Schottky anomaly of Reissner-Nordström-de Sitter spacetime*

Hai-Long Zhen (甄海龙)[1,2] Yu-Bo Ma (马宇波)[1,2] Huai-Fan Li (李怀繁)[2,3]
Li-Chun Zhang (张丽春)[1,2] Yun-Zhi Du (杜云芝)[1,2†]

[1]Department of Physics, Shanxi Datong University, Datong 037009, China
[2]Institute of Theoretical Physics, Shanxi Datong University, Datong, 037009, China
[3]Shanxi College of Technology, College of General Education, Shuozhou 036000, China

Abstract: In the extended thermodynamics of black holes, there exists a thermodynamical pressure whose dual thermodynamical quantity is volume. Extensive studies have been conducted on the phase structure of numerous black holes, demonstrating striking similarities to the phase structures of various ordinary matter systems. From a comparison between the thermodynamic properties of spherically symmetric AdS black holes and ordinary thermodynamic systems, we know that the isovolumetric heat capacity of the former is zero, whereas that of the latter is non-zero. The intrinsic reason for this discrepancy is a topic of interest. For Reissner-Nördstrom-de Sitter (RN-dS) spacetime with the coexistence of the black hole and cosmological horizons, the effective thermodynamic quantities are presented alongside the interaction between two horizons. The heat capacity in RN-dS spacetime is then investigated, and it is demonstrated that the behavior of the heat capacity in RN-dS spacetime is analogous to that of Schottky specific heat. Treating two horizons in RN-dS spacetime as two distinct energy levels in a two-energy-level system, we investigate the thermodynamic properties in RN-dS spacetime by studying the thermodynamic properties in an ordinary two-energy system, thereby elucidating the intrinsic reasons for the occurrence of Schottky specific heat in RN-dS spacetime. The heat capacity observed in RN-dS spacetime is not only consistent with that of Schottky specific heat described by the effective thermodynamic quantities in RN-dS spacetime but also with that of an ordinary two-energy-level system. These results not only reveal the quantum properties of RN-dS spacetime but also provide a new avenue for further in-depth study of the quantum properties of black holes and dS spacetime.

Keywords: de Sitter spacetime, effective thermodynamic quantities, heat capacity, Schottky specific heat

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I. INTRODUCTION

On 10 April 2019, the Event Horizon Telescope (EHT) research team released the first photograph of the supermassive black hole at the centre of the galaxy M87 [1, 2]. The morphology of the photograph is consistent with the Kerr black hole model predicted by general relativity. This observation provides compelling evidence for the presence of a supermassive black hole in the central engine of active galactic nuclei. A black hole is not only a strong gravitational system but also a thermodynamic system. As early as the 1970s, physicists such as Hawking and Bekenstein established the four laws of black hole thermodynamics [3, 4]. The investigation of the thermodynamic properties of black holes not only contributes to a deeper understanding of their nature but also provides insights into their internal microscopic state.

To investigate the microstructure of black holes as well as the formation and evolution of black holes in the Universe, the study of the thermodynamic properties of de Sitter (dS) and anti-de Sitter (AdS) spacetime has attracted significant attention. Recently, in the context of n-dimensional AdS spacetime, a potential correlation between the cosmological constant, Λ, and thermodynamical pressure, P, has been proposed. This hypothesis, which can be found in [5, 6], has attracted significant attention. This relation is defined as $P=n(n-1)/(16\pi l^2)$, $\Lambda=n(n-1)/(2l^2)$, where the thermodynamical volume, V, is the conjugate quantity to the pressure, i.e.,

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† E-mail: duyzh13@lzu.edu.cn

$V=(\partial M/\partial P)_{S,Q_i,J_k}$. Based on these considerations, the thermodynamic properties of AdS black holes have been studied from different perspectives. First, by treating the AdS black hole as an ordinary thermodynamic system, its phase transition has been studied in comparison with that of an ordinary system. It has been demonstrated that the phase transition of the AdS black hole is analogous to that of the ordinary van der Waals system [7−31]. Second, the John-Thomson coefficients of AdS black holes have been investigated. The effects of different parameters and dimensions of AdS black holes on the John-Thomson coefficient have been explored [32−36]. Recently, the study of the topological properties of AdS black holes has attracted interest. The properties of black holes at the phase transition point are classified through the study of topological numbers [37−41]. The microstructure of black holes has been one of the most significant theoretical problems. Notable advances have been made in exploring the microstructure of black holes using various methods [42−50]. Nevertheless, the microstructure of black holes remains unknown. To gain insight into the quantum properties and microstructure of black holes, it is necessary to have a comprehensive understanding of their thermodynamic properties. Consequently, the discussion of the various thermodynamic properties of black holes remains a significant topic in contemporary theoretical physics.

During its early inflationary epoch, our universe is regarded as a quasi-dS spacetime. If the cosmological constant corresponds to dark energy, our universe will evolve into a new dS phase [51]. To construct the entire history of the evolution of the Universe and to identify the intrinsic cause of the accelerated expansion of the Universe, it is necessary to have a clear understanding of the classical, quantum, and thermodynamic properties of dS spacetime. Consequently, the investigation of the thermodynamic characteristics of dS spacetime has attracted considerable attention [52−71]. It is well established that in dS spacetime, the black hole and cosmological horizons coexist if the spacetime parameters satisfy certain conditions. In this region, the requirement of thermodynamic equilibrium stability in spacetime is not satisfied due to the existence of two horizon interfaces with different radiation temperatures. Consequently, to discuss the thermodynamic properties of this region, it is necessary to establish a thermodynamic system that satisfies the thermodynamic equilibrium stability requirement. The thermodynamic system must satisfy the universal relation for thermodynamic systems, which is the first law of thermodynamics. First, the state parameters of the effective thermodynamic system established in dS spacetime satisfy the first law of thermodynamics. Second, the effective thermodynamic quantities satisfy the boundary conditions when the values of the state parameter in spacetime are those of the boundary of the coexistence region between the two horizon interfaces. Consequently, the effective thermodynamic system, based on the aforementioned conditions, provides a more comprehensive representation for the thermodynamic properties in spacetime.

Based on the consideration of the interaction between the two horizon interfaces in dS spacetime, a differential equation for the entropy of an effective thermodynamic system is presented. This is for the state parameter satisfying the universal first law of thermodynamics. The solution for the entropy of the effective thermodynamic system is obtained by the condition satisfied by the effective thermodynamic system at the boundary. Subsequently, the effective temperature and pressure are determined, and the equation of state of the effective thermodynamic system satisfying the boundary conditions is found. This provides a foundation for further in-depth study of the thermodynamic effect in dS spacetime. This study reveals that when the thermodynamic properties are described by the effective thermodynamic quantities in Reissner-Nördstrom-de Sitter (RN-dS) spacetime, the behavior of the heat capacity in RN-dS spacetime is analogous to that of Schottky specific heat. This is highly consistent with the behavior of the heat capacity when the two horizons in RN-dS spacetime are treated as a two-energy-level system. This conclusion contributes to a deeper understanding of dS spacetime, which can be employed to investigate the state of motion of microscopic particles within black holes and to simulate the evolution of the Universe. Furthermore, this discovery offers a novel avenue for investigating the intrinsic causes of the accelerated expansion of the Universe.

The remainder of this paper is arranged as follows. In Sec. II, the conditions for the presence of the black hole and cosmological horizons in RN-dS spacetime, along with the effects of parameters in spacetime on the existence of spacetime with the black hole and cosmological horizons, are discussed. Furthermore, the interval of the position ratio, x, between the two horizons in spacetime is determined. In Sec. III, the effective thermodynamic quantities in RN-dS spacetime satisfying the boundary conditions are presented. The behavior of the effective temperature, $T_{\rm eff}$, radiation temperature of the black hole horizon, T_+, and radiation temperature of the cosmological horizon, T_c, with respect to x in spacetime is analysed. The Smarr relation is expressed by treating effective quantities as state parameters. In Sec. IV, the behavior of the heat capacity, $C_{Q,l}$, is investigated in RN-dS spacetime under the conditions where both the charge, Q, and cosmological constant, l, are fixed. The results demonstrate that the behavior of the heat capacity, $C_{Q,l}$, in RN-dS spacetime with effective temperature $T_{\rm eff}$ and position ratio x exhibits Schottky peaks. This is similar to the behavior of the heat capacity, $\hat{C}_{Q,l}$, in a two-level system treating the black hole and cosmological horizons as two distinct energy levels within an effective thermodynamic system. This indicates that the Schottky peaks in RN-dS spacetime are determined by a two-energy-level system comprising two distinct horizons. Finally, a brief summary is presented in Sec. V.

II. RN-DS SPACETIME

The RN-dS black hole is a static solution of the Einstein-Maxwell equations, and it is also the solution obtained when incorporating an additional charge to the Schwarzschild-dS case (see Ref. [67]). The metric of the RN-dS black hole in a four-dimensional dS spacetime is

$$\mathrm{d}s^2 = -g(r)\mathrm{d}t^2 + g^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2 \tag{1}$$

with the horizon function

$$g(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{r^2}{l^2}. \tag{2}$$

M and Q represent the mass and charge of the black hole, respectively. The curvature radius of dS space is denoted by l. The black hole and cosmological horizons (r_+ and r_c, respectively) satisfy the equation $g(r_{+,c}) = 0$. Substituting Eq. (2) into this relation, we get

$$M = \frac{r}{2}\left(1 + \frac{Q^2}{r^2} - \frac{r^2}{l^2}\right). \tag{3}$$

The $M-r$ curve can then be drawn from Eq. (3).

As shown in Fig. 1, the $M-r$ curve demonstrates that RN-dS spacetime possesses an inner black hole horizon r_-, a black hole horizon r_+, and a cosmological horizon r_c in case of parameters $Q = 1$, $l^2 = 60$. The corresponding mass parameter of RN-dS spacetime with the coexistence of the black hole and cosmological horizons satisfies the expression $M_A \le M \le M_c$. At the point of local maximum, denoted as C, the black hole and cosmological horizons coincide. This point corresponds to the maximum energy, $M = M_c$, where both the black hole and cosmological horizons exist. At the point of the local minimum, denoted as A, the inner and outer horizons of the black hole coincide. This point corresponds to the minimum energy, $M = M_A$, where both the black hole and inner black hole horizons exist in spacetime. In RN-dS spacetime, a black hole does not exist when $M > M_c$ or $M < M_A$ [72]. The Nariai radius and mass are given in Ref. [72]

$$r_N^2 = \frac{l^2}{6}\left(1 + \sqrt{1 - \frac{12Q^2}{l^2}}\right),$$
$$M_N = \frac{l}{\sqrt{6}\left(1 + \sqrt{1 - \frac{12Q^2}{l^2}}\right)} \times \left(\frac{1}{3}\left(1 + \sqrt{1 - \frac{12Q^2}{l^2}}\right) + \frac{4Q^2}{l^2}\right). \tag{4}$$

When $Q^2 \ll l^2$, $M_A \to 0$ and $M_C \to \frac{1}{3\sqrt{3}}$. From Eq. (4), we can see that the charge Q^2 and cosmological constant l^2 should satisfy the condition $\frac{12Q^2}{l^2} \le 1$. When the local minimum and maximum points of the black hole mass curve merge into an inflection point, the three horizons coincide. In this case, the black hole is referred to as an ultracold black hole, whose horizon radius and mass are as follows:

$$r_{AC}^4 = \frac{Q^2 l^2}{3}, \quad r_{AC}^4 = 4Q^4, \quad M_{AC} = \frac{4Q}{3\sqrt{2}}. \tag{5}$$

From Eq. (4), it can be seen that when $Q = 0$, $r_{A,C}^2 = \frac{l^2}{3}$, there is no region of two coexisting horizons. By treating the black hole and cosmological horizons as independent thermodynamic systems, it can be demonstrated that the thermodynamic quantities of the two horizons each satisfy the first law of thermodynamics [51, 60, 72].

For the black hole horizon, the associated thermodynamic quantities are

$$T_+ = \frac{g'(r_+)}{4\pi} = \frac{1}{4\pi r_+}\left(1 - \frac{3r_+^2}{l^2} - \frac{Q^2}{r_+^2}\right), \quad S_+ = \pi r_+^2, \quad \phi_+ = \frac{Q}{r_+}. \tag{6}$$

which satisfy the following expression:

$$\mathrm{d}M = T_+\mathrm{d}S_+ + V_+\mathrm{d}P + \phi_+\mathrm{d}Q. \tag{7}$$

The corresponding thermodynamic quantities associated with the cosmological horizon are

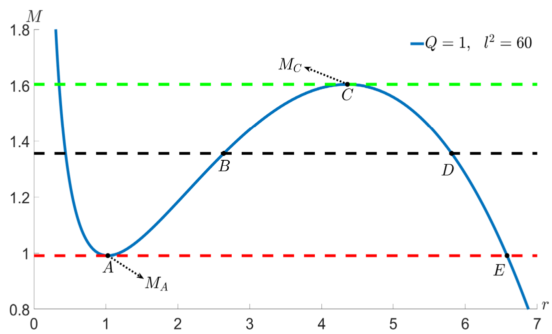

Fig. 1. (color online) $M-r$ curve.

$$T_c = -\frac{g'(r_c)}{4\pi} = -\frac{1}{4\pi r_c}\left(1 - \frac{3r_c^2}{l^2} - \frac{Q^2}{r_c^2}\right), \; S_c = \pi r_c^2, \; \phi_c = \frac{Q}{r_c}. \tag{8}$$

and these quantities also satisfy the first law of thermodynamics:

$$\mathrm{d}M = -T_c \mathrm{d}S_c + V_c \mathrm{d}P + \phi_c \mathrm{d}Q. \tag{9}$$

From the expression $g(r_{+,c}) = 0$, we can derive

$$M = \frac{r_c x(1+x)}{2(1+x+x^2)} + \frac{Q^2(1+x)(1+x^2)}{2r_c x(1+x+x^2)},$$
$$\frac{1}{l^2} = \frac{1}{r_c^2(1+x+x^2)} - \frac{Q^2}{r_c^4(1+x+x^2)}, \tag{10}$$

where $x = r_+/r_c$, which denotes the ratio of the position of the black hole horizon to that of the cosmological horizon. By solving Eq. (10), we can obtain two solutions of r_c^2: a "+" sign before one radical (a larger root) and a "-" sign before the other (a smaller one), which corresponds to the point C when $x \to 1$. Consequently, only the "+" sign root corresponding to the larger root is the physical solution, and it has the following form:

$$r_c^2 = \frac{l^2}{6}\left(1 + \sqrt{1 - \frac{12Q^2}{l^2}}\right). \tag{11}$$

As $l^2 = \alpha 12Q^2$ (with $\alpha \geq 1$being constant), from Eq. (10), we get the position of the ultracold black hole horizon

$$r_c^2 = \frac{6\alpha Q^2}{(1+x+x^2)}\left(1 \pm \sqrt{1 - \frac{(1+x+x^2)}{3x\alpha}}\right),$$
$$\frac{1}{r_c^2} = \frac{x}{2Q^2}\left(1 - \sqrt{1 - \frac{(1+x+x^2)}{3x\alpha}}\right), \tag{12}$$

where α=1 and $x \to 1$.

Substituting Eq. (10) into Eqs. (6) and (8) yields the radiation temperatures of the black hole and cosmological horizons, respectively:

$$T_+ = \frac{(1-x)}{4\pi r_c x}\left(\frac{(1+2x)}{(1+x+x^2)} - \frac{Q^2(1+2x+3x^2)}{r_c^2 x^2(1+x+x^2)}\right), \tag{13}$$

$$T_c = \frac{(1-x)}{4\pi r_c}\left(\frac{(2+x)}{(1+x+x^2)} - \frac{Q^2(3+2x+x^2)}{r_c^2 x(1+x+x^2)}\right). \tag{14}$$

From Fig. 1, it can be observed that the black hole and cosmological horizons appear simultaneously in spacetime in the case of the electric charge, Q, and cosmological constant, l, fixed, while the energy in spacetime meets the constraints $M_A \leq M \leq M_C$. The energy, M, electric charge, Q, and cosmological constant, l, corresponding to the two points B and D are identical. Nevertheless, in general, the radiation temperature of the black hole horizon (B point), T_+, is not equal to the radiation temperature of the cosmological horizon (D point), T_c. In this region, two thermodynamic subsystems with different temperatures emerge in spacetime, corresponding to the thermodynamic system on the black hole and the cosmological horizons, respectively. Consequently, under the same parameters, the spacetime in this region is in a state of thermodynamic equilibrium that is unstable. At point A, the radiation temperature of the black hole is zero, which also marks the end of the coexistence region of two different radiation temperatures in spacetime. Point E, which corresponds to point A, represents the endpoint of spacetime from two thermodynamic subsystems to the thermodynamic system only including the cosmological horizon, as well as the endpoint of spacetime with the two thermodynamic subsystems. The RN-dS spacetime exhibits disparate thermodynamic characteristics across distinct regions. Of particular interest is the thermodynamic behavior in the coexistence region, where two disparate temperatures, T_+ and T_c, are of great theoretical interest [51, 53–72].

In recent years, a search has been conducted for the state quantities of thermodynamic systems in spacetime in the region of coexistence of two subthermodynamic systems under various conditions. In [53, 62], the sum of the entropies corresponding to the two horizons is defined as the total entropy of spacetime. The effective temperature of the thermodynamic system in spacetime is then obtained by satisfying the first law of thermodynamics, which requires that the considered system be in equilibrium. This concept paves the way for the investigation of the thermodynamic properties of non-equilibrium states in spacetime. However, the interaction between two horizons is not considered, and the conclusions obtained are therefore not comprehensive. In [64, 66], the effective temperature and interaction entropy of spacetime are presented based on the interaction between the two horizons. This provides a foundation for the investigation of the thermodynamic properties of dS spacetime and the entropic force between the two horizons. It is crucial to acknowledge that this approach does not account for the condition that when the spacetime is at the endpoint of the two subthermodynamic systems, E, only the temperature of the cosmological horizon, T_c, exists in spacetime, with the radiation temperature of the black hole horizon, T_+, being zero. This paper presents an extension of the study of the thermodynamic properties of dS spacetime, as previously outlined in [64, 66], to encompass the effective thermodynamic quantities of the RN-dS spacetime under various known boundary conditions.

