"Conserved charges" of the Bondi-Metzner-Sachs algebra in the Brans-Dicke theory*

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Abstract: The asymptotic symmetries in the Brans-Dicke theory are analyzed using Penrose's conformal completion method, which is independent of the coordinate system used. These symmetries, indeed, include supertranslations and Lorentz transformations for an asymptotically flat spacetime. With the Wald-Zoupas formalism, "conserved charges" and fluxes of the Bondi-Metzner-Sachs algebra are computed. The scalar degree of freedom contributes only to the Lorentz boost charge, even though it plays a role in various fluxes. The flux-balance laws are further applied to constrain the displacement memory, spin memory, and center-of-mass memory effects.

Keywords: Bondi-Metzner-Sachs symmetry, conserved charges, Brans-Dicke theory, gravitational waves, memory effects

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I. INTRODUCTION

The detection of gravitational waves (GWs) by LIGO/Virgo collaborations [1-13] confirmed Einstein's prediction based on general relativity (GR) [14, 15]. GWs now constitute a probe into the nature of gravity in the strong-field and high-speed regime. With GWs, several methods have been developed to elucidate whether gravity is described by GR or its alternatives. For example, one may examine whether a GW waveform agrees with GR's prediction precisely; one could also count how many GW polarizations are detected [16, 17]. The GW memory effect is probably the most intriguing phenomenon because of its intimate relation with asymptotic symmetries.

The memory effect and asymptotic symmetries have been investigated by numerous studies in the field of GR [18-24]. This effect usually refers to the permanent change in the relative distance between test particles far away from the source, approximately at the null infinity $\mathscr{I}$, due to the passage of GWs. It is also called the displacement memory. The asymptotic symmetries are diffeomorphisms preserving the geometry of $\mathscr{I}$ and form the Bondi-Metzner-Sachs (BMS) group, which is a semi-direct product of an infinite dimensional commutative supertranslation group and the Lorentz group. The energy flux of a GW induces a transition among degenerate vacua, which are associated with each other by the action of supertranslations. This explains the memory effect in GR [24]. In addition, the spin memory and center-of-mass (CM) memory are related to the angular momentum flux arriving at $\mathscr{I}$ [25, 26].

Alternative theories of gravity also include the memory effect, as discussed in Refs. [27-33]. In particular, Ref. [34] discussed the memory effect and BMS symmetries in the Brans-Dicke theory (BD) [35] using the fully nonlinear equations of motion, as opposed to the post-Newtonian formalism in Refs. [27, 28]. It was discovered that there are also asymptotic symmetries at $\mathscr{I}$ in BD, similar to those in GR. Because of the presence of the plus and cross polarizations in BD, the displacement memory effect also exists in BD and is related to the energy flux and supertranslations. The breathing polarization also causes the displacement memory; it was named S memory by Du and Nishizawa [29]. The angular momentum flux penetrating $\mathscr{I}$ and the Lorentz transformations cause the vacuum transitions in the scalar sector. Utilizing a slightly different coordinate system, Ref. [36] obtained similar results. In the present study, the asymptotic symmetries of an asymptotically flat spacetime in BD were analyzed again using Penrose's conformal completion method [37, 38]. This method is covariant and independent of the coordinate system used.

It is well-known that the existence of symmetries im-

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plies the existence of some conserved charges, according to Noether's theorem. Thus, the BMS symmetries on $\mathscr{I}$ prompt us to search for such quantities defined on $\mathscr{I}$. However, in general, there are GWs passing through $\mathscr{I}$; hence, it is difficult to obtain them, and worse, none of these quantities are actually conserved. These quantities vary along $\mathscr{I}$, and the changes should be provided by some fluxes. All the "conserved charges" and associated fluxes can be calculated using the Hamiltonian formalism devised by Wald and Zoupas [39]. This is a general method applicable to any theory of gravity. The procedure starts by specifying the phase space with certain boundary conditions, computing the presymplectic potential current, θ_{abc}, and symplectic current, ω_{abc}, and obtaining the Neother charge 2-form, $Q_{ab}(\xi)$, associated with an infinitesimal BMS transformation, ξ^a. Then, to find the "conserved charges" and fluxes on $\mathscr{I}$, the asