^2+2M\ell_*} < 2M, \\ r_h^{fp} = 2M + 2\sqrt{\ell_*^2+2M\ell_*} - 2\ell_* > 2M. \tag{9}$$

Additionally, in the stochastic interpretation (for completeness), the horizon radius is given by

$$ r_h = 2M \pm \delta(r), \quad \delta(r) := 2\sqrt{\ell_*^2+2M\ell_*} - 2\ell_*. \tag{10}$$

Here, $\delta(r)$ represents quantum fluctuations due to the multi-fractional structure. This introduces a degree of intrinsic uncertainty in the horizon radius, which becomes particularly significant for microscopic black holes where the Schwarzschild radius approaches the multi-fractional scale ℓ_*.

A different approach to multi-fractional theory is called weighted derivatives. In the Einstein frame, the gravitational action in the theory with weighted derivatives is

$$S_g = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-\bar{g}}\left(\bar{R} - \omega\partial_\mu\Phi\bar{\partial}^\mu\Phi - \mathrm{e}^{-\Phi}U\right), \tag{11}$$

where $\Phi(x) = \ln v(x)$, and U is a potential related to the measure weight $v(x)$. Assuming an Ansatz,

$$\mathrm{d}s^2 = -\gamma_1(r)\mathrm{d}t^2 + \gamma_2(r)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{12}$$

the field equations can be solved:

$$\begin{aligned} 0 &= (\gamma_1\gamma_2)', \\ 0 &= \gamma_1'' - \gamma_1'\left(\frac{\gamma_2'}{2\gamma_2} + \frac{1}{r} - \frac{\gamma_1'}{2\gamma_1}\right) - \frac{\gamma_1}{r^2}\left(r\frac{\gamma_2'}{\gamma_2} - 2\gamma_2 + 2\right), \end{aligned} \tag{13}$$

where the solution is obtained as

$$\begin{aligned} \gamma_1(r) &= 1 - \frac{2M}{r} - \frac{\chi r^2}{6}, \\ \gamma_2(r) &= \gamma_1(r)^{-1}. \end{aligned} \tag{14}$$

Here, χ can be treated as positive (dS-like) or negative (AdS-like). It is the simplest black hole solution where $\Lambda = (1/2)\chi$, an increase by a factor of 1/2 owing to the multi-scaling nature of the geometry - an alternative perspective to the cosmological constant as explained in [18].

III. ANALYSIS OF SHADOW WITH EFFECTS FROM MULTI-FRACTIONAL THEORY

For the calculations to be sufficiently clear, we can rewrite Eq. (4) as

$${}^q ds^2 = A(q)\mathrm{d}t^2 + B(q)\mathrm{d}q^2 + C(q)\mathrm{d}\phi^2, \tag{15}$$

where we specialize at $\theta = \pi/2$, converting it to a $(2+1)$ metric. To derive the black hole shadow radius, we require only the expression for the photon sphere radius and the critical impact parameter. The formalism we follow is well-known and used widely in the literature (see Refs. [20, 21, 85]). Using Eq. (15), the photon sphere radius can be solved simply as

$$\frac{A(q)}{dq}q^2 - 2A(q)q = (2q - 6M) = 0, \tag{16}$$

resulting in

$$q(r_{\mathrm{ph}}) = 3M. \tag{17}$$

Indeed, as we can observe, we need not implement some coordinate transformation in Eq. (15) because the expression directly results from the Einstein-Hilbert action in the q-derivative formalism [18]. As we set $\alpha = 1/2$ and select the upper sign in Eq. (5), we find two solutions for the photon sphere:

$$r_{\mathrm{ph}} = 2\ell_* \mp 2\sqrt{\ell_*(3M + \ell_*)} + 3M. \tag{18}$$

We remark that if we select the fuzziness feature of spacetime in Eq. (5), we still obtain the same expression above. As we follow the reconciliation of the so-called problem of *presentation* in [18], particularly in the treatment of the black horizon, the upper sign in Eq. (18) represents the initial point of the photon sphere, whereas the lower sign represents that of the final point. Remarkably, both of these cases are subjected to certain approximations that we can study. These are (a) $\ell_*/M \to 0$ and (b) $\ell_*/M \to \infty$. This would mean that for the former, $M \gg \ell_*$, and for the latter, $\ell_* \gg M$. Finally, a further examination of Eq. (18) shows that it is reducible to the standard Schwarzschild case without the influence of the multi-scale length ($\ell_* = 0$).

Selecting the upper sign in Eq. (18), we determine the approximations for $\ell_*/M \to 0$ and $\ell_*/M \to \infty$ as

$$\begin{aligned} r_{\mathrm{ph}}^- &\sim 3M - 2\sqrt{3M\ell_*} + 2\ell_* - O(\ell_*^{3/2}), \\ r_{\mathrm{ph}}^- &\sim \frac{9M^2}{4\ell_*} - \frac{27M^3}{8\ell_*^2} + O(\ell_*^{-3}), \end{aligned} \tag{19}$$

respectively. When the lower sign is selected, the two above approximations become

$$\begin{aligned} r_{\mathrm{ph}}^+ &\sim 3M + 2\sqrt{3M\ell_*} + 2\ell_* + O(\ell_*), \\ r_{\mathrm{ph}}^+ &\sim 4\ell_* + 6M - \frac{9M^2}{4\ell} + O(\ell_*^{-2}), \end{aligned} \tag{20}$$

With these simplified expressions for the photon sphere radii, we can now derive the impact parameter as

$$b(r)^2 = \frac{C(q)}{A(q)} = \frac{\left(2\sqrt{r\ell} + r\right)^3}{2\sqrt{r\ell} \mp 2M \pm r}, \tag{21}$$

and the critical impact parameter is simply determined using Eqs. (19) and (20) substituted into Eq. (21). That is, $b_{\mathrm{crit}} = b(r = r_{\mathrm{ph}})$. If we use Eq. (20), we must use the upper sign in Eq. (5). For the lower sign, Eq. (19) is used.

We observe that it produces the same critical impact parameters. For $\ell_*/M \to 0$ and $\ell_*/M \to \infty$, we find

$$
\begin{aligned}
b_{\rm crit}^2 &\sim 27M^2 + \frac{3\ell_*^2}{M} + O(\ell_*^{7/2}),\\
b_{\rm crit}^2 &\sim 27M^2 + \frac{164025M^6}{1024\ell^4} + O(\ell_*^{-5}),
\end{aligned} \tag{22}
$$

respectively.

Finally, we find the exact expression for the q-version of the shadow radius [85] as

$$
R_{\rm sh} = \frac{b_{\rm crit} r_{\rm obs}}{q(r_{\rm obs})}\sqrt{A(q(r_{\rm obs}))}. \tag{23}
$$

Again, for $\ell_*/M \to 0$ and $\ell_*/M \to \infty$, we find

$$
\begin{aligned}
R_{\rm sh}^{\rm far} &\sim 3\sqrt{3}M - 6\sqrt{3\ell_*}M\left(\frac{1}{r_{\rm obs}}\right)^{1/2} + O(r_{\rm obs}^{-3/2}),\\
R_{\rm sh}^{\rm far} &\sim \frac{3\sqrt{3}M r_{\rm obs}^{3/2}}{8\ell^{\frac{3}{2}}} - \frac{3\sqrt{3}M r_{\rm obs}}{4\ell} + O(r_{\rm obs}^{1/2}).
\end{aligned} \tag{24}
$$

which is the one we require to constraint ℓ_* using the EHT data/results for Sgr. A* and M87*. Indeed, because of the uncertainty of the photon sphere's position caused by the intrinsic stochasticity of spacetime, the behavior of the shadow radius is also affected. However, an interesting behavior to note is the coupling of the multi-length scale to the observer's position relative to the black hole.

The Schwarzschild shadow radius is bounded by the following uncertainties: $4.209M \leq R_{\rm Sch} \leq 5.560M$ for Sgr. A* at the 2σ level of significance [86] and $4.313M \leq R_{\rm Sch} \leq 6.079M$ for M87* at the 1σ level of significance [87]. Let δ represent these upper and lower bounds. Thus, we obtain

$$
{}^qR_{\rm sh}^{\rm far} = R_{\rm Schw} + \delta. \tag{25}
$$

Here, we can determine the parameter estimation for the multi-scale length ℓ_*, again, for $\ell_*/M \to 0$ and $\ell_*/M \to \infty$ as

$$
\begin{aligned}
\ell_* &\sim \frac{\delta^2 r_{\rm obs}^3}{108M^4},\\
\ell_* &\sim \frac{(3\sqrt{3}M)^{2/3} r_{\rm obs}}{4\left(3\sqrt{3}M + \delta\right)^{2/3}},
\end{aligned} \tag{26}
$$

respectively. For the second equation above, we only used the first term as this is the largest contribution. It shows its dependence on the observer's distance from the black hole in terms of the usual r-coordinate and cannot be negative. Now, for Sgr. A* ($M = 6.40\times10^9$ m), $\delta = \pm0.987M$ and $r_{\rm obs} = 4.02\times10^{10}M$, yielding $\ell_*/M = 5.86\times10^{29}$. For M87* ($M = 9.60\times10^{12}$ m), $\delta = \pm0.883M$ and $r_{\rm obs} = 5.4\times10^{10}M$, which leads to $\ell_*/M = 1.14\times10^{30}$.

The violation of the proposed universal bound on black hole size in both 4D Einstein-Gauss-Bonnet (EGB) gravity and multi-fractional gravity suggests a deeper underlying modification to spacetime structure beyond classical GR. For 4D EGB gravity, the negative Gauss-Bonnet (GB) coupling constant leads to a modification of the photon sphere and shadow size, breaking the classical inequalities that relate these radii to the event horizon and mass of the black hole [88]. Similarly, in multi-fractional gravity, the introduction of a scale-dependent radial coordinate transformation via the parameter ℓ_* creates an intrinsic quantum fuzziness in black hole geometry, altering the event horizon, photon sphere, and shadow radius. This suggests that the classical limits on black hole size may not hold in alternative gravity theories where quantum or fractal-like structures emerge at small scales. The key insight here is that the breakdown of these bounds is not limited to any single alternative theory but rather may be a common feature of quantum-corrected or modified spacetime geometries, pointing towards a potential observational signature of new physics. Future studies using black hole shadow observations from the EHT could provide empirical constraints on these effects, which would aid in distinguishing different quantum gravity models.

For comparison, the cosmological horizon is only $\sim 9.51\times10^{25}$ m, and the obtained values of ℓ_* significantly exceed it. Furthermore, the first expression in Eq. (26) does not apply to astrophysical black holes because the condition $M \gg \ell_*$ is not satisfied. However, if we examine the second expression, we find that $8.95\times10^9 \leq \ell_*/M \leq 1.16\times10^{10}$ for Sgr. A*, and $1.22\times10^{10} \leq \ell_*/M \leq 1.53\times10^{10}$ for M87*. These results show that the second expression in Eq. (26) is the one applicable for constraining ℓ_* in astrophysical black holes because the condition $\ell_* \gg M$ is satisfied. The implication of this result is such that the shadow radius is completely modified as shown in the second expression in Eq. (24). Note that the Schwarzschild case is recovered if $\ell_* \sim 0.1164 r_{\rm obs}$.

Finally, when the cosmological constant is considered an effect of a multi-scaling nature of geometry in the weighted derivative formalism, then Eq. (14) applies. Basically, the shadow analysis would be the same as that of the standard dS/AdS black hole spacetime, where the photon sphere is found not to be affected by the cosmological constant: $r_{\rm ph} = 3M$. Thus, the expression for the critical impact parameter is

$$
b_{\rm crit}^2 = \frac{54M^2}{1-9\chi M^2}. \tag{27}
$$

Again, as we follow the formalism of deriving the shadow radius, we find that the far approximation is

$$ {}^{w}R_{\rm sh}^{\rm far} \sim 3\sqrt{3}M - \frac{M\sqrt{3}\chi r_{\rm obs}^2}{4} - O(r_{\rm obs}). \tag{28} $$

Thus, using the EHT results, we can possibly constrain χ:

$$ \chi = -\frac{4\delta\sqrt{3}}{3Mr_{\rm obs}^2}. \tag{29} $$

In dS spacetime where $\chi > 0$, we select $\delta < 0$, whereas, in AdS spacetime, $\delta > 0$ must be selected. For Sgr. A*, we find that $\chi \sim 3.49\times10^{-41}$ m^{-2}, whereas for M87*, $\chi \sim 7.59\times10^{-48}$ m^{-2}. We observe that constraints on M87* yield a closer value to $\Lambda \sim 10^{-52}$ m^{-2}, thus providing a small but significant deviation. We may attribute the deviation to the multi-fractional effects of spacetime under the weighted derivative formalism.

IV. DEFLECTION ANGLE IN THE WEAK FIELD REGIME

In this section, we use the version of the GBT that enables us to calculate the weak deflection angle even if the spacetime is non-asymptotically flat. The $1+2$ dimensional metric in Eq. (15) is utilized for the general version of GBT given by [77]

$$ \Theta = \iint_{{}_{q(r_{\rm ph})}^{\rm R}\square_{q(r_{\rm ph})}^{\rm S}} K {\rm d}S + \phi_{\rm RS}. \tag{30} $$

Here, the quadrilateral, as symbolized by the box, represents the integration domain used in the generalized GBT. Specifically, it refers to the quadrilateral domain formed by the source of light, the receiver, the photon sphere, and the observer's location. Such an integration domain must be specified to compute the weak deflection angle in the presence of non-asymptotic spacetime modifications.

Note that, because we are dealing with multi-fractional theory, the GBT above must be rewritten in its q-version. The integral above is evaluated in a quadrilateral spanning from the position of the source S, the photon sphere $q(r_{\rm ph})$, the position of the receiver R, and back to the position of the source. Additionally, $\phi_{\rm RS}$ is the separation between the source and receiver, as given by

$$ \phi_{\rm RS} = \phi_{\rm R} - \phi_{\rm S}, \tag{31} $$

where $\phi_{\rm R} = \pi - \phi_{\rm S}$. Furthermore, ${\rm d}S = \sqrt{g}{\rm d}q{\rm d}\phi$ is the infinitesimal curve surface, and g is the determinant of the Jacobi metric in its q-version as

$$ {\rm d}l^2 = g_{ij}{\rm d}x^i{\rm d}x^j = (E^2 - \mu^2 A(q))\left[\frac{B(q)}{A(q)}{\rm d}q^2 + \frac{C(q)}{A(q)}{\rm d}\phi^2\right], $$
$$ g = \frac{(E^2-\mu^2A(q))B(q)C(q)}{A(q)^2}. \tag{32} $$

In the above expression, μ is the mass of the time-like particle, and E is the energy per unit mass. In terms of these variables, we can rewrite Eq. (30) as

$$ \Theta = \int_{\phi_{\rm S}}^{\phi_{\rm R}}\int_{q(r_{\rm ph})}^{q(r(\phi))} K\sqrt{g}\,{\rm d}q\,{\rm d}\phi + \phi_{\rm RS}, \tag{33} $$

where the Gaussian curvature is given by

$$ K = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\phi}\left(\frac{\sqrt{g}}{g_{qq}}\Gamma^{\phi}_{qq}\right) - \frac{\partial}{\partial q}\left(\frac{\sqrt{g}}{g_{qq}}\Gamma^{\phi}_{q\phi}\right)\right] $$
$$ = -\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial q}\left(\frac{\sqrt{g}}{g_{qq}}\Gamma^{\phi}_{q\phi}\right)\right]. \tag{34} $$

The Jacobi metric implies that $\Gamma^{\phi}_{qq} = 0$, and using the photon sphere as the integration domain,

$$ \left[\int K\sqrt{g}{\rm d}q\right]\Bigg|_{q=q(r_{\rm ph})} = 0, \tag{35} $$

resulting in

$$ \int_{q(r_{\rm ph})}^{q(r(\phi))} K\sqrt{g}{\rm d}q = -\frac{A(q)\left(E^2 - A(q)\right)C' - E^2C(q)A(q)'}{2A(q)(E^2-A(q))\sqrt{B(q)C(q)}}\Bigg|_{q=q(r(\phi))} \tag{36} $$

where prime denotes differentiation with respect to q.

As indicated by $q(r(\phi))$, the orbit equation is required. By allowing $u(q) = q^{-1}$,

$$ F(u) \equiv \left(\frac{{\rm d}u(q)}{{\rm d}\varphi}\right)^2 = \frac{C(u(q))^2u(q)^4}{A(u(q))B(u(q))} $$
$$ \times\left[\left(\frac{E}{J}\right)^2 - A(u(q))\left(\frac{1}{J^2} + \frac{1}{C(u(q))}\right)\right], \tag{37} $$

where $J = Evb$ is the angular momentum of the particle, and b is the impact parameter. The above results in

$$ {}^{w}u(\phi) = \frac{1}{b}\sin(\phi) + \frac{M}{b^2v^2}\left(1 + v^2\cos(\phi)\right). \tag{38} $$

Using the above expression on Eq. (36),

$$ \int_{q(r_{\rm ph})}^{q(r(\phi))} K\sqrt{g}{\rm d}q \sim -1 + \frac{\left(2E^2-1\right)\sin(\phi)M}{b(E^2-1)}. \tag{39} $$

Thus,

$$\int_{\phi_S}^{\phi_R}\int_{q(r_{\rm ph})}^{q(r(\phi))} K\sqrt{g}\,{\rm d}q\,{\rm d}\phi \sim -\phi_{\rm RS} - \frac{\left(2E^2-1\right)M}{(E^2-1)b}\cos(\phi)\Big|_{\phi_S}^{\phi_R}. \tag{40}$$

We solve ϕ as

$$\phi = \arcsin(bu(q)) - \frac{M\left[v^2\left(b^2u(q)^2-1\right)-1\right]}{2\sqrt{1-b^2u(q)^2}\,b\,v^2}, \tag{41}$$

and it follows that

$$\cos(\phi) = \sqrt{1-b^2u(q)^2} + \frac{u(q)\left[v^2\left(b^2u(q)^2-1\right)-1\right]M}{2\sqrt{1-b^2u(q)^2}\,v^2}. \tag{42}$$

Owing to the relations

$$\cos(\pi-\phi) = -\cos(\phi), \qquad \phi_{\rm RS} = \pi - 2\phi, \tag{43}$$

we can find the weak deflection angle in its q-version as

$${}^q\Theta = \frac{2\left(2E^2-1\right)M\cos(\phi)}{b(E^2-1)} = \frac{2\left(v^2+1\right)M\sqrt{1-b^2u(q)^2}}{b\,v^2}. \tag{44}$$

Using Eq. (5), we can finally recast the above equation as

$${}^q\Theta = \frac{2\left(v^2+1\right)M}{b\,v^2}\sqrt{1-\frac{b^2u^2}{\left(2\sqrt{lu}+1\right)^2}}, \tag{45}$$

where u is now defined as r^{-1}. In the far approximation,

$${}^q\Theta^{\rm far}_{\rm timelike} \sim \frac{2\left(v^2+1\right)M}{b\,v^2}\left(1-\frac{b^2u^2}{2}+2b^2\sqrt{\ell_*}\,u^{\frac{5}{2}}-O(u^3)\right). \tag{46}$$

With photons, where $v=1$,

$${}^q\Theta^{\rm far}_{\rm null} \sim \frac{4M}{b}\left(1-\frac{b^2u^2}{2}+2b^2\sqrt{\ell_*}\,u^{\frac{5}{2}}-O(u^3)\right), \tag{47}$$

where u can now be associated with $r_{\rm obs}$.

Let us constrain ℓ_* through the Solar System test. If the photon only grazes the surface of the Sun, then $b=R_\odot$, and $M=M_\odot$. The results from the parametrized post-Newtonian (PPN) formalism equation for the deflection of light are given by [89]

$$\Theta^{\rm PPN} \simeq \frac{4M_\odot}{R_\odot}\left(\frac{1+\gamma}{2}\right). \tag{48}$$

According to the astrometric observation of the Very Long Baseline Array (VLBA), γ is the PPN deflection parameter, equal to $0.9998\pm\Delta$, where $\Delta=0.0003$ [89, 90]. Subsequently, substituting Eq. (5) into Eq. (47), we find the relation

$$\frac{4M_\odot}{R}\left(1-\frac{R^2}{2r_{\rm obs}^2}+\frac{2R^2\sqrt{l}}{r_{\rm obs}^{5/2}}\right) = \frac{4M_\odot}{R}\left(\frac{1.9998\pm\Delta}{2}\right), \tag{49}$$

Through comparison,

$$\ell_* \sim \frac{(-0.0002\pm\Delta)^2 r_{\rm obs}^5}{16R_\odot^4} + O(r_{\rm obs}^3). \tag{50}$$

Next, because $M_\odot = 1476.6148$ m, $R_\odot = 696340000$ m, and $r_{\rm obs} = 1.4899\times10^{11}$ m, we find constraints for l_*

$$1.32\times10^8 \le \ell_*/M_\odot \le 3.30\times10^9. \tag{51}$$

In the weighted derivative formalism, where we use Eq. (14), we find the weak deflection angle as

$${}^w\Theta = \frac{2\left(v^2+1\right)M\sqrt{1-b^2u^2}}{v^2b} + \frac{b\left(v^2-2\right)\chi\sqrt{1-b^2u^2}}{6v^2u}. \tag{52}$$

Now, if $u\to0$,

$${}^w\Theta^{\rm far}_{\rm timelike} = \frac{2\left(v^2+1\right)M}{v^2b} + \frac{b\left(v^2-2\right)\chi r_{\rm obs}}{6v^2}. \tag{53}$$

For photons,

$${}^w\Theta^{\rm far}_{\rm null} = \frac{4M}{b} - \frac{b\chi r_{\rm obs}}{6} \tag{54}$$

Given the distance of Earth from the Sun, $r_{\rm obs} = 148.99$ million km, constraints for χ using the Solar System test can be determined as

$$\chi = -\frac{12M(-0.0002\pm\Delta)}{R_\odot^2 r_{\rm obs}}, \tag{55}$$

which yields the bounds

$$-5.35\times10^{-23} \le \chi M_\odot^2 \le 2.67\times10^{-22}. \tag{56}$$

These are the only bounds for potentially detecting the deviation caused by the weighted derivative formalism in the multi-fractional theory of spacetime. Thus, the weak gravitational field of the Sun provides a significantly large deviation to the accepted value of the cosmological constant Λ.

V. CONCLUSION

In this study, we explore the properties of black holes within the multi-fractional theory, particularly through the formalism of q-derivatives and weighted derivatives. By incorporating scale-dependent geometric structures, we find that the presence of a multi-scale length ℓ_* introduces notable modifications to the shadow radius and weak deflection angle. These deviations leave traces of the quantum gravitational effects in the context of multi-fractional theory in astrophysical settings.

The analysis of black hole shadows, particularly for the supermassive black holes M87* and Sgr A*, reveal constraints on ℓ_* that are on the order of 10^{10}, implying that multi-fractional effects, while difficult to detect, might be present at large scales. The weak deflection angle computed using the generalized Gauss-Bonnet theorem further supports the sensitivity of gravitational lensing to these multi-fractional corrections, offering a secondary observational test. We show that, in the weak field regime, such as those in the Solar System, constraints on ℓ_* are similarly placed within tight bounds.

The findings of this study not only extend our understanding of black holes within the multi-fractional theory framework but also demonstrate that this approach can provide testable predictions in gravitational lensing and black hole shadow phenomena. Future work should focus on refining observational techniques and applying these theoretical models to upcoming datasets from projects such as the EHT. Additionally, the role of the cosmological constant and its interpretation within the weighted derivative formalism deserves further investigation, particularly in its effects on large-scale structures and horizon formation. The multi-fractional theory of spacetime then offers a promising pathway for bridging classical GR with quantum gravity, providing testable predictions for future astrophysical observations. While current constraints suggest that these effects are small, ongoing advancements in observational precision may soon reveal their measurable imprints on black hole environments.

The associated black hole geometry and properties where the most general q-version of the black hole spacetime is affected by a modulation term would be interesting to explore further in future works [18]. Given these complications, the analysis is expected to interestingly require some numerical approach. While the q derivatives are interesting in a phenomenological sense, q-versions of more general spacetime metrics can be studied, such as those of charged black holes and rotating black holes.

VI. ACKNOWLEDGEMENTS

R. P. would like to acknowledge the networking support of the COST Action CA18108 - Quantum gravity phenomenology in the multi-messenger approach (QG-MM), COST Action CA21106 - COSMIC WISPers in the Dark Universe: Theory, astrophysics and experiments (CosmicWISPers), the COST Action CA22113 - Fundamental challenges in theoretical physics (THEORY-CHALLENGES), and the COST Action CA21136 - Addressing observational tensions in cosmology with systematics and fundamental physics (CosmoVerse).

References

[1] R. C. Myers and M. J. Perry, Annals Phys. **172**, 304 (1986)
[2] R. Emparan and H. S. Reall, Living Rev. Rel. **11**, 6 (2008), arXiv: 0801.3471[hep-th]
[3] E. Witten, Nucl. Phys. B **443**, 85 (1995), arXiv: hep-th/9503124
[4] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998), arXiv: hep-th/9711200
[5] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999), arXiv: hep-th/9906064
[6] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996), arXiv: hep-th/9601029
[7] A. Chodos and S. L. Detweiler, Gen. Rel. Grav. **14**, 879 (1982)
[8] D. Y. Xu, Class. Quant. Grav. **5**, 871 (1988)
[9] Y. G. Shen and Z. Q. Tan, Phys. Lett. A **142**, 341 (1989)
[10] B. R. Iyer and C. V. Vishveshwara, Pramana **32**, 749 (1989)
[11] C. Cadeau and E. Woolgar, Class. Quant. Grav. **18**, 527 (2001), arXiv: gr-qc/0011029
[12] Y. S. Myung, Phys. Rev. D **88**, 084006 (2013), arXiv: 1308.3907[gr-qc]
[13] Z. W. Feng, H. L. Li, X. T. Zu *et al.*, Eur. Phys. J. C **76**, 212 (2016), arXiv: 1604.04702[hep-th]
[14] T. Q. Do, EPJ Web Conf. **206**, 08002 (2019)
[15] X. C. Cai and Y. G. Miao, Eur. Phys. J. C **81**, 559 (2021), arXiv: 2011.05542[gr-qc]
[16] B. C. Paul, arXiv: 2301.00358[gr-qc]
[17] G. Calcagni, JHEP **2017**, 138 (2017), [Erratum: JHEP 2017, 020 (2017)], arXiv: 1612.05632[hep-th]
[18] G. Calcagni, D. Rodríguez Fernández, and M. Ronco, Eur. Phys. J. C **77**, 335 (2017), arXiv: 1703.07811[gr-qc]
[19] R. Alves Batista *et al.*, Class. Quant. Grav. **42**, 032001 (2025), arXiv: 2312.00409[gr-qc]
[20] C. M. Claudel, K. S. Virbhadra, and G. F. R. Ellis, J. Math. Phys. **42**, 818 (2001), arXiv: gr-qc/0005050
[21] K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D **65**, 103004 (2002)
[22] J. L. Synge, Mon. Not. Roy. Astron. Soc. **131**, 463 (1966)
[23] J. P. Luminet, Astron. Astrophys. **75**, 228 (1979)
[24] H. Falcke, F. Melia, and E. Agol, Astrophys. J. Lett. **528**, L13 (2000), arXiv: astro-ph/9912263
[25] K. Akiyama *et al.* (Event Horizon Telescope), Astrophys. J. Lett. **875**, L1 (2019), arXiv: 1906.11238[astro-ph.GA]
[26] K. Akiyama *et al.* (Event Horizon Telescope), Astrophys. J. Lett. **875**, L4 (2019), arXiv: 1906.11241[astro-ph.GA]
[27] K. Akiyama *et al.* (Event Horizon Telescope), Astrophys. J. Lett. **930**, L17 (2022), arXiv: 2311.09484[astro-ph.HE]

[28] K. Akiyama *et al*. (Event Horizon Telescope), Astrophys. J. Lett. **930**, L12 (2022), arXiv: 2311.08680[astro-ph.HE]
[29] K. Akiyama *et al*. (Event Horizon Telescope), Astrophys. J. Lett. **930**, L14 (2022), arXiv: 2311.09479[astro-ph.HE]
[30] Y. Hou, Z. Zhang, H. Yan *et al*., Phys. Rev. D **106**, 064058 (2022), arXiv: 2206.13744[gr-qc]
[31] X. Wang, Y. Hou, and M. Guo, JCAP **05**, 036 (2023), arXiv: 2301.04851[gr-qc]
[32] R. Kumar and S. G. Ghosh, Astrophys. J. **892**, 78 (2020), arXiv: 1811.01260[gr-qc]
[33] S. W. Wei, Y. C. Zou, Y. X. Liu *et al*., JCAP **08**, 030 (2019), arXiv: 1904.07710[gr-qc]
[34] Z. Zhong, Z. Hu, H. Yan *et al*., Phys. Rev. D **104**, 104028 (2021), arXiv: 2108.06140[gr-qc]
[35] H. C. D. L. Junior, P. V. P. Cunha, C. A. R. Herdeiro *et al*., Phys. Rev. D **104**, 044018 (2021), arXiv: 2104.09577[gr-qc]
[36] Z. Hu, Z. Zhong, P.-C. Li *et al*., Phys. Rev. D **103**, 044057 (2021), arXiv: 2012.07022[gr-qc]
[37] C. Sun, Y. Liu, W.-L. Qian *et al*., Chin. Phys. C **46**, 065103 (2022), arXiv: 2201.01890[gr-qc]
[38] G. Mustafa, F. Atamurotov, I. Hussain *et al*., Chin. Phys. C **46**, 125107 (2022), arXiv: 2207.07608[gr-qc]
[39] F. Atamurotov, I. Hussain, G. Mustafa *et al*., Chin. Phys. C **47**, 025102 (2023)
[40] F. Atamurotov, I. Hussain, G. Mustafa *et al*., Eur. Phys. J. C **82**, 831 (2022), arXiv: 2209.01652[gr-qc]
[41] A. Ditta, X. Tiecheng, F. Atamurotov *et al*., Commun. Theor. Phys. **75**, 125404 (2023)
[42] T. Mirzaev, S. Li, B. Narzilloev *et al*., Eur. Phys. J. Plus **138**, 47 (2023)
[43] B. Narzilloev, S. Shaymatov, I. Hussain *et al*., Eur. Phys. J. C **81**, 849 (2021), arXiv: 2109.02816[gr-qc]
[44] G. Mustafa and I. Hussain, Eur. Phys. J. C **81**, 419 (2021)
[45] S. Sau and J. W. Moffat, Phys. Rev. D **107**, 124003 (2023), arXiv: 2211.15040[gr-qc]
[46] J. Rayimbaev, R. C. Pantig, A. Övgün *et al*., Annals Phys. **454**, 169335 (2023), arXiv: 2206.06599[gr-qc]
[47] B. Puliçe, R. C. Pantig, A. Övgün *et al*., Class. Quant. Grav. **40**, 195003 (2023), arXiv: 2308.08415[gr-qc]
[48] B. Puliçe, R. C. Pantig, A. Övgün *et al*., Fortsch. Phys. **72**, 2300138 (2024), arXiv: 2403.02373[gr-qc]
[49] Y. B. Zineb, F. Ozel, and D. Psaltis, arXiv: 2412.01904 [astro-ph.IM]
[50] U. Papnoi, F. Atamurotov, S. G. Ghosh *et al*., Phys. Rev. D **90**, 024073 (2014), arXiv: 1407.0834[gr-qc]
[51] B. P. Singh and S. G. Ghosh, Annals Phys. **395**, 127 (2018), arXiv: 1707.07125[gr-qc]
[52] K. Nozari, S. Saghafi, and A. Mohammadpour, Eur. Phys. J. C **84**, 778 (2024), arXiv: 2407.06393[gr-qc]
[53] A. S. Lemos, J. A. V. Campos, and F. A. Brito, Phys. Rev. D **110**, 064079 (2024), arXiv: 2407.04609[gr-qc]
[54] S. Roy, S. Chatterjee, and R. Koley, Eur. Phys. J. C **84**, 47 (2024), arXiv: 2309.02709[gr-qc]
[55] K. Nozari and S. Saghafi, Eur. Phys. J. C **83**, 588 (2023), arXiv: 2305.17237[gr-qc]
[56] B. P. Singh, R. Kumar, and S. G. Ghosh, New Astron. **99**, 101945 (2023)
[57] S. Mandal, S. Upadhyay, Y. Myrzakulov *et al*., Int. J. Mod. Phys. A **38**, 2350047 (2023), arXiv: 2207.10085[gr-qc]