III. EFFECTIVE THERMODYNAMIC QUANTITIES IN RN-DS SPACETIME

The analyses presented in Sec. II demonstrate that, to establish a global thermodynamic system in a region with two subsystems in spacetime, certain boundary conditions must be satisfied by the system and be universal. The thermodynamic properties in RN-dS spacetime, exhibiting black hole and cosmological horizons, are considered. It is therefore of interest to consider the interval $M_A \le M \le M_C$. The exact values of M_A and M_C are related to the spacetime parameters, such as charge, Q, and the cosmological constant, l, which are given by Eq. (10). Considering the connection between the black hole and cosmological horizons, the effective thermodynamic quantities and corresponding first law of black hole thermodynamics are derived as follow [64, 66]:

$$\mathrm{d}M = T_{\rm eff}\mathrm{d}S - P_{\rm eff}\mathrm{d}V + \phi_{\rm eff}\mathrm{d}Q. \tag{15}$$

Here, the thermodynamic volume is that between the black hole and cosmological horizons, namely [53, 60, 62],

$$V = V_c - V_+. \tag{16}$$

From Eq. (13), it can be seen that, when the potential on the black hole horizon at point A, denoted by ϕ_A, satisfies the expression

$$\frac{Q^2}{r_A^2} = \frac{1+2x}{1+2x+3x^2}, \tag{17}$$

the radiation temperature of the black hole horizon, $T_+ = 0$. The radiation temperature of the cosmological horizon, which corresponds to the point E, is

$$T_E = \frac{(1-x)^2(1+x)}{2\pi r_c(1+2x+3x^2)}. \tag{18}$$

When the radiation temperature of the black hole is taken as that at point A, i.e., the black hole temperature is zero, under the same parameters, the effective temperature of RN-dS spacetime should be T_E, while the potential on the black hole horizon in the RN-dS spacetime simultaneously satisfies Eq. (17). In this context, the effective temperature in spacetime, $T_{\rm eff}$, can be derived from Eqs. (15) and (16) and the boundary condition of Eq. (18) as follows:

$$\begin{aligned} T_{\rm eff} = \frac{1-x}{4\pi r_c x^5}\Big\{&\left[(1+x)\left(1+x^3\right)-2x^2\right] \\ &-\frac{Q^2}{r_c^2x^2}\left[\left(1+x+x^2\right)\left(1+x^4\right)-2x^3\right]\Big\}. \end{aligned} \tag{19}$$

The effective pressure in spacetime, $P_{\rm eff}$, is

$$\begin{aligned} P_{\rm eff} =& -\frac{(1-x)}{16\pi r_c^2x^5}\left[F'(x)\left(x(1+x)-\frac{Q^2(1+x+x^2+x^3)}{r_c^2x}\right)-\frac{2F(x)}{(1+x+x^2)}\left((1+2x)-\frac{Q^2(1+2x+3x^2)}{r_c^2x^2}\right)\right] \\ =& -\frac{(1-x)}{8\pi r_c^2x^5(1-x^3)}\left[\frac{x^6}{(1-x^3)}\left((1+x)-\frac{Q^2(1+x+x^2+x^3)}{r_c^2x^2}\right)\right. \\ &\left.-F(x)\left(\left[(1+x)(1+x^3)-2x^2\right]-\frac{Q^2\left[(1+x+x^2)(1+x^4)-2x^3\right]}{r_c^2x^2}\right)\right] \end{aligned} \tag{20}$$

with

$$F(x) = \frac{8}{5}(1-x^3)^{2/3}+\frac{2}{5(1-x^3)}-1 = 1+x^2+f_0(x),$$

$$f_0(x) = \frac{8}{5}(1-x^3)^{2/3}-\frac{8+5x^2-10x^3-5x^5}{5(1-x^3)}. \tag{21}$$

The thermodynamic volume, V, entropy, S and potential, $\phi_{\rm eff}$, of the effective thermodynamic system are, respectively,

$$V = \frac{4\pi}{c}r_c^3(1-x^3),\ \ S = \pi r_c^2F(x),\ \ \phi_{\rm eff} = \frac{Q(1+x)(1+x^2)}{r_cx(1+x+x^2)}. \tag{22}$$

Note that the effective thermodynamic temperature, pressure, and entropy of RN-dS spacetime are different from those given in Ref. [28]. This is due to the fact that we have considered different boundary conditions, although the thermodynamical volume has the same form. From Eqs. (10) and (17), given charge, Q, and cosmological constant, l, the minimum value of $x = x_{\min}$ is given by

$$\frac{6\alpha}{(1+x+x^2)}\left(1\pm\sqrt{1-\frac{(1+x+x^2)}{3x\alpha}}\right)=\frac{1+2x+3x^2}{x^2(1+2x)},\quad (23)$$

i.e., $(1+2x=3x^2)^2-12\alpha x^2(1+2x)=0$.

From Fig. 2, it can be observed that $x_{\min}$ tends to zero with increasing α. This is demonstrated by the values of $x_{\min}$, which are 0.0560513957, 0.042749089, and 0.029789688 when the values of α are 30, 50, and 100, respectively. It can be concluded that the position ratio x assumes a value within the interval $x_{\min}\le x\le 1$ when the coexistence region of the two horizons exists in spacetime.

When the potential on the black hole horizon in spacetime satisfies the expression [73]

$$\frac{Q^2}{r_+^2}=\frac{1}{(1+x)^2},\quad (24)$$

the radiation temperature of the black hole horizon, $\bar{T}_+$, is equal to that of the cosmological horizon, $\bar{T}_c$, i.e.,

$$\bar{T}_+=\bar{T}_c=\frac{1-x}{2\pi r_c(1+x)^2}.\quad (25)$$

Substituting Eq. (24) into Eq. (19), we obtain

$$T_{eff}=\bar{T}_{eff}=\bar{T}_++\frac{(1-x)}{2\pi r_c x^4(1+x)^2}=\bar{T}_c+\frac{(1-x)}{2\pi r_c x^4(1+x)^2}.\quad (26)$$

From Eq. (26), it can be observed that treating the thermodynamic system corresponding to the two horizons with the same radiation temperature in RN-dS spacetime as an independent thermodynamic system leads to a discrepancy in the effective radiation temperature, which takes into account the conditions of interaction between the two horizons. It can be demonstrated that the effective temperature, $\bar{T}_{\rm eff}$, of an effective thermodynamic system consisting of two subsystems with the same temperature in RN-dS spacetime is not equal to the temperature of the subsystems, i.e., $\bar{T}_{\rm eff}>\bar{T}_+=\bar{T}_c$. This phenomenon can be attributed to the interaction between the two horizons. This phenomenon can be further attributed to the existence of an interaction between the two subsystems. Despite the temperatures of the two subsystems being identical, namely, $\bar{T}_+=\bar{T}_c$, the interaction between them results in the effective temperature, $\bar{T}_{\rm eff}$, being distinct from that of the respective subsystems. This illustrates the pivotal role of the interaction between the two horizons in RN-dS spacetime in the effective thermodynamic system. This demonstrates that the thermodynamic system in RN-dS spacetime is distinct from the ordinary thermodynamic system. The discovery of this phenomenon suggests that it is necessary to take into account the behavior of

