

# Phenomenological studies on $\bar{B}^0 \rightarrow [K^- \pi^+]_{S/V} [\pi^+ \pi^-]_{V/S} \rightarrow K^- \pi^+ \pi^+ \pi^-$ decay\*

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**Abstract:** Within the quasi-two-body decay model, we study the localized CP violation and branching fraction of the four-body decay  $\bar{B}^0 \rightarrow [K^- \pi^+]_{S/V} [\pi^+ \pi^-]_{V/S} \rightarrow K^- \pi^+ \pi^+ \pi^-$  when the  $K^- \pi^+$  and  $\pi^- \pi^+$  pair invariant masses are  $0.35 < m_{K^- \pi^+} < 2.04$  GeV and  $0 < m_{\pi^- \pi^+} < 1.06$  GeV, with the pairs being dominated by the  $\bar{K}_0^*(700)^0$ ,  $\bar{K}^*(892)^0$ ,  $\bar{K}^*(1410)^0$ ,  $\bar{K}_0^*(1430)$  and  $\bar{K}^*(1680)^0$ , and  $f_0(500)$ ,  $\rho^0(770)$ ,  $\omega(782)$  and  $f_0(980)$  resonances, respectively. When dealing with the dynamical functions of these resonances,  $f_0(500)$ ,  $\rho^0(770)$ ,  $f_0(980)$  and  $\bar{K}_0^*(1430)$  are modeled with the Bugg model, Gounaris-Sakurai function, Flatté formalism and LASS lineshape, respectively, while the others are described by the relativistic Breit-Wigner function. Adopting the end point divergence parameters  $\rho_A \in [0, 0.5]$  and  $\phi_A \in [0, 2\pi]$ , our predicted results are  $\mathcal{A}_{CP}(\bar{B}^0 \rightarrow K^- \pi^+ \pi^+ \pi^-) \in [-0.365, 0.447]$  and  $\mathcal{B}(\bar{B}^0 \rightarrow K^- \pi^+ \pi^+ \pi^-) \in [6.11, 185.32] \times 10^{-8}$ , based on the hypothetical  $q\bar{q}$  structures for the scalar mesons in the QCD factorization approach. Meanwhile, we calculate the CP violating asymmetries and branching fractions of the two-body decays  $\bar{B}^0 \rightarrow SV(VS)$  and all the individual four-body decays  $\bar{B}^0 \rightarrow SV(VS) \rightarrow K^- \pi^+ \pi^+ \pi^-$ , respectively. Our theoretical results for the two-body decays  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 f_0(980)$ ,  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \omega(782)$ ,  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 f_0(980)$ ,  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \rho$ , and  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \omega$  are consistent with the available experimental data, with the remaining predictions await testing in future high precision experiments.

**Keywords:** CP violation,  $B$  meson, QCD factorization

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## I. INTRODUCTION

Four-body decays of heavy mesons are hard to investigate because of their complicated phase spaces and relatively small branching fractions. This leads to much less research on four-body decays than on two- and three-body decays [1-11]. We have discussed localized CP violation and branching fractions of the four-body decays  $\bar{B}^0 \rightarrow K^- \pi^+ \pi^- \pi^+$  in Ref. [12], focusing on the  $\pi\pi$  and  $K\pi$  invariant masses near the masses of the  $f_0(500)$  and  $K_0^*(700)$  mesons. The more resonance states there are, the more abundant physical mechanisms are available to us. We now further expand our research to include more contributions from different resonances in our study of CP violation and branching fractions in  $\bar{B}^0$  four-body de-

cays. Specifically, the invariant mass of the  $K^- \pi^+$  pair lies in the range  $0.35 < m_{K^- \pi^+} < 2.04$  GeV, which is dominated by the  $\bar{K}_0^*(700)^0$ ,  $\bar{K}^*(892)^0$ ,  $\bar{K}^*(1410)^0$ ,  $\bar{K}_0^*(1430)$  and  $\bar{K}^*(1680)^0$  resonances, and that of the  $\pi^- \pi^+$  pair is in the range  $0 < m_{\pi^- \pi^+} < 1.06$  GeV, which includes the  $f_0(500)$ ,  $\rho^0(770)$ ,  $\omega(782)$  and  $f_0(980)$  resonances. Meanwhile, studying the multibody decays can provide rich information about their intermediate resonances, especially about the compositions of scalar mesons, which are still unclear. The basic structure of the scalar meson is not well established because it is very difficult to identify experimentally [13, 14]. In the  $B \rightarrow f_0(980)K$  channel,  $B$  decay into a scalar meson was first observed and updated in Ref. [15], and confirmed by BaBar [16]. In Refs. [17, 18], there are two typical scenarios for scalar mesons based on

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their mass spectra and strong or electromagnetic decays. In Scenario 1 (S1), the light scalar mesons (such as  $f_0(500)$ ,  $\bar{K}_0^*(700)^0$ ,  $f_0(980)$  and  $a_0(980)$  mesons) are regarded as the lowest-lying  $q\bar{q}$  states, and some others (their masses near 1.5 GeV, including  $a_0(1450)$ ,  $K_0^*(1430)$ ,  $f_0(1370)$  and  $f_0(1500)$  [19-21]) are treated as the first corresponding excited states. In Scenario 2 (S2), the heavier nonet mesons are regarded as the ground states of  $q\bar{q}$ , while the lighter nonet mesons are not regular mesons and might be four-quark states. To further improve our understanding of the QCD mechanism and quark confinement, it is necessary for us to study the structural composition of the scalar mesons and related content.

In 2019, the LHCb collaboration studied the  $B^0 \rightarrow \rho(770)^0 K^*(892)^0$  decay within a quasi-two-body decay mode,  $B^0 \rightarrow (\pi^+ \pi^-)(K^+ \pi^-)$  [22]. In our work, we adopt this mechanism to study the four-body decay  $\bar{B}^0 \rightarrow K^- \pi^+ \pi^- \pi^+$ , i.e.  $\bar{B}^0 \rightarrow \bar{k}\rho \rightarrow K^- \pi^+ \pi^- \pi^+$ ,  $\bar{B}^0 \rightarrow \bar{k}\omega \rightarrow K^- \pi^+ \pi^- \pi^+$ ,  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 \sigma \rightarrow K^- \pi^+ \pi^- \pi^+$ ,  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 f_0(980) \rightarrow K^- \pi^+ \pi^- \pi^+$ ,  $\bar{B}^0 \rightarrow \bar{B}^0 \rightarrow \bar{K}^*(1410)^0 \sigma \rightarrow K^- \pi^+ \pi^- \pi^+$ ,  $\bar{B}^0 \rightarrow \bar{K}^*(1410)^0 f_0(980) \rightarrow K^- \pi^+ \pi^- \pi^+$ ,  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \rho \rightarrow K^- \pi^+ \pi^- \pi^+$ ,  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \omega \rightarrow K^- \pi^+ \pi^- \pi^+$ ,  $\bar{B}^0 \rightarrow \bar{K}^*(1680)^0 \sigma \rightarrow K^- \pi^+ \pi^- \pi^+$  and  $\bar{B}^0 \rightarrow \bar{K}^*(1680)^0 f_0(980) \rightarrow K^- \pi^+ \pi^- \pi^+$ , where the scalar mesons will be treated using S1 as mentioned above. We can then calculate the localized  $CP$  violations and branching fractions of the four-body decay  $\bar{B}^0 \rightarrow K^- \pi^+ \pi^- \pi^+$ . We can also calculate the  $CP$  violations and branching fractions of the two-body decays  $\bar{B}^0 \rightarrow SV(VS)$  and all the individual four-body decays  $\bar{B}^0 \rightarrow SV(VS) \rightarrow K^- \pi^+ \pi^- \pi^+$ . In fact, with the further development of the LHCb and Belle II experiments, more and more decay modes involving one or two scalar states in the  $B$  and  $D$  meson decays are expected to be measured with high precision in the future.

The remainder of this paper is organized as follows. Our theoretical framework is presented in Sect. II. In Sect. III, we give our numerical results. We summarize our work in Sect. IV. Appendix A collects the explicit formulas for all the four-body decay amplitudes. The dynamical functions for the corresponding resonances are summarized in Appendix B. We also consider the  $f_0(500)$ – $f_0(980)$  mixing in Appendix C. Related theoretical parameters are listed in Appendix D.

## II. THEORETICAL FRAMEWORK

### A. $B$ decay in the QCD factorization approach

In the framework of the QCD factorization approach [4, 23], the effective Hamiltonian matrix elements can be written as

$$\langle M_1 M_2 | \mathcal{H}_{\text{eff}} | B \rangle = \sum_{p=u,c} \lambda_p^{(D)} \langle M_1 M_2 | \mathcal{T}_A^p + \mathcal{T}_B^p | B \rangle, \quad (1)$$

where  $\mathcal{H}_{\text{eff}}$  is the effective weak Hamiltonian,  $\lambda_p^{(D)} = V_{pb} V_{pD}^*$ ,  $V_{pb}$  and  $V_{pD}$  are the CKM matrix elements, and  $\mathcal{T}_A^p$  and  $\mathcal{T}_B^p$  describe the contributions from non-annihilation and annihilation amplitudes, respectively; they can be expressed in terms of  $a_i^p$  and  $b_i^p$ .

Generally,  $a_i^p$  includes the contributions from naive factorization, vertex correction, penguin amplitude and spectator scattering, and can be expressed as follows [4]:

$$a_i^p(M_1 M_2) = \left( c_i + \frac{c_{i+1}}{N_c} \right) N_i(M_2) + \frac{c_{i+1}}{N_c} \frac{C_F \alpha_s}{4\pi} \times \left[ V_i(M_2) + \frac{4\pi^2}{N_c} H_i(M_1 M_2) \right] + P_i^p(M_2), \quad (2)$$

where  $c_i$  are the Wilson coefficients,  $N_i(M_2)$  is the leading-order coefficient, and  $V_i(M_2)$ ,  $H_i(M_1 M_2)$  and  $P_i^p(M_1 M_2)$  are one-loop vertex corrections, hard spectator interactions with a hard gluon exchange between the emitted meson and the spectator quark of the  $B$  meson, and penguin contractions, respectively.  $C_F = (N_c^2 - 1)/2N_c$ , with  $N_c = 3$  [4].

The weak annihilation contributions can be expressed in terms of  $b_i$  and  $b_{i,EW}$ , which are:

$$\begin{aligned} b_1 &= \frac{C_F}{N_c^2} c_1 A_1^i, & b_2 &= \frac{C_F}{N_c^2} c_2 A_1^i, \\ b_3^p &= \frac{C_F}{N_c^2} \left[ c_3 A_1^i + c_5 (A_3^i + A_3^f) + N_c c_6 A_3^f \right], \\ b_4^p &= \frac{C_F}{N_c^2} \left[ c_4 A_1^i + c_6 A_2^i \right], \\ b_{3,EW}^p &= \frac{C_F}{N_c^2} \left[ c_9 A_1^i + C_7 (A_3^i + A_3^f) + N_c c_8 A_3^f \right], \\ b_{4,EW}^p &= \frac{C_F}{N_c^2} \left[ c_{10} A_1^i + c_8 A_2^i \right], \end{aligned} \quad (3)$$

where the subscripts 1, 2, 3 of  $A_n^{i,f}$  ( $n = 1, 2, 3$ ) stand for the annihilation amplitudes induced from  $(V-A)(V-A)$ ,  $(V-A)(V+A)$ , and  $(S-P)(S+P)$  operators, respectively, and the superscripts  $i$  and  $f$  refer to gluon emission from the initial- and final-state quarks, respectively. The explicit expressions for  $A_n^{i,f}$  can be found in Ref. [24].

In the expressions for the spectator and annihilation corrections, there are end-point divergences  $X = \int_0^1 dx/(1-x)$ , which can be parametrized as [17]

$$X_{H,A} = (1 + \rho_{H,A} e^{i\phi_{H,A}}) \ln \frac{m_B}{\Lambda_h}, \quad (4)$$

with  $\Lambda_h$  being a typical scale of order 500 MeV,  $\rho_{H,A}$  an unknown real parameter and  $\phi_{H,A}$  the free strong phase in

the range  $[0, 2\pi]$ .

### B. Four-body decay amplitudes

For the four-body decay  $\bar{B}^0 \rightarrow K^- \pi^+ \pi^+ \pi^-$ , we con-

sider the two-body cascade decay mode  $\bar{B}^0 \rightarrow [K^- \pi^+]_{S/V} [\pi^- \pi^+]_{V/S} \rightarrow K^- \pi^+ \pi^- \pi^+$ . Within the QCDF framework in Ref. [4], we can deduce the two-body weak decay amplitudes of  $\bar{B}^0 \rightarrow [K^- \pi^+]_{S/V} [\pi^- \pi^+]_{V/S}$ , which are:

$$\begin{aligned} \mathcal{M}(\bar{B}^0 \rightarrow \bar{K}_{0i}^{*0} \rho) = & iG_F \sum_{p=u,c} \lambda_p^{(s)} \left\{ \left[ \delta_{pu} \alpha_2(\bar{K}_{0i}^{*0} \rho) + \frac{3}{2} \alpha_{3,EW}^p(\bar{K}_{0i}^{*0} \rho) \right] f_{\rho} m_{\rho} \varepsilon_{\rho}^* \cdot p_B F_1^{\bar{B}^0 \bar{K}_{0i}^{*0}}(m_{\rho}^2) \right. \\ & + \left[ \alpha_4^p(\rho \bar{K}_{0i}^{*0}) - \frac{1}{2} \alpha_{4,EW}^p(\rho \bar{K}_{0i}^{*0}) \right] \bar{f}_{\bar{K}_{0i}^{*0}} m_{\rho} \varepsilon_{\rho}^* \cdot p_B A_0^{\bar{B}^0 \rho}(m_{\bar{K}_{0i}^{*0}}^2) \\ & \left. + \left[ \frac{1}{2} b_3^p(\rho \bar{K}_{0i}^{*0}) - \frac{1}{4} b_{3,EW}^p(\rho \bar{K}_{0i}^{*0}) \right] f_{\bar{B}^0} f_{\rho} \bar{f}_{\bar{K}_{0i}^{*0}} \right\}, \end{aligned} \quad (5)$$

$$\begin{aligned} \mathcal{M}(\bar{B}^0 \rightarrow \bar{K}_{0i}^{*0} \omega) = & iG_F \sum_{p=u,c} \lambda_p^{(s)} \left\{ \left[ \delta_{pu} \alpha_2(\bar{K}_{0i}^{*0} \omega) + 2\alpha_3^p(\bar{K}_{0i}^{*0} \omega) + \frac{1}{2} \alpha_{3,EW}^p(\bar{K}_{0i}^{*0} \omega) \right] f_{\omega} m_{\omega} \varepsilon_{\omega}^* \cdot p_B F_1^{\bar{B}^0 \bar{K}_{0i}^{*0}}(m_{\omega}^2) \right. \\ & + \left[ \frac{1}{2} \alpha_{4,EW}^p(\omega \bar{K}_{0i}^{*0}) - \alpha_4^p(\omega \bar{K}_{0i}^{*0}) \right] \bar{f}_{\bar{K}_{0i}^{*0}} m_{\omega} \varepsilon_{\omega}^* \cdot p_B A_0^{\bar{B}^0 \omega}(m_{\bar{K}_{0i}^{*0}}^2) \\ & \left. + \left[ \frac{1}{4} b_{3,EW}^p(\omega \bar{K}_{0i}^{*0}) - \frac{1}{2} b_3^p(\omega \bar{K}_{0i}^{*0}) \right] f_{\bar{B}^0} f_{\omega} \bar{f}_{\bar{K}_{0i}^{*0}} \right\}, \end{aligned} \quad (6)$$

with  $\bar{K}_{0i}^{*0} = \bar{K}_0^*(700)^0$ ,  $\bar{K}_0^*(1430)^0$  corresponding to  $i = 1, 2$ , respectively, and

$$\begin{aligned} \mathcal{M}(\bar{B}^0 \rightarrow \bar{K}_i^{*0} f_{0j}) = & -iG_F \sum_{p=u,c} \lambda_p^{(s)} \left\{ \left[ \delta_{pu} \alpha_2(\bar{K}_i^{*0} f_{0j}) + 2\alpha_3^p(\bar{K}_i^{*0} f_{0j}) + \frac{1}{2} \alpha_{3,EW}^p(\bar{K}_i^{*0} f_{0j}) \right] \right. \\ & \times \bar{f}_{f_{0j}} m_{\bar{K}_i^{*0}} \varepsilon_{\bar{K}_i^{*0}}^* \cdot p_B A_0^{\bar{B}^0 \bar{K}_i^{*0}}(m_{f_{0j}}^2) + \left[ \sqrt{2} \alpha_3^p(\bar{K}_i^{*0} f_{0j}) + \sqrt{2} \alpha_4^p(\bar{K}_i^{*0} f_{0j}) \right. \\ & - \frac{1}{\sqrt{2}} \alpha_{3,EW}^p(\bar{K}_i^{*0} f_{0j}) - \frac{1}{\sqrt{2}} \alpha_{4,EW}^p(\bar{K}_i^{*0} f_{0j}) \left. \right] \bar{f}_{f_{0j}} m_{\bar{K}_i^{*0}} \varepsilon_{\bar{K}_i^{*0}}^* \cdot p_B A_0^{\bar{B}^0 \bar{K}_i^{*0}}(m_{f_{0j}}^2) \\ & + \left[ \frac{1}{2} \alpha_{4,EW}^p(f_{0j} \bar{K}_i^{*0}) - \alpha_4^p(f_{0j} \bar{K}_i^{*0}) \right] f_{\bar{K}_i^{*0}} m_{\bar{K}_i^{*0}} \varepsilon_{\bar{K}_i^{*0}}^* \cdot p_B F_1^{\bar{B}^0 f_{0j}}(m_{\bar{K}_i^{*0}}^2) + \left[ \frac{1}{\sqrt{2}} b_3^p(\bar{K}_i^{*0} f_{0j}) \right. \\ & \left. - \frac{1}{2\sqrt{2}} b_{3,EW}^p(\bar{K}_i^{*0} f_{0j}) \right] f_{\bar{B}^0} f_{\bar{K}_i^{*0}} \bar{f}_{f_{0j}}^S + \left[ \frac{1}{2} b_3^p(f_{0j} \bar{K}_i^{*0}) - \frac{1}{4} b_{3,EW}^p(f_{0j} \bar{K}_i^{*0}) \right] f_{\bar{B}^0} f_{\bar{K}_i^{*0}} \bar{f}_{f_{0j}}^n \left. \right\}, \end{aligned} \quad (7)$$

with  $\bar{K}_i^{*0} = \bar{K}^*(892)^0$ ,  $\bar{K}^*(1410)^0$ ,  $\bar{K}^*(1680)^0$  corresponding to  $i = 1, 2, 3$ , respectively, and  $f_{0j} = f_0(500)$ ,  $f_0(980)$  when  $j = 1, 2$ , respectively. In Eqs. (5)-(7),  $F_1^{\bar{B}^0 \rightarrow S}(m_{\bar{K}_i^{*0}}^2)$  and  $A_0^{\bar{B}^0 \rightarrow V}(m_{\bar{K}_i^{*0}}^2)$  are the form factors for  $\bar{B}^0$  to scalar and vector meson transitions, respectively,  $f_V$ ,  $\bar{f}_S$ , and  $f_{\bar{B}^0}$  are the decay constants of the vector, scalar, and  $\bar{B}^0$  mesons,

respectively,  $\bar{f}_{f_{0j}}^S$  and  $\bar{f}_{f_{0j}}^n$  are the decay constants of the  $f_{0j}$  mesons coming from the up and strange quark components, respectively.

In the framework of the two two-body decays, the four-body decay can be factorized into three pieces as follows:

$$\mathcal{M}(\bar{B}^0 \rightarrow [K^- \pi^+]_S [\pi^- \pi^+]_V \rightarrow K^- \pi^+ \pi^- \pi^+) = \frac{\langle S V | \mathcal{H}_{\text{eff}} | \bar{B}^0 \rangle \langle K^- \pi^+ | \mathcal{H}_{S K^- \pi^+} | S \rangle \langle \pi^- \pi^+ | \mathcal{H}_{V \pi \pi^+} | V \rangle}{s_S s_V}, \quad (8)$$

and

$$\mathcal{M}(\bar{B}^0 \rightarrow [K^- \pi^+]_V [\pi^- \pi^+]_S \rightarrow K^- \pi^+ \pi^- \pi^+) = \frac{\langle V S | \mathcal{H}_{\text{eff}} | \bar{B}^0 \rangle \langle K^- \pi^+ | \mathcal{H}_{V K^- \pi^+} | V \rangle \langle \pi^- \pi^+ | \mathcal{H}_{S \pi \pi^+} | S \rangle}{s_V s_S}, \quad (9)$$

where  $\mathcal{H}_{\text{eff}}$  is the effective weak Hamiltonian,

$\langle M_1 M_2 | \mathcal{H}_S | V \rangle = g_V M_1 M_2 (p_{M_1} - p_{M_2}) \cdot \epsilon_V$  and  $\langle M_1 M_2 | \mathcal{H}_S | S \rangle =$

$g_{SM,M_2}$ ,  $g_{VM,M_2}$  and  $g_{SM,M_2}$  are the strong coupling constants of the corresponding vector and scalar mesons decays, and  $s_{S/V}$  are the reciprocals of the dynamical functions  $T_{S/V}$  for the corresponding resonances. The specific kinds and expressions of  $T_{S/V}$  are given in the fifth column of Table 1 and Appendix C, respectively.

**Table 1.** Masses, widths and decay models of the intermediate resonances [25].

Resonance	Mass/MeV	Width/MeV	$J^P$	Model
$\sigma$	$475 \pm 75$	$550 \pm 150$	$0^+$	BUGG
$\rho$	$775.26 \pm 0.25$	$149.1 \pm 0.8$	$1^-$	GS
$\omega$	$782.65 \pm 0.12$	$8.49 \pm 0.08$	$1^-$	RBW
$f_0(980)$	$990 \pm 20$	$65 \pm 45$	$0^+$	FLATTÉ
$\bar{\kappa}$	$824 \pm 30$	$478 \pm 50$	$0^+$	RBW
$\bar{K}^*(892)^0$	$895.5 \pm 0.20$	$47.3 \pm 0.5$	$1^-$	RBW
$\bar{K}^*(1410)^0$	$1421 \pm 9$	$236 \pm 18$	$1^-$	RBW
$\bar{K}_0^*(1430)^0$	$1425 \pm 50$	$270 \pm 80$	$0^+$	LASS
$\bar{K}^*(1680)^0$	$1718 \pm 18$	$322 \pm 110$	$1^-$	RBW

When considering the contributions from the  $\bar{B}^0 \rightarrow [K^-\pi^+]_S[\pi^-\pi^+]_V \rightarrow K^-\pi^+\pi^-\pi^+$  and  $\bar{B}^0 \rightarrow [K^-\pi^+]_V[\pi^-\pi^+]_S \rightarrow K^-\pi^+\pi^-\pi^+$  channels as listed in Eqs. (8) and (9), the total decay amplitude of the  $\bar{B}^0 \rightarrow K^-\pi^+\pi^-\pi^+$  decay can be written as (As for the relative strong phase  $\delta$  between these two interference amplitudes, we set  $\delta = 0$  as in Refs. [5, 30, 31])

$$\begin{aligned} \mathcal{M} = & \mathcal{M}(\bar{B}^0 \rightarrow [K^-\pi^+]_S[\pi^-\pi^+]_V \rightarrow K^-\pi^+\pi^-\pi^+) \\ & + \mathcal{M}(\bar{B}^0 \rightarrow [K^-\pi^+]_V[\pi^-\pi^+]_S \rightarrow K^-\pi^+\pi^-\pi^+). \end{aligned} \quad (10)$$

### C. Kinematics of the four-body decay and localized CP violation

One can use the five variables  $s_{\pi\pi}$ ,  $s_{K\pi}$ ,  $\phi$ ,  $\theta_\pi$  and  $\theta_K$  to describe the kinematics of the four-body decay  $\bar{B}^0 \rightarrow K^-(p_1)\pi^+(p_2)\pi^-(p_3)\pi^+(p_4)$  [26-29], where  $s_{\pi\pi}$  and  $s_{K\pi}$  are the invariant mass squared of the  $\pi\pi$  system and  $K\pi$  system, respectively,  $\phi$  is the angle between the  $\pi\pi$  and  $K\pi$  planes, and  $\theta_\pi$  (or  $\theta_K$ ) is the angle of the  $\pi^+$  (or  $K^-$ ) in the  $\pi\pi$  (or  $K\pi$ ) center-of-mass system with respect to the  $\pi\pi$  (or  $K\pi$ ) line of flight in the  $\bar{B}^0$  rest frame. Their specific physical ranges can be found in detail in Refs. [12, 26-29].

For presentation and calculation, it is more convenient to replace the individual momenta  $p_1, p_2, p_3, p_4$  with the following kinematic variables:

$$\begin{aligned} P &= p_1 + p_2, & Q &= p_1 - p_2, \\ L &= p_3 + p_4, & N &= p_3 - p_4. \end{aligned} \quad (11)$$

Using the above formula, we can get:

$$\begin{aligned} P^2 &= s_{K\pi}, & Q^2 &= 2(p_K^2 + p_\pi^2) - s_{K\pi}, & L^2 &= s_{\pi\pi}, \\ P \cdot L &= \frac{1}{2}(m_{\bar{B}^0}^2 - s_{K\pi} - s_{\pi\pi}), & P \cdot N &= X \cos \theta_\pi, \\ L \cdot Q &= \sigma(s_{K\pi})X \cos \theta_K, \end{aligned} \quad (12)$$

where

$$\sigma(s_{K\pi}) = \sqrt{1 - (m_K^2 + m_\pi^2)/s_{K\pi}}. \quad (13)$$

With the decay amplitude, one can get the decay rate of the four-body decay [32],

$$d^5\Gamma = \frac{1}{4(4\pi)^6 m_{\bar{B}^0}^3} \sigma(s_{\pi\pi}) X(s_{\pi\pi}, s_{K\pi}) \sum_{\text{spins}} |\mathcal{M}|^2 d\Omega, \quad (14)$$

where  $\sigma(s_{\pi\pi}) = \sqrt{1 - 4m_\pi^2/s_{\pi\pi}}$ , and  $\Omega$  represents the phase space with  $d\Omega = ds_{\pi\pi} ds_{K\pi} d\cos\theta_\pi d\cos\theta_K d\phi$ .

The differential CP asymmetry parameter and the localized integrated CP asymmetry take the following forms:

$$\mathcal{A}_{CP} = \frac{|\mathcal{M}|^2 - |\bar{\mathcal{M}}|^2}{|\mathcal{M}|^2 + |\bar{\mathcal{M}}|^2}, \quad (15)$$

and

$$\mathcal{A}_{CP}^\Omega = \frac{\int d\Omega (|\mathcal{M}|^2 - |\bar{\mathcal{M}}|^2)}{\int d\Omega (|\mathcal{M}|^2 + |\bar{\mathcal{M}}|^2)}, \quad (16)$$

respectively.

## III. NUMERICAL RESULTS

When dealing with the scalar mesons, we adopt Scenario 1 in Ref. [17], in which those with masses below or near 1 GeV ( $\sigma$ ,  $f_0(980)$ ,  $\kappa$ ) and near 1.5 GeV ( $\bar{K}_0^*(1430)$ ) are suggested as the lowest-lying  $q\bar{q}$  states and the first excited state, respectively. For the decay constants of the  $f_{0j}$  mesons, we consider the  $f_0(500) - f_0(980)$  mixing with the mixing angle  $|\varphi_m| = 17^\circ$  (see Appendix A for details). For the decay constants and Gegenbauer moments of the  $\bar{K}^*(1410)^0$  and the  $\bar{K}^*(1680)^0$  mesons, we assume they have the same central values as that of  $\bar{K}^*(892)^0$  and assign their uncertainties to be  $\pm 0.1$  [33]. With the QCDF approach, we have obtained the amplitudes of the two-body decays  $\bar{B}^0 \rightarrow SV$  and  $\bar{B}^0 \rightarrow VS$ , which are listed in Eqs. (5)-(7). Generally, the end-point divergence parameter  $\rho_A$  is constrained in the range  $[0, 1]$  and  $\phi_A$  is treated as a free strong phase. The experimental data for  $B$  two-body decays can provide important information to

restrict the ranges of these two parameters. In fact, compared with the  $B \rightarrow PV/VP/PP$  decays, there is much less experimental data for the  $B \rightarrow VS/PS$  and  $B \rightarrow SV/SP$  decays, so the values of  $\rho_A$  and  $\phi_A$  for these decays are not well-determined. Therefore, we adopt  $\rho_{A,H} < 0.5$  and  $0 \leq \phi_{A,H} \leq 2\pi$ , as in Refs. [17, 24]. With more experimental data, both of these could be defined in small regions in the future.

Substituting Eqs. (5)-(7) into Eq. (15), we obtain the  $CP$ -violating asymmetries of the two-body decays  $\bar{B}^0 \rightarrow SV$  and  $\bar{B}^0 \rightarrow VS$  with the parameters given in Table 1 and Appendix F, which are listed in Table 2. From Table 2, one can see our theoretical results for the  $CP$  asymmetries of  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 f_0(980)$  and  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \omega$  are consistent with the data from the BaBar collaboration. However, the predicted central values of the  $CP$  asymmetries of  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \rho$  and  $\bar{B}^0 \rightarrow$

$\bar{K}_0^*(1430)^0 \omega$  are larger than those in Ref. [18]. The main difference between our work and Ref. [18] is the structure of the  $\bar{K}_0^*(1430)^0$  meson, which is explored in S1 in our work and S2 in Ref. [18]. Furthermore, we predict the  $CP$  asymmetries of some other decay channels. We find the signs of the  $CP$  asymmetries are negative in  $\bar{B}^0 \rightarrow \bar{\kappa} \rho$ ,  $\bar{B}^0 \rightarrow \bar{K}^*(1410)^0 f_0(980)$  and  $\bar{B}^0 \rightarrow \bar{K}^*(1680)^0 f_0(980)$  decays, with the first of these being one order of magnitude larger than the other two. For the positive values of the  $CP$  asymmetries in our work, those for the  $\bar{B}^0 \rightarrow \bar{\kappa} \omega$  and  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 \sigma$  decays are also one order of magnitude larger than the others. We have also calculated the branching fractions of the two-body decays  $\bar{B}^0 \rightarrow SV$  and  $\bar{B}^0 \rightarrow VS$  which are listed in Table 3. Our results are consistent with the available experimental data for the  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 f_0(980)$ ,  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \rho$  and  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \omega$  decays. Meanwhile, we find the mag-

**Table 2.** Direct  $CP$  violations (in units of  $10^{-2}$ ) of the two-body decays  $\bar{B}^0 \rightarrow [K^- \pi^+]_{S/V} [\pi^+ \pi^-]_{V/S}$ . The experimental branching fractions are taken from Ref. [34]. The theoretical errors come from the uncertainties of the form factors, decay constants, Gegenbauer moments and divergence parameters.

Decay mode	BaBar	PDG [25]	[18]	This work
$\bar{\kappa} \rho$	–	–	–	$-10.66 \pm 3.14$
$\bar{\kappa} \omega$	–	–	–	$17.43 \pm 6.53$
$\bar{K}^*(892)^0 \sigma$	–	–	–	$25.57 \pm 10.42$
$\bar{K}^*(892)^0 f_0(980)$	$7 \pm 10 \pm 2$	$7 \pm 10$	–	$9.31 \pm 1.04$
$\bar{K}^*(1410)^0 \sigma$	–	–	–	$0.43 \pm 0.13$
$\bar{K}^*(1410)^0 f_0(980)$	–	–	–	$-2.01 \pm 0.19$
$\bar{K}_0^*(1430)^0 \rho$	–	–	$0.54^{+0.45+0.02+3.76}_{-0.46-0.02-1.80}$	$6.03 \pm 0.97$
$\bar{K}_0^*(1430)^0 \omega$	$-7 \pm 9 \pm 2$	–	$0.03^{+0.37+0.01+0.29}_{-0.35-0.01-3.00}$	$-9.53 \pm 3.88$
$\bar{K}^*(1680)^0 \sigma$	–	–	–	$3.03 \pm 0.77$
$\bar{K}^*(1680)^0 f_0(980)$	–	–	–	$-2.76 \pm 0.20$

**Table 3.** Branching fractions (in units of  $10^{-6}$ ) of the two-body decays  $\bar{B}^0 \rightarrow [K^- \pi^+]_{S/V} [\pi^+ \pi^-]_{V/S}$ . We have used  $\mathcal{B}(f_0(980) \rightarrow \pi^+ \pi^-) = 0.5$  to obtain the experimental branching fractions for  $f_0(980)V$ . The theoretical errors come from the uncertainties of the form factors, decay constants, Gegenbauer moments and divergence parameters.

Decay mode	BaBar	Belle	LHCb [24]	PDG [25]	QCDF [18]	pQCD [35, 36]	This work
$\bar{\kappa} \rho$	–	–	–	–	–	–	$1.35 \pm 0.47$
$\bar{\kappa} \omega$	–	–	–	–	–	–	$3.87 \pm 1.65$
$\bar{K}^*(892)^0 \sigma$	–	–	–	–	–	–	$0.11 \pm 0.04$
$\bar{K}^*(892)^0 f_0(980)$	$11.4 \pm 1.4$	$< 4.4$	–	$7.8^{+4.2}_{-3.6}$	$9.1^{+1.0+1.0+5.3}_{-0.4-0.5-0.7}$	$11.2 \sim 13.7$	$9.48 \pm 2.88$
$\bar{K}^*(1410)^0 \sigma$	–	–	–	–	–	–	$25.41 \pm 9.13$
$\bar{K}^*(1410)^0 f_0(980)$	–	–	–	–	–	–	$14.39 \pm 4.22$
$\bar{K}_0^*(1430)^0 \rho$	$27 \pm 4 \pm 2 \pm 3$	–	$10.0^{+2.4+0.5+12.1}_{-2.0-0.4-3.1}$	$27.0 \pm 6.0$	$4.1^{+1.1+0.2+2.6}_{-1.0-0.2-0.1}$	$4.8^{+1.1+1.0+0.3}_{-0.0-1.0-0.3}$	$8.13 \pm 2.03$
$\bar{K}_0^*(1430)^0 \omega$	$6.4^{+1.4+0.3+4.0}_{-1.2-0.2-0.9}$	–	–	$16.0 \pm 3.4$	$9.3^{+2.7+0.3+3.9}_{-2.2-0.3-1.3}$	$9.3^{+2.1+3.6+1.2}_{-2.0-2.9-1.0}$	$5.02 \pm 1.06$
$\bar{K}^*(1680)^0 \sigma$	–	–	–	–	–	–	$27.64 \pm 8.59$
$\bar{K}^*(1680)^0 f_0(980)$	–	–	–	–	–	–	$21.76 \pm 8.33$



nitudes of the branching fractions are of order  $10^{-5}$  for  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 f_0(980)$ ,  $\bar{B}^0 \rightarrow \bar{K}^*(1410)^0 \sigma$  and  $\bar{B}^0 \rightarrow \bar{K}^*(1410)^0 f_0(980)$ , but of order  $10^{-6}$  for  $\bar{B}^0 \rightarrow \bar{\kappa} \rho$ ,  $\bar{B}^0 \rightarrow \bar{\kappa} \omega$ ,  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \rho$  and  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \omega$ . We note that the predicted branching fraction of  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 \sigma$  is the smallest, of the order of  $10^{-7}$ .

For different intermediate resonance states, we use different models to deal with their dynamical functions. These are listed in detail in Table 1 and Appendix D;  $\sigma$ ,  $\rho^0(770)$ ,  $f_0(980)$  and  $\bar{K}_0^*(1430)$  are modeled with the Bugg model [37], Gounaris-Sakurai function [38], Flatté formalism [39] and LASS lineshape [40-42], respectively, while the others are described by the relativistic Breit-Wigner function [43]. Inserting Eqs. (A1)-(A3) into Eqs. (16) and (14), we can directly obtain the  $CP$  asymmetries and branching fractions of all the individual four-body decay channels  $\bar{B}^0 \rightarrow [K^-\pi^+]_{S/V}[\pi^+\pi^-]_{V/S} \rightarrow K^-\pi^+\pi^+\pi^-$  by integrating the phase space of Eq. (14), both of which are summarized in Table 4. From this table, we can conclude that the ranges of these  $CP$  asymmetries and branching fractions are about  $[-7.03, 24.33] \times 10^{-2}$  and  $[0.11, 27.3] \times 10^{-6}$ , respectively. Considering the contributions from all the four-body decays listed in Table 4, we can obtain the localized integrated  $CP$  asymmetries and branching fractions of the  $\bar{B}^0 \rightarrow K^-\pi^+\pi^+\pi^-$  decay by integrating the phase space. Our results are in the ranges  $\mathcal{A}_{CP}(\bar{B}^0 \rightarrow K^-\pi^+\pi^+\pi^-) = [-0.365, 0.447]$  and  $\mathcal{B}(\bar{B}^0 \rightarrow K^-\pi^+\pi^+\pi^-) = [6.11, 185.32] \times 10^{-8}$  when the invariant masses of  $K^-\pi^+$  and  $\pi^-\pi^+$  are in the ranges  $0.35 < m_{K^-\pi^+} < 2.04 \text{ GeV}$  and  $0 < m_{\pi^-\pi^+} < 1.06 \text{ GeV}$ , where the  $K\pi$  channel is dominated by the  $\kappa$ ,  $\bar{K}^*(892)^0$ ,  $\bar{K}^*(1410)^0$ ,  $\bar{K}_0^*(1430)$  and  $\bar{K}^*(1680)^0$  resonances, the  $\pi\pi$  channel is dominated by the  $\sigma$ ,  $\rho^0(770)$ ,  $\omega(782)$  and  $f_0(980)$  resonances, and the ranges of  $\rho_A$  and  $\phi_A$  are taken

**Table 4.** Direct  $CP$  violations (in units of  $10^{-2}$ ) and branching fractions (in units of  $10^{-6}$ ) of the four-body decays  $\bar{B}^0 \rightarrow [K^-\pi^+]_{S/V}[\pi^+\pi^-]_{V/S} \rightarrow K^-\pi^+\pi^+\pi^-$ . The theoretical errors come from the uncertainties of the form factors, decay constants, Gegenbauer moments and divergence parameters.

Decay mode	$CP$ asymmetries	Branching fractions
$\bar{\kappa} \rho (\rightarrow K^-\pi^+\pi^+\pi^-)$	$-10.03 \pm 5.01$	$1.46 \pm 0.51$
$\bar{\kappa} \omega (\rightarrow K^-\pi^+\pi^+\pi^-)$	$18.34 \pm 5.17$	$4.10 \pm 0.63$
$\bar{K}^*(892)^0 \sigma (\rightarrow K^-\pi^+\pi^+\pi^-)$	$24.33 \pm 9.01$	$0.11 \pm 0.05$
$\bar{K}^*(892)^0 f_0(980) (\rightarrow K^-\pi^+\pi^+\pi^-)$	$-3.85 \pm 1.01$	$9.22 \pm 4.15$
$\bar{K}^*(1410)^0 \sigma (\rightarrow K^-\pi^+\pi^+\pi^-)$	$0.41 \pm 0.53$	$21.18 \pm 6.32$
$\bar{K}^*(1410)^0 f_0(980) (\rightarrow K^-\pi^+\pi^+\pi^-)$	$-2.38 \pm 0.49$	$16.01 \pm 4.04$
$\bar{K}_0^*(1430)^0 \rho (\rightarrow K^-\pi^+\pi^+\pi^-)$	$-7.03 \pm 2.47$	$2.03 \pm 0.41$
$\bar{K}_0^*(1430)^0 \omega (\rightarrow K^-\pi^+\pi^+\pi^-)$	$10.39 \pm 3.42$	$2.55 \pm 0.87$
$\bar{K}^*(1680)^0 \sigma (\rightarrow K^-\pi^+\pi^+\pi^-)$	$8.05 \pm 3.01$	$27.30 \pm 7.05$
$\bar{K}^*(1680)^0 f_0(980) (\rightarrow K^-\pi^+\pi^+\pi^-)$	$-5.03 \pm 0.62$	$19.89 \pm 4.01$

as  $[0, 0.5]$  and  $[0, 2\pi]$ , respectively. Both of them are expected to be tested experimentally in the near future.

#### IV. SUMMARY

In this work, we have revisited the four-body decay  $\bar{B}^0 \rightarrow K^-\pi^+\pi^-\pi^+$  in the framework of the two two-body decays. We have considered more contributions from different resonances. We have also updated the model when dealing with the dynamical function for the  $\rho$  resonance. The most important thing is that we have added the relevant calculations to further test the rationality of the two-quark model for scalar mesons in the two-body decay of the  $\bar{B}^0$  meson. In this analysis, we first calculated the direct  $CP$ -violating asymmetries and branching fractions of the two-body decays  $\bar{B}^0 \rightarrow [K^-\pi^+]_{S/V}[\pi^+\pi^-]_{V/S}$  within the QCDF approach, as listed in Table 2 and Table 3, respectively. From these two tables, we can see that our theoretical results are consistent with the available experimental data for the  $CP$  asymmetries of the  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 f_0(980)$  and  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \omega$  decays and the branching fractions of the  $\bar{B}^0 \rightarrow \bar{K}^*(892)^0 f_0(980)$ ,  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \rho$  and  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \omega$  decays. Because of different structures of the  $\bar{K}_0^*(1430)^0$  meson, our predicted central values for the  $CP$  asymmetries are larger than those given in Ref. [18] for the  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \rho$  and  $\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0 \omega$  decays. It is found that the signs of the  $CP$  asymmetries are negative for the  $\bar{B}^0 \rightarrow \bar{\kappa} \rho$ ,  $\bar{B}^0 \rightarrow \bar{K}^*(1410)^0 f_0(980)$  and  $\bar{B}^0 \rightarrow \bar{K}^*(1680)^0 f_0(980)$  decays and are positive for other decays. The magnitudes of the branching fractions for the two-body decays considered,  $\bar{B}^0 \rightarrow [K^-\pi^+]_{S/V}[\pi^+\pi^-]_{V/S}$ , are of orders  $10^{-7} \sim 10^{-5}$ . Then, under the assumption of the quasi-two-body decay mode, we regard the  $\bar{B}^0 \rightarrow K^-\pi^+\pi^-\pi^+$  decay as happening through  $\bar{B}^0 \rightarrow [K^-\pi^+]_{S/V}[\pi^+\pi^-]_{V/S} \rightarrow K^-\pi^+\pi^-\pi^+$  and calculate the direct  $CP$  asymmetries and branching fractions of all the individual four-body decay channels  $\bar{B}^0 \rightarrow [K^-\pi^+]_{S/V}[\pi^+\pi^-]_{V/S} \rightarrow K^-\pi^+\pi^+\pi^-$ . Their ranges are about  $[-7.03, 24.33] \times 10^{-2}$  and  $[0.11, 27.3] \times 10^{-6}$ , respectively. Finally, considering the contributions from all these decay channels, we obtain the localized integrated  $CP$  asymmetries and the branching fraction of  $\bar{B}^0 \rightarrow K^-\pi^+\pi^-\pi^+$  when  $0.35 < m_{K^-\pi^+} < 2.04 \text{ GeV}$  and  $0 < m_{\pi^-\pi^+} < 1.06 \text{ GeV}$ , which are dominated by the  $\bar{K}_0^*(700)^0$ ,  $\bar{K}^*(892)^0$ ,  $\bar{K}^*(1410)^0$ ,  $\bar{K}_0^*(1430)$  and  $\bar{K}^*(1680)^0$ , and  $f_0(500)$ ,  $\rho^0(770)$ ,  $\omega(782)$  and  $f_0(980)$  resonances, respectively. The predicted results are  $\mathcal{A}_{CP}(\bar{B}^0 \rightarrow K^-\pi^+\pi^+\pi^-) = [-0.365, 0.447]$  and  $\mathcal{B}(\bar{B}^0 \rightarrow K^-\pi^+\pi^+\pi^-) = [6.11, 185.32] \times 10^{-8}$ . In our analysis, the errors come from the uncertainties of the form factors, decay constants, Gegenbauer moments and divergence parameters. These theoretical predictions await testing in future high-precision experiments. If our predictions are confirmed,

the viewpoint that scalars have a  $q\bar{q}$  composition may be supported. However, to exclude other possible structures, more investigations will be needed due to uncertainties from both theory and experiments.

## APPENDIX A: FOUR-BODY DECAY AMPLITUDES

Considering the related weak and strong decays, one can obtain the four-body decay amplitudes of the  $\bar{B}^0 \rightarrow [K^- \pi^+]_{S/V} [\pi^+ \pi^-]_{V/S} \rightarrow K^- \pi^+ \pi^+ \pi^-$  channels as follows:

$$\begin{aligned} \mathcal{M}(\bar{B}^0 \rightarrow \bar{K}_{0i}^{*0} \rho \rightarrow K^- \pi^+ \pi^+ \pi^-) &= \frac{iG_F g_{\bar{K}_{0i}^{*0} K \pi} g_{\rho \pi \pi}}{S_{\bar{K}_{0i}^{*0}} S_\rho} \left[ (P \cdot N) + (L \cdot N) + \frac{1}{m_\rho^2} (L \cdot P + L^2)(L \cdot N) \right] \\ &\times \sum_{p=u,c} \lambda_p^{(s)} \left\{ \left[ \delta_{pu} \alpha_2(\bar{K}_{0i}^{*0} \rho) + \frac{3}{2} \alpha_{3,EW}^p(\bar{K}_{0i}^{*0} \rho) \right] f_\rho m_{\bar{B}^0} p_c F_1^{\bar{B}^0 \bar{K}_{0i}^{*0}}(m_\rho^2) \right. \\ &+ \left[ \alpha_4^p(\rho \bar{K}_{0i}^{*0}) - \frac{1}{2} \alpha_{4,EW}^p(\rho \bar{K}_{0i}^{*0}) \right] \bar{f}_{\bar{K}_{0i}^{*0}} m_{\bar{B}^0} p_c A_0^{\bar{B}^0 \rho}(m_{\bar{K}_{0i}^{*0}}^2) \\ &\left. + \left[ \frac{1}{2} b_3^p(\rho \bar{K}_{0i}^{*0}) - \frac{1}{4} b_{3,EW}^p(\rho \bar{K}_{0i}^{*0}) \right] \frac{f_{\bar{B}^0} f_\rho \bar{f}_{\bar{K}_{0i}^{*0}} m_{\bar{B}^0} p_c}{m_\rho} \right\}, \end{aligned} \quad (A1)$$

$$\begin{aligned} \mathcal{M}(\bar{B}^0 \rightarrow \bar{K}_{0i}^{*0} \omega \rightarrow K^- \pi^+ \pi^+ \pi^-) &= \frac{iG_F g_{\bar{K}_{0i}^{*0} K \pi} g_{\omega \pi \pi}}{S_{\bar{K}_{0i}^{*0}} S_\omega} \left[ (P \cdot N) + (L \cdot N) + \frac{1}{m_\omega^2} (L \cdot P + L^2)(L \cdot N) \right] \\ &\times \sum_{p=u,c} \lambda_p^{(s)} \left\{ \left[ \delta_{pu} \alpha_2(\bar{K}_{0i}^{*0} \omega) + 2\alpha_3^p(\bar{K}_{0i}^{*0} \omega) + \frac{1}{2} \alpha_{3,EW}^p(\bar{K}_{0i}^{*0} \omega) \right] \right. \\ &\times f_\omega m_{\bar{B}^0} p_c F_1^{\bar{B}^0 \bar{K}_{0i}^{*0}}(m_\omega^2) + \left[ \frac{1}{2} \alpha_{4,EW}^p(\omega \bar{K}_{0i}^{*0}) - \alpha_4^p(\omega \bar{K}_{0i}^{*0}) \right] \bar{f}_{\bar{K}_{0i}^{*0}} m_{\bar{B}^0} p_c \\ &\left. \times A_0^{\bar{B}^0 \omega}(m_{\bar{K}_{0i}^{*0}}^2) + \left[ \frac{1}{4} b_{3,EW}^p(\omega \bar{K}_{0i}^{*0}) - \frac{1}{2} b_3^p(\omega \bar{K}_{0i}^{*0}) \right] \frac{f_{\bar{B}^0} f_\omega \bar{f}_{\bar{K}_{0i}^{*0}} m_{\bar{B}^0} p_c}{m_\omega} \right\}, \end{aligned} \quad (A2)$$

and

$$\begin{aligned} \mathcal{M}(\bar{B}^0 \rightarrow \bar{K}_i^{*0} f_{0j} \rightarrow K^- \pi^+ \pi^+ \pi^-) &= -\frac{iG_F g_{\bar{K}_i^{*0} K \pi} g_{f_{0j} \pi \pi}}{S_{\bar{K}_i^{*0}} S_{f_{0j}}} \left[ -(P \cdot Q) - (L \cdot Q) + \frac{1}{m_{\bar{K}_i^{*0}}^2} (P^2 + P \cdot L)(P \cdot Q) \right] \\ &\times \sum_{p=u,c} \lambda_p^{(s)} \left\{ \left[ \delta_{pu} \alpha_2(\bar{K}_i^{*0} f_{0j}) + 2\alpha_3^p(\bar{K}_i^{*0} f_{0j}) + \frac{1}{2} \alpha_{3,EW}^p(\bar{K}_i^{*0} f_{0j}) \right] \right. \\ &\times \bar{f}_{f_{0j}} m_{\bar{B}^0} p_c A_0^{\bar{B}^0 \bar{K}_i^{*0}}(m_{f_{0j}}^2) + \left[ \sqrt{2} \alpha_3^p(\bar{K}_i^{*0} f_{0j}) + \sqrt{2} \alpha_4^p(\bar{K}_i^{*0} f_{0j}) \right. \\ &\left. - \frac{1}{\sqrt{2}} \alpha_{3,EW}^p(\bar{K}_i^{*0} \sigma) - \frac{1}{\sqrt{2}} \alpha_{4,EW}^p(\bar{K}_i^{*0} \sigma) \right] \bar{f}_{\sigma} m_{\bar{B}^0} p_c A_0^{\bar{B}^0 \bar{K}_i^{*0}}(m_{f_{0j}}^2) \\ &+ \left[ \frac{1}{2} \alpha_{4,EW}^p(f_{0j} \bar{K}_i^{*0}) - \alpha_4^p(f_{0j} \bar{K}_i^{*0}) \right] \bar{f}_{\bar{K}_i^{*0}} m_{\bar{B}^0} p_c F_1^{\bar{B}^0 f_{0j}}(m_{\bar{K}_i^{*0}}^2) \\ &+ \left[ \frac{1}{\sqrt{2}} b_3^p(\bar{K}_i^{*0} f_{0j}) - \frac{1}{2\sqrt{2}} b_{3,EW}^p(\bar{K}_i^{*0} f_{0j}) \right] \frac{f_{\bar{B}^0} \bar{f}_{\bar{K}_i^{*0}} \bar{f}_{f_{0j}}^s m_{\bar{B}^0} p_c}{m_{\bar{K}_i^{*0}}} \\ &\left. + \left[ \frac{1}{2} b_3^p(f_{0j} \bar{K}_i^{*0}) - \frac{1}{4} b_{3,EW}^p(f_{0j} \bar{K}_i^{*0}) \right] \frac{f_{\bar{B}^0} \bar{f}_{\bar{K}_i^{*0}} \bar{f}_{f_{0j}}^n m_{\bar{B}^0} p_c}{m_{\bar{K}_i^{*0}}} \right\}. \end{aligned} \quad (A3)$$

## APPENDIX B: DYNAMICAL FUNCTIONS FOR THE CORRESPONDING RESONANCES

### B.1. BUGG MODEL

We adopt the Bugg model [37] to parameterize the  $\sigma$  resonance:

$$T_R(m_{\pi\pi}) = 1/[M^2 - s_{\pi\pi} - g_1^2(s_{\pi\pi}) \frac{s_{\pi\pi} - s_A}{M^2 - s_A} z(s_{\pi\pi}) - iM\Gamma_{\text{tot}}(s_{\pi\pi})], \quad (\text{B1})$$

where  $z(s_{\pi\pi}) = j_1(s_{\pi\pi}) - j_1(M^2)$  with  $j_1(s_{\pi\pi}) = \frac{1}{\pi} [2 + \rho_1 \times \ln(\frac{1 - \rho_1}{1 + \rho_1})]$ ,  $\Gamma_{\text{tot}}(s_{\pi\pi}) = \sum_{i=1}^4 \Gamma_i(s_{\pi\pi})$  with:

$$\begin{aligned} M\Gamma_1(s_{\pi\pi}) &= g_1^2(s_{\pi\pi}) \frac{s_{\pi\pi} - s_A}{M^2 - s_A} \rho_1(s_{\pi\pi}), \\ M\Gamma_2(s_{\pi\pi}) &= 0.6g_1^2(s_{\pi\pi})(s_{\pi\pi}/M^2) \exp(-\alpha|s_{\pi\pi} - 4m_K^2|) \rho_2(s_{\pi\pi}), \\ M\Gamma_3(s_{\pi\pi}) &= 0.2g_1^2(s_{\pi\pi})(s_{\pi\pi}/M^2) \exp(-\alpha|s_{\pi\pi} - 4m_\eta^2|) \rho_3(s_{\pi\pi}), \\ M\Gamma_4(s_{\pi\pi}) &= Mg_4\rho_{4\pi}(s_{\pi\pi})/\rho_{4\pi}(M^2), \end{aligned} \quad (\text{B2})$$

and:

$$\begin{aligned} g_1^2(s_{\pi\pi}) &= M(b_1 + b_2s) \exp[-(s_{\pi\pi} - M^2)/A], \\ \rho_{4\pi}(s_{\pi\pi}) &= 1.0/[1 + \exp(7.082 - 2.845s_{\pi\pi})]. \end{aligned} \quad (\text{B3})$$

In the above two formulas, the relevant parameters are specifically fixed as  $M = 0.953$  GeV,  $g_{4\pi} = 0.011$  GeV,  $s_A = 0.14m_\pi^2$ ,  $A = 2.426$  GeV<sup>2</sup>,  $b_1 = 1.302$  GeV<sup>2</sup>, and  $b^2 = 0.340$  in Ref. [37]. The phase-space factor parameters  $\rho_1$ ,  $\rho_2$  and  $\rho_3$  have the following forms:

$$\rho_i(s_{\pi\pi}) = \sqrt{1 - 4\frac{m_i^2}{s_{\pi\pi}}}, \quad (\text{B4})$$

with  $m_1 = m_\pi$ ,  $m_2 = m_K$  and  $m_3 = m_\eta$ .

### B.2. GOUNARIS-SAKURAI FUNCTION

In the framework of the Gounaris-Sakurai model, which includes an analytic dispersive term, the propagator of the  $\rho^0(770)$  resonance can be expressed as [38]

$$T_R(m_{\pi\pi}) = \frac{1 + D\Gamma_0/m_0}{m_0^2 - s_{\pi\pi} + f(m_{\pi\pi}) - im_0\Gamma(m_{\pi\pi})}, \quad (\text{B5})$$

where  $m_0$  and  $\Gamma_0$  are the the mass and decay width of the  $\rho^0(770)$  meson, respectively, and  $f(m_{\pi\pi})$  is given by

$$f(m_{\pi\pi}) = \Gamma_0 \frac{m_0^2}{q_0^3} \left[ q^2 [h(m_{\pi\pi}) - h(m_0)] + (m_0^2 - m_{\pi\pi}^2) q_0^2 \frac{dh}{dm_{\pi\pi}^2} \Big|_{m_0} \right], \quad (\text{B6})$$

where  $q_0$  is the value of  $q = |\vec{q}|$  when the mass of the  $\pi\pi$  pair satisfies  $m_{\pi\pi} = m_{\rho^0(770)}$ , with:

$$h(m_{\pi\pi}) = \frac{2}{\pi} \frac{q}{m_{\pi\pi}} \log\left(\frac{m_{\pi\pi} + 2q}{2m_\pi}\right), \quad (\text{B7})$$

$$\frac{dh}{dm_{\pi\pi}^2} \Big|_{m_0} = h(m_0) [(8q_0^2)^{-1} - (2m_0^2)^{-1}] + (2\pi m_0^2)^{-1}. \quad (\text{B8})$$

In Eq. (B5), the concrete form of the constant parameter  $D$  is

$$D = \frac{3}{\pi} \frac{m_\pi^2}{q_0^2} \log\left(\frac{m_0 + 2q_0}{2m_\pi}\right) + \frac{m_0}{2\pi q_0} - \frac{m_\pi^2 m_0}{\pi q_0^3}. \quad (\text{B9})$$

### B.3. FLATTÉ MODEL

In Refs. [39, 44], when studying the  $f_0(980)$  resonance, we can use the Flatté model to deal with it, which has the following form:

$$T_R(m_{\pi\pi}) = \frac{1}{m_R^2 - s_{\pi\pi} - im_R(g_{\pi\pi}\rho_{\pi\pi} + g_{KK}F_{KK}^2\rho_{KK})}, \quad (\text{B10})$$

where  $m_R$  is the mass of the  $f_0(980)$  meson, and  $g_{\pi\pi}$  (or  $g_{KK}$ ) is the coupling constant of the  $f_0(980)$  resonance decay to a  $\pi^+\pi^-$  (or  $K^+K^-$ ) pair. Within the Lorentz-invariant phase space, the phase-space  $\rho$  factors are given by:

$$\begin{aligned} \rho_{\pi\pi} &= \frac{2}{3} \sqrt{1 - \frac{4m_{\pi^\pm}^2}{s_{\pi\pi}}} + \frac{1}{3} \sqrt{1 - \frac{4m_{\pi^0}^2}{s_{\pi\pi}}}, \\ \rho_{KK} &= \frac{1}{2} \sqrt{1 - \frac{4m_{K^\pm}^2}{s_{\pi\pi}}} + \frac{1}{2} \sqrt{1 - \frac{4m_{K^0}^2}{s_{\pi\pi}}}. \end{aligned} \quad (\text{B11})$$

Compared to the normal Flatté function, a form factor  $F_{KK} = \exp(-\alpha k^2)$  in Eq. (B10) is introduced above the  $KK$  threshold and serves to reduce the  $\rho_{KK}$  factor as  $s_{\pi\pi}$  increases, where  $k$  is the momentum of each kaon in the  $KK$  rest frame, and  $\alpha = (2.0 \pm 0.25)$  GeV<sup>-2</sup> [44]. This parametrization slightly decreases the  $f_0(980)$  width above the  $KK$  threshold. The parameter  $\alpha$  is fixed to be 2.0 GeV<sup>-2</sup>, which is not very sensitive to the fit.

### B.4. LASS MODEL

Generally, the LASS model can describe the low



mass of the  $K^+\pi^-$  resonance. It has been used widely in theories and experiments [40-42], and has been written as

$$T(m_{K\pi}) = \frac{m_{K\pi}}{|\vec{q}| \cot \delta_B - i|\vec{q}|} + e^{2i\delta_B} \frac{m_0 \Gamma_0 \frac{m_0}{|q_0|}}{m_0^2 - s_{K\pi}^2 - im_0 \Gamma_0 \frac{|\vec{q}|}{m_{K\pi}} \frac{m_0}{|q_0|}}, \quad (\text{B12})$$

where  $m_0$  and  $\Gamma_0$  are the mass and width of the  $K_0^*(1430)$  state, respectively,  $|q_0|$  is the value of  $|\vec{q}|$  when  $m_{K\pi} = m_{K_0^*(1430)}$ ,  $|\vec{q}|$  is the momentum vector of the resonance decay product measured in the resonance rest frame, and  $\cot \delta_B$  has two terms,  $\cot \delta_B = \frac{1}{a|\vec{q}|} + \frac{1}{2}r|\vec{q}|$ , with  $a = (3.1 \pm 1.0) \text{ GeV}^{-1}$  and  $r = (7.0 \pm 2.3) \text{ GeV}^{-1}$  being the scattering length and effective range [42], respectively.

### B.5. RELATIVISTIC BREIT-WIGNER

We adopt the relativistic Breit-Wigner function to describe the distributions of the  $\bar{K}_0^*(700)^0$ ,  $\bar{K}^*(892)^0$ ,  $\bar{K}^*(1410)^0$  and  $\bar{K}^*(1680)^0$  resonances [43],

$$T_R(m_{K\pi}) = \frac{1}{M_R^2 - s_{K\pi} - iM_R \Gamma_{K\pi}} \quad (R = \bar{\kappa}, \bar{K}^*), \quad (\text{B13})$$

with

$$\Gamma_{K\pi} = \Gamma_0^R \left( \frac{p_{K\pi}}{p_R} \right)^{2J+1} \left( \frac{M_R}{m_{K\pi}} \right) F_R^2, \quad (\text{B14})$$

where  $M_R$  and  $\Gamma_0^R$  are the mass and width, respectively,  $m_{K\pi}$  is the invariant mass of the  $K\pi$  pair,  $p_{K\pi}(p_R)$  is the momentum of either daughter in the  $K\pi$  (or  $R$ ) rest frame, and  $F_R$  is the Blatt-Weisskopf centrifugal barrier factor [45], which is listed in Table B1 and depends on a single parameter  $R_r$ , which can be taken as  $R_r = 1.5 \text{ GeV}^{-1}$  [46].

**Table B1.** Summary of the Blatt-Weisskopf penetration form factors.

Spin	$F_R$
0	1
1	$\frac{\sqrt{1 + (R_r p_R)^2}}{\sqrt{1 + (R_r p_{AB})^2}}$

### APPENDIX C: $f_0(500) - f_0(980)$ MIXING

Analogous to the  $\eta - \eta'$  mixing, using a  $2 \times 2$  rotation matrix, the  $f_0(500) - f_0(980)$  mixing can be parameterized as

$$\begin{pmatrix} f_0(980) \\ f_0(500) \end{pmatrix} = \begin{pmatrix} \cos \varphi_m & \sin \varphi_m \\ -\sin \varphi_m & \cos \varphi_m \end{pmatrix} \begin{pmatrix} f_s \\ f_q \end{pmatrix}, \quad (\text{C1})$$

where  $f_s \equiv s\bar{s}$  and  $f_q \equiv \frac{u\bar{u} + d\bar{d}}{\sqrt{2}}$ , and  $\varphi_m$  is the mixing angle, which has been summarized in Refs. [18, 47]. However, based on the measurement by the LHCb collaboration, the range of  $\varphi_m$  is  $|\varphi_m| < 31^\circ$  [48]. In our calculation, we adopt  $|\varphi_m| = 17^\circ$  [18].

### APPENDIX D: THEORETICAL INPUT PARAMETERS

The predictions obtained in the QCDF approach depend on many input parameters. The values of the Wolfenstein parameters are taken from Ref. [49]:  $\bar{\rho} = 0.117 \pm 0.021$ ,  $\bar{\eta} = 0.353 \pm 0.013$ .

For the masses used in the  $\bar{B}^0$  decays, we use the following values, except for those listed in Table 1 (in GeV) [49]:

$$\begin{aligned} m_u = m_d = 0.0035, \quad m_s = 0.119, \quad m_b = 4.2, \\ m_{\pi^\pm} = 0.14, \quad m_{K^-} = 0.494, \quad m_{\bar{B}^0} = 5.28, \end{aligned} \quad (\text{D1})$$

while for the widths we shall use (in units of GeV) [49]:

$$\begin{aligned} \Gamma_{\rho \rightarrow \pi\pi} = 0.149, \quad \Gamma_{\omega \rightarrow \pi\pi} = 0.00013, \quad \Gamma_{\sigma \rightarrow \pi\pi} = 0.3, \\ \Gamma_{f_0(980) \rightarrow \pi\pi} = 0.33, \quad \Gamma_{\bar{K}^*(892)^0 \rightarrow K\pi} = 0.0487, \\ \Gamma_{\bar{K}^*(1410)^0 \rightarrow K\pi} = 0.015, \quad \Gamma_{\bar{K}^*(1680)^0 \rightarrow K\pi} = 0.10, \\ \Gamma_{K_0^*(1430) \rightarrow K\pi} = 0.251. \end{aligned} \quad (\text{D2})$$

The Wilson coefficients used in our calculations are taken from Refs. [50-53]:

$$\begin{aligned} c_1 = -0.3125, \quad c_2 = 1.1502, \quad c_3 = 0.0174, \\ c_4 = -0.0373, \quad c_5 = 0.0104, \quad c_6 = -0.0459, \\ c_7 = -1.050 \times 10^{-5}, \quad c_8 = 3.839 \times 10^{-4}, \\ c_9 = -0.0101, \quad c_{10} = 1.959 \times 10^{-3}. \end{aligned} \quad (\text{D3})$$

The following relevant decay constants (in GeV) are used [17, 54, 55]:

$$\begin{aligned} f_{\pi^\pm} = 0.131, \quad f_{\bar{B}^0} = 0.21 \pm 0.02, \quad f_{K^-} = 0.156 \pm 0.007, \\ \bar{f}_\sigma^s = -0.21 \pm 0.093, \quad \bar{f}_\sigma^u = 0.4829 \pm 0.076, \\ \bar{f}_{\bar{K}} = 0.34 \pm 0.02, \quad f_\rho = 0.216 \pm 0.003, \\ f_\rho^\perp = 0.165 \pm 0.009, \quad f_\omega = 0.187 \pm 0.005, \\ f_\omega^\perp = 0.151 \pm 0.009, \quad f_{\bar{K}^*(892)^0} = 0.22 \pm 0.005, \\ f_{\bar{K}^*(892)^0}^\perp = 0.185 \pm 0.010, \quad \bar{f}_{\bar{K}_0^*(1430)^0} = -0.300 \pm 0.030, \\ \bar{f}_{f_0(980)}^s = 0.325 \pm 0.016, \quad \bar{f}_{f_0(980)}^u = 0.1013 \pm 0.005. \end{aligned} \quad (\text{D4})$$

As for the form factors, we use [17, 33, 55, 56]:

$$\begin{aligned} F_0^{\bar{B}^0 \rightarrow K}(0) &= 0.35 \pm 0.04, & F_0^{\bar{B}^0 \rightarrow \sigma}(0) &= 0.45 \pm 0.15, & F_0^{\bar{B}^0 \rightarrow \kappa}(0) &= 0.3 \pm 0.1, \\ A_0^{\bar{B}^0 \rightarrow \bar{K}^*(892)^0}(0) &= 0.374 \pm 0.034, & F_0^{\bar{B}^0 \rightarrow \pi}(0) &= 0.25 \pm 0.03, & F_0^{\bar{B}^0 \rightarrow \bar{K}_0^*(1430)^0}(0) &= 0.21, \\ A_0^{\bar{B}^0 \rightarrow \bar{K}^*(1410)^0}(0) &= 0.26 \pm 0.0275, & A_0^{\bar{B}^0 \rightarrow \bar{K}^*(1680)^0}(0) &= 0.2154 \pm 0.0281 & A_0^{\bar{B}^0 \rightarrow \rho}(0) &= 0.303 \pm 0.029, \end{aligned} \quad (D5)$$

The values of the Gegenbauer moments at  $\mu = 1$  GeV are taken from [17, 54, 55]:

$$\begin{aligned} \alpha_1^\rho &= 0, & \alpha_2^\rho &= 0.15 \pm 0.07, & \alpha_{1,\perp}^\rho &= 0, & \alpha_{2,\perp}^\rho &= 0.14 \pm 0.06, & \alpha_1^\omega &= 0, & \alpha_2^\omega &= 0.15 \pm 0.07, & \alpha_{1,\perp}^\omega &= 0, & \alpha_{2,\perp}^\omega &= 0.14 \pm 0.06, \\ \alpha_1^{\bar{K}^*(892)^0} &= 0.03 \pm 0.02, & \alpha_{1,\perp}^{\bar{K}^*(892)^0} &= 0.04 \pm 0.03, & \alpha_2^{\bar{K}^*(892)^0} &= 0.11 \pm 0.09, & \alpha_{2,\perp}^{\bar{K}^*(892)^0} &= 0.10 \pm 0.08, \\ B_{1,\sigma}^u &= -0.42 \pm 0.074, & B_{3,\sigma}^u &= -0.58 \pm 0.23, & B_{1,\sigma}^s &= -0.35 \pm 0.061, & B_{3,\sigma}^s &= -0.43 \pm 0.18, \\ B_{1,f_0(980)}^u &= -0.92 \pm 0.08, & B_{3,f_0(980)}^u &= -0.74 \pm 0.064, & B_{1,f_0(980)}^s &= -1 \pm 0.05, & B_{3,f_0(980)}^s &= -0.8 \pm 0.04, \\ B_{1,\bar{\kappa}} &= -0.92 \pm 0.11, & B_{3,\bar{\kappa}} &= 0.15 \pm 0.09, & B_{1,\bar{K}_0^*(1430)^0} &= 0.58 \pm 0.07, & B_{3,\bar{K}_0^*(1430)^0} &= -1.20 \pm 0.08. \end{aligned} \quad (D6)$$

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