[58] A. Uniyal, S. Kanzi, and I. Sakallı, Eur. Phys. J. C **83**, 668 (2023), arXiv: 2207.10122[hep-th]
[59] I. Banerjee, S. Chakraborty, and S. SenGupta, Phys. Rev. D **106**, 084051 (2022), arXiv: 2207.09003[gr-qc]
[60] Y. Younesizadeh and S. Jokar, Nucl. Phys. B **982**, 115884 (2022)
[61] Z. Y. Tang, X.-M. Kuang, B. Wang *et al*., Sci. Bull. **67**, 2272 (2022), arXiv: 2206.08608[gr-qc]
[62] A. Belhaj, M. Benali, and Y. Hassouni, Eur. Phys. J. C **82**, 619 (2022), arXiv: 2203.06774[hep-th]
[63] Y. Guo and Y. G. Miao, Phys. Rev. D **102**, 084057 (2020), arXiv: 2007.08227[hep-th]
[64] A. Belhaj, M. Benali, A. El Balali *et al*., Class. Quant. Grav. **37**, 215004 (2020), arXiv: 2006.01078[gr-qc]
[65] F. Ahmed, D. V. Singh, and S. G. Ghosh, Gen. Rel. Grav. **54**, 21 (2022), arXiv: 2002.12031[gr-qc]
[66] N. J. L. S. Lobos, A. M. Ncube, R. C. Pantig *et al*., Annals Phys. **473**, 169906 (2025), arXiv: 2406.08078[gr-qc]
[67] J. a. P. A. Novo, P. V. P. Cunha, and C. A. R. Herdeiro, arXiv: 2410.05390[gr-qc]
[68] K. S. Virbhadra, D. Narasimha, and S. M. Chitre, arXiv: astro-ph/9801174
[69] K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D **62**, 084003 (2000), arXiv: astro-ph/9904193
[70] K. S. Virbhadra and C. R. Keeton, Phys. Rev. D **77**, 124014 (2008), arXiv: 0710.2333[gr-qc]
[71] K. S. Virbhadra, Phys. Rev. D **79**, 083004 (2009), arXiv: 0810.2109[gr-qc]
[72] S. L. Adler and K. S. Virbhadra, Gen. Rel. Grav. **54**, 93 (2022), arXiv: 2205.04628[gr-qc]
[73] K. S. Virbhadra, arXiv: 2204.01792[gr-qc]
[74] K. S. Virbhadra, Phys. Rev. D **106**, 064038 (2022), arXiv: 2204.01879[gr-qc]
[75] G. W. Gibbons and M. C. Werner, Class. Quant. Grav. **25**, 235009 (2008), arXiv: 0807.0854[gr-qc]
[76] A. Ishihara, Y. Suzuki, T. Ono *et al*., Phys. Rev. D **94**, 084015 (2016), arXiv: 1604.08308[gr-qc]
[77] Z. Li, G. Zhang, and A. Övgün, Phys. Rev. D **101**, 124058 (2020), arXiv: 2006.13047[gr-qc]
[78] Z. Li, G. He, and T. Zhou, Phys. Rev. D **101**, 044001 (2020), arXiv: 1908.01647[gr-qc]
[79] Z. Li and J. Jia, Eur. Phys. J. C **80**, 157 (2020), arXiv: 1912.05194[gr-qc]
[80] G. He, X. Zhou, Z. Feng *et al*., Eur. Phys. J. C **80**, 835 (2020)
[81] Z. Li, Y. Duan, and J. Jia, Class. Quant. Grav. **39**, 015002 (2022), arXiv: 2012.14226[gr-qc]
[82] Z. Li, H. Liu, and J. Jia, Phys. Rev. D **104**, 084027 (2021), arXiv: 2107.11616[gr-qc]
[83] H. Liu, J. Liang, and J. Jia, Class. Quant. Grav. **39**, 195013 (2022), arXiv: 2204.04519[gr-qc]
[84] G. Amelino-Camelia, G. Calcagni, and M. Ronco, Phys. Lett. B **774**, 630 (2017), arXiv: 1705.04876[hep-th]
[85] V. Perlick and O. Y. Tsupko, Phys. Rept. **947**, 1 (2022), arXiv: 2105.07101[gr-qc]
[86] S. Vagnozzi *et al*., Class. Quant. Grav. **40**, 165007 (2023), arXiv: 2205.07787[gr-qc]
[87] P. Kocherlakota *et al*. (Event Horizon Telescope), Phys. Rev. D **103**, 104047 (2021), arXiv: 2105.09343[gr-qc]
[88] M. Guo and P. C. Li, Eur. Phys. J. C **80**, 588 (2020), arXiv: 2003.02523[gr-qc]
[89] R. T. Chen, S. Li, L. G. Zhu *et al*., Phys. Rev. D **109**, 024010 (2024), arXiv: 2311.12270[gr-qc]
[90] E. Fomalont, S. Kopeikin, G. Lanyi *et al*., Astrophy. J. **699**, 1395 (2009)