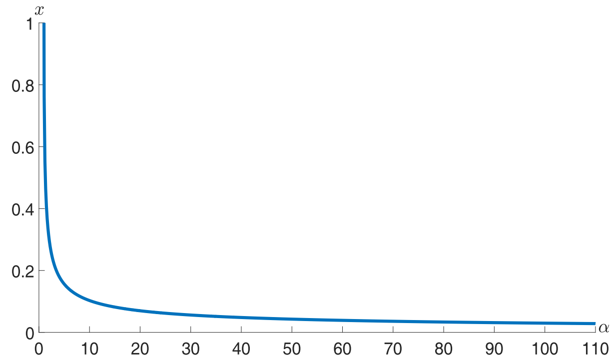

Fig. 2. (color online) $x_{\min}-\alpha$ curve

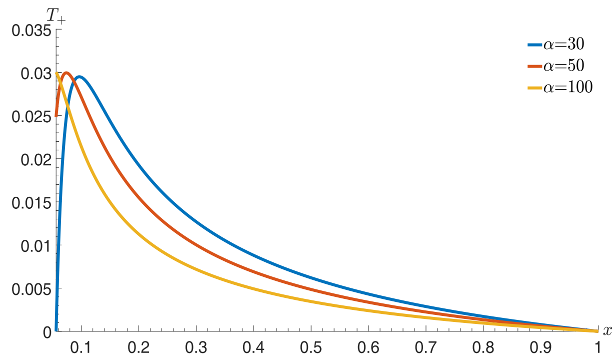

Fig. 3. (color online) T_+-x curve for different α, $Q=1$.

thermodynamic quantities due to the interaction between black holes and the external environment when the thermodynamic properties of black holes are studied by analogy with the method used to study ordinary thermodynamic systems. Consequently, it is important to investigate the intrinsic causes of the interaction between the two horizons in greater depth.

The behaviors of T_+, T_c, and $T_{\rm eff}$ can be plotted against the ratio between the two horizon positions, x, for different values of the parameter α by substituting Eq. (12) into Eqs. (13), (14), and (19), respectively (taking a + sign in Eq. (12)).

Figures 3, 4, and 5 illustrate that T_+, T_c, and $T_{\rm eff}$ decrease as α increases at a fixed value of x. The behaviors of T_+, T_c, and $T_{\rm eff}$ as a function of x are plotted in Fig. 6.

Figure 6 demonstrates that the temperature in question,which is a consequence of the interaction between the two horizons, is constrained to satisfy $T_{\rm eff}\ge T_+\ge T_c$ under a fixed x. The boundary condition of the effective thermodynamic system is satisfied when the effective temperature, $T_{\rm eff}$, and radiation temperature of the cosmological horizon, T_c, are equal and the radiation temperature of the black hole horizon, T_+, is zero. In a small region of the smaller ratio between two horizons, the black hole temperature increases with increasing x, while the cosmological temperature remains constant until they are equal; this may be from the interplay between the two ho-

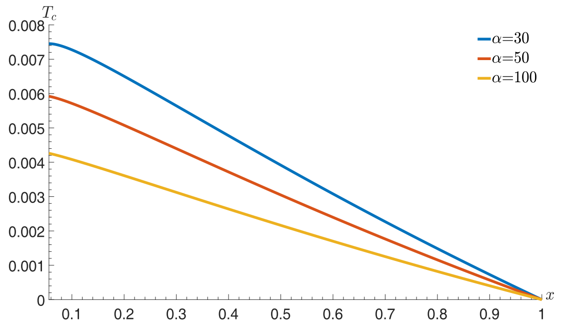

Fig. 4. (color online) T_c-x curve for different α, $Q=1$.

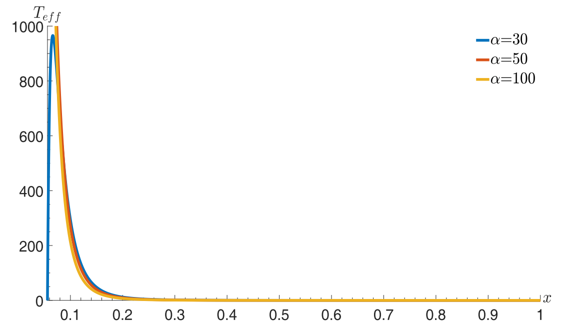

Fig. 5. (color online) $T_{\rm eff}-x$ curve for different α, $Q=1$.

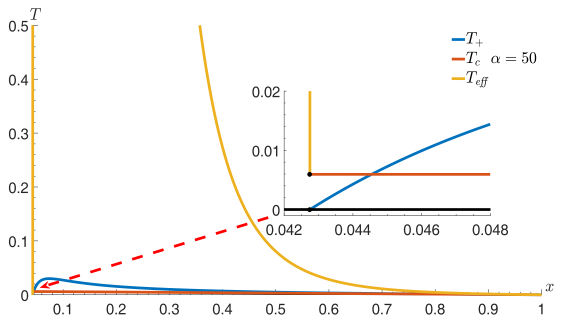

Fig. 6. (color online) $T_+,T_c,T_{\rm eff}-x$ curves for $\alpha=50$, $Q=1$.

rizons. The black hole temperature is always higher than the cosmological temperature. Furthermore, the Smarr relation of the effective thermodynamic system is as follows:

$$M=2T_{\rm eff}S+\varphi_{\rm eff}Q-3P_{\rm eff}V. \tag{27}$$

IV. SCHOTTKY ANOMALY OF RN-DS BLACK HOLES

The heat capacity of a thermal system is typically an increasing function of temperature. However, a peak in the heat capacity, known as a Schottky anomaly, can occur in a system that has a maximum energy. It is well established that, for a paramagnetic mass system with j = 1/2, which is a two-level system, the partition function reads

$$z={\rm e}^{\beta\varepsilon}+{\rm e}^{-\beta\varepsilon}, \tag{28}$$

where $\beta=1/kT$, and $\pm\varepsilon$represents the energy of two energy levels. From Eq. (28), the internal energy of the system has the following form:

$$U=F+TS=-N\varepsilon\tanh(\beta\varepsilon). \tag{29}$$

Neglecting the interaction between particles, the specific heat can be expressed in the following form:

$$\hat{C}=Nk\left(\frac{\Delta}{T}\right)^2\frac{{\rm e}^{\frac{\Delta}{T}}}{(1+{\rm e}^{\frac{\Delta}{T}})^2}. \tag{30}$$

This is a characteristic of a two-energy-level system with an energy gap $\Delta=2\varepsilon$, which is just the well-known Schottky specific heat. This is of great importance for the treatment of a multitude of systems as two-level systems. The behaviour of heat capacity as a function of temperature is presented in Fig. 7 (see more details in Refs. [68, 70]).

The effective thermodynamic system, comprising two thermodynamic subsystems, was obtained through the consideration of the interaction between the two horizons with disparate temperatures in dS spacetime. The interaction between the two subsystems occurs through the two horizons. It is therefore necessary to determine whether the thermodynamic properties between the two horizons can be described as those of a two-level system. This is one of the key issues to be addressed in the study of thermodynamic properties in dS spacetime. In recent years, Schottky specific heat has been investigated, with some promising results emerging in dS spacetime [68, 70, 74]. Nevertheless, the conclusions obtained are not exhaustive in the absence of the interaction between the black hole and cosmological horizons. The Schottky specific heat in RN-dS spacetime based on the interaction between the two horizons is investigated in this section.

When the charge, Q, and cosmological constant, l, in spacetime are held constant, based on these effective thermodynamical quantities, the heat capacity of this RN-dS spacetime can be obtained by the following expression:

$$C_{Q,l}=T_{\rm eff}\left(\frac{\partial S}{\partial T_{\rm eff}}\right)_{Q,l}=T_{\rm eff}\left(\frac{{\rm d}S/{\rm d}x}{{\rm d}T_{\rm eff}/{\rm d}x}\right)_{Q,l}. \tag{31}$$

From Eq. (31), the behavior of $C_{Q,l}$ with respect to the ratio between two horizons and the effective temperature can be plotted under different values of α.

As illustrated in Figs. 8 and 9, the heat capacity of RN-dS spacetime described by the effective quantities ex-

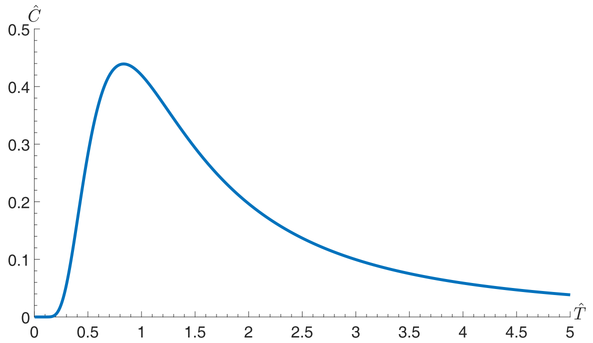

Fig. 7. (color online) $\hat{C}-\hat{T}$ curve with $\hat{T}=T/\varepsilon$.

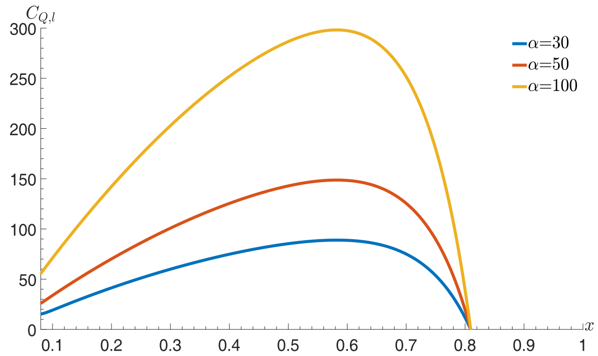

Fig. 8. (color online) $C_{Q,l}-x$ curve for different α.

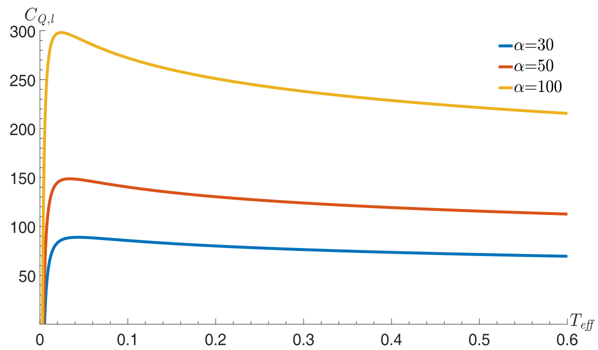

Fig. 9. (color online) $C_{Q,l}-T_{\rm eff}$ curve for different α.

hibits similarities to that of the two-level system shown in Fig. 7, namely, both exhibit certain behavioral patterns. The heat capacity has an extreme value within certain parameter ranges. Note that, when the space between two horizons in RN-dS spacetime is depicted by the corresponding effective quantities, we call it the effective thermodynamic system. The effective thermodynamic system in RN-dS spacetime is constituted by the subsystems that correspond to the two horizons. Consequently, the two horizons in RN-dS spacetime can be regarded as two distinct energy levels of a two-level system. When the particles are situated on different horizons, they exhibit distinct energy levels. According to this hypothesis, the two horizons in RN-dS spacetime are regarded as two different energy levels. Thus, the energies of the particles on the black hole and cosmological horizons read

$$\varepsilon_+=\varepsilon T_+,\quad \varepsilon_c=\varepsilon T_c, \tag{32}$$

where ε is a positive constant, and the term δ in Eq. (30) becomes

$$\Delta=\varepsilon_+-\varepsilon_c=\varepsilon(T_+-T_c),\quad T=T_{\rm eff}/\varepsilon. \tag{33}$$

Because the behavior of heat capacity for RN-dS spacetime regarded as a two-level system is independent of the parameter ε, we set it to $\varepsilon=1$. Substituting Eqs. (13), (14), and (19) into Eq. (33), we can obtain the following:

$$\frac{\Delta}{T_{\rm eff}}=\frac{\varepsilon x^4(1-x)^2(1+x)\left[1-\frac{1}{2}\left(1-\sqrt{1-\frac{(1+x+x^2)}{3x\alpha}}\right)\frac{(1+x)^2}{x}\right]}{(1-x^3)\left\{\left[(1+x)(1+x^3)-2x^2\right]-\frac{1}{2x}\left(1-\sqrt{1-\frac{(1+x+x^2)}{3x\alpha}}\right)\left[(1+x+x^2)(1+x^4)-2x^3\right]\right\}}. \tag{34}$$

To differentiate this from a general two-level system, we denote the heat capacity of RN-dS spacetime as a two-level system in the following form:

$$\hat{C}_{Q,l}=Nk\left(\frac{\varepsilon\Delta}{T_{\rm eff}}\right)^2\frac{e^{\frac{\varepsilon\Delta}{T_{\rm eff}}}}{\left(1+e^{\frac{\varepsilon\Delta}{T_{\rm eff}}}\right)^2}. \tag{35}$$

The corresponding behaviors with respect to the ratio between two horizons and effective temperature are shown in Figs. 10 and 11.

A comparison of Figs. 8 and 10 and Figs. 9 and 11 reveals that the two horizons in RN-dS spacetime are indeed two distinct energy levels of the effective thermodynamic system. Therefore, RN-dS spacetime is a quantum system with two separate energy levels, and its heat capacity $\hat{C}_{Q,l}$ approaches zero as the temperature tends to zero. With increasing temperature, $\hat{C}_{Q,l}$ reaches a maximum and then decreases to zero, which is the famous Schottky behavior. It is significant that numerous systems can be

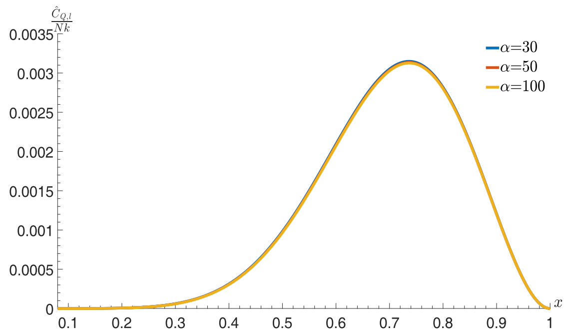

Fig. 10. (color online) $\hat{C}_{Q,l}/Nk-x$ curve for different α, $\varepsilon=1$.

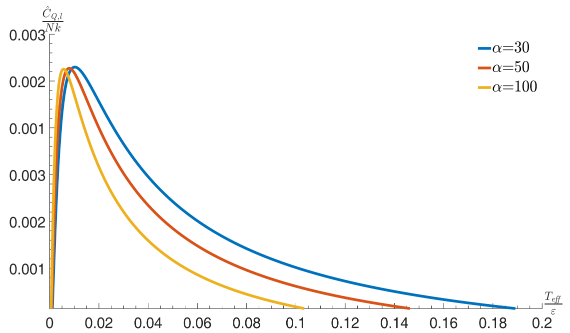

Fig. 11. (color online) $\hat{C}_{Q,l}/Nk-T_{\rm eff}/\varepsilon$ curve for different α.

treated as two-level systems. This result demonstrates a novel quantum property of RN-dS spacetime.

V. CONCLUSION

In this study, we considered a new boundary condition of RN-dS spacetime with the interplay between the black hole and cosmological horizons and provided the corresponding thermodynamic effective quantities. The Smarr relation and system free energy were expressed as effective thermodynamic quantities. These are the foundation for further study of the thermodynamic properties in RN-dS spacetime. By analyzing the heat capacity of RN-dS spacetime with the coexistence of two horizons, we found that the heat capacity of RN-dS spacetime described by the effective thermodynamic quantities has similar behavior to Schottky specific heat. To elucidate the underlying cause of this phenomenon, we proposed that the black hole and cosmological horizons can be regarded as two distinct energy levels of RN-dS spacetime, which is an effective thermodynamic system, *i.e.*, a two-level system. The heat capacity of this two-level system was discussed by a comparison between Figs. 8 and 10 and Figs. 9 and 11. It was found that the behavior of the heat capacity of this two-level system is highly similar to that of the effective thermodynamic system. Consequently, it can be concluded that the behavior of the heat capacity, $C_{Q,l}$, in RN-dS spacetime is primarily determined by the heat capacity, $\hat{C}_{Q,l}$, of the two-level system.

In addition, it was shown that RN-dS spacetime with two separate energy levels is a quantum system, and when the effective temperature tends to zero, the corresponding heat capacity $\hat{C}$ approaches zero. The heat capacity $\hat{C}$ also has an extreme value and is in accordance with that in a two-level system, i.e., the well-known Schottky specific heat. Furthermore, we investigated the effect of different parameters in RN-dS spacetime on the behavior of heat capacity. A comprehensive investigation of this topic would provide a deeper understanding of the quantum properties of black holes. At the same time, it would also suggest a new avenue for research on the thermodynamic properties of dS spacetime. This result encapsulates the quantum properties of RN-dS spacetime and offers a novel avenue for further in-depth investigation into the quantum properties of black holes and dS spacetime. In future work, we will discuss whether the behavior of heat capacity of thermodynamic systems in RN-dS spacetime is universal to that in other dS spacetimes with black holes and whether the parameters in other dS spacetimes can affect the property of heat capacity.

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References

[1] The event horizon telescope collaboration *et al.*, ApJL **875**, L1 (2019)
[2] The event horizon telescope collaboration *et al.*, ApJL **875**, L2 (2019)
[3] S. W. Hawking, Commun. Math. Phys **43**(3), 199 (1975)
[4] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973)
[5] D. Kubiznak and R. B. Mann, JHEP **07**, 033 (2012)
[6] R. G. Cai, L. M. Cao, L. Li *et al.*, JHEP **09**, 005 (2013)
[7] R. G. Cai, S. M. Ruan, S. J. Wang *et al.*, JHEP **09**, 161 (2016)
[8] J. L. Zhang, R. G. Cai, and H. W. Yu, Phys. Rev. D **91**, 044028 (2015)
[9] S. Gunasekaran, D. Kubiznak, and R. B. Mann, JHEP **11**, 110 (2012)
[10] A. M. Frassino, D. Kubiznak, R. B. Mann *et al.*, JHEP **09**, 080 (2014)
[11] N. Altamirano, D. Kubiznak, R. B. Mann *et al.*, Class. Quant. Grav. **31**, 042001 (2014)
[12] J. Y. Yang and R. B. Mann, JHEP **08**, 028 (2023)
[13] J. Wu and R. B. Mann, Phys. Rev. D **107**, 084035 (2023)
[14] S. W. Wei, Y. P. Zhang, Y. X. Liu *et al.*, Phys. Rev. Res. **5**, 043050 (2023)

[15] T. Durgut, R. A. Hennigar, H. K. Kunduri *et al.*, JHEP **03**, 114 (2023)
[16] J. Wu and R. B. Mann, Class. Quan. Grav. **40**, 06LT01 (2023)
[17] Y. Xiao, Y. Tian, and Y.X. Liu, Phys. Rev. L **132**, 021401 (2024)
[18] M. T. Wang, H. Y. Jiang, and Y. X. Liu, JHEP **07**, 178 (2023)
[19] R. Zhao, H. H. Zhao, M. S. Ma *et al.*, Eur. Phy. J. C **73**, 2645 (2013)
[20] S. H. Hendi, R. B. Mann, S. Panahiyan *et al.*, Phys. Rev. D **95**, 021501(R) (2017)
[21] S. H. Hendi, S. Hajkhalili, M. Jamil *et al.*, Fort. der. Phys. **71**(8-9), 202200101 (2023)
[22] M. Momennia and S. H. Hendi, Phys. Lett. B **822**, 136692 (2021)
[23] S. H. Hendi and K. Jafarzade, Phys. Rev. D **103**, 104011 (2021)
[24] H. Laassiri, A. Daassou, and R. Benbrik, Int. J. Mod. Phys. A **39**(24), 2450089 (2024)
[25] X. Q. Li, H. P. Yan, L. L. Xing *et al.*, Phys. Rev. D **107**, 104055 (2023)
[26] S. I. Kruglov, Eur. Phys. J. C **82**(4), 292 (2022)
[27] S. I. Kruglov, Annals Phys. **441**, 168894 (2022)
[28] Y. Z. Du, H. F. Li, and R. Zhao, Eur. Phys. J. C **82**, 859 (2022)
[29] Y. Z. Du, H. F. Li, F. Liu *et al.*, JHEP **01**, 137 (2023)
[30] Z. Y. Gao, X. Q. Kong, and L. Zhao, Eur. Phys. J. C **82**, 112 (2022)
[31] S. Ghaffari, G. G. Luciano, and A. Sheykhi, Phys. Dark Univ. **44**, 101447 (2024)
[32] M. Y. Zhang, H. Chen, H. Hassanabadi *et al.*, Chin. Phys. C **48**(6), 065101 (2024)
[33] D. J. Gogoi, Y. Sekhmani, D. Kalita *et al.*, Fort. der. Phys. **71**, 2300010 (2023)
[34] M. Yasir, X. Tiecheng, F. Javed *et al.*, Chin. Phys. C **48**, 015103 (2024)
[35] J. Barrientos and J. Mena, Phys. Rev. D **106**, 044064 (2022)
[36] J. Sadeghi, M. R. Alipour, S. N. Gashti *et al.*, arXiv: 2402.02257 [hep-th]
[37] T. N. Hunga and C. H. Nam, Eur. Phys. J. C **83**, 582 (2023)
[38] N. J. Gogoi and P. Phukon, Phys. Rev. D **108**, 066016 (2023)
[39] C. Fairoos and T. Sharqui, Int. J. Mod. Phys. A **38**, 2350133 (2023)
[40] D. Wu, S.Y. Gu, X.D. Zhu *et al.*, JHEP **06**, 213 (2024)
[41] P. K. Yerra, C. Bhamidipati, and S. Mukherji, JHEP **03**, 138 (2024)
[42] S. W. Wei and Y. X. Liu, Phys. Rev. D **91**, 044018 (2015)
[43] S. W. Wei and Y. X. Liu, Phys. Rev. Lett. **115**, 111302 (2015)
[44] S. W. Wei, Y. X. Liu, and R. B. Mann, Phys. Rev. Lett. **123**, 071103 (2019)
[45] D. Y. Chen, J. Tao, and X. T. Yang, Phys. Dark. Universe **42**, 101379 (2023)
[46] O. Sokoliuk, S. Pradhan, A. Baransky *et al.*, Fort. Phys. **72**, 2300043 (2024)
[47] G. Ruppeiner and A. M. Sturzu, Phys. Rev. D **108**, 086004 (2023)
[48] S. Masood A. S. Bukhari, B. Pourhassan, H. Aounallah *et al.*, Class. Quant. Grav. **40**, 225007 (2023)
[49] G. R. Li, G. P. Li, and S. Guo, Class. Quant. Grav. **39**, 195011 (2022)
[50] X. Y. Guo, H. F. Li, L. C. Zhang *et al.*, Phys. Rev. D **100**, 064036 (2019)
[51] R. G. Cai, Nucl. Phys. B **628**, 375 (2002)
[52] G. A. Marks, F. Simovic, and R. B. Mann, Phys. Rev. D **104**, 104056 (2021)
[53] Y. Sekiwa, Phys. Rev. D **73**, 084009 (2006)
[54] M. Urano, A. Tomimatsu, and H. Saida, Class. Quant. Grav. **26**, 105010 (2009)
[55] S. Mbarek and R. B. Mann, JHEP **02**, 103 (2019)
[56] D. Kubiznak and F. Simovic, Class. Quan. Grav **33**, 245001 (2016)
[57] F. Simovic and R. B. Mann, JHEP **05**, 136 (2019)
[58] S. Haroon, R. A. Hennigar, R. B. Mann *et al.*, Phys. Rev. D **101**, 084051 (2020)
[59] F. Simovic, D. Fusco, and R. B. Mann, JHEP **02**, 219 (2021)
[60] B. P. Dolan, D. Kastor, D. Kubiznak *et al.*, Phys. Rev. D. **87**, 104017 (2013)
[61] S. Bhattacharya, Eur. Phys. J. C **76**, 112 (2016), arXiv: 1506.07809
[62] J. McInerney, G. Satishchandrana, and J. Traschena, Class. Quantum Grav **33**, 105007 (2016)
[63] P. Kanti and T. Pappas, Phys. Rev. D **96**, 024038 (2017)
[64] L. C. Zhang, R. Zhao, and M.S. Ma, Phys. Lett. B **761**, 74 (2016)
[65] L. C. Zhang and R. Zhao, Phys. Lett. B **797**, 134798 (2019)
[66] Y. B. Ma, Y. Zhang, L. C. Zhang *et al.*, Eur. Phys. J. C **80**, 213 (2020)
[67] J. Ko and B. Gwak, JHEP **03**, 072 (2024)
[68] V. Chakrabhavi, M. Etheredge, Y. Qiu *et al.*, JHEP **02**, 231 (2024)
[69] P. R. Anderson and J. Traschen, JHEP **01**, 192 (2022)
[70] J. Dinsmore, P. Draper, D. Kastor *et al.*, Class. Quan. Grav **37**, 054001 (2020)
[71] Y. B. Du and X. D. Zhang, Eur. Phys. J. C **83**, 927 (2023)
[72] C. H. Nam, Eur. Phys. J. C **78**, 418 (2018)
[73] F. Mellor and I. Moss, Class. Quan. Grav. **6**, 1379 (1989)
[74] C. V. Johnson, Class. Quant. Grav. **37**, 054003 (2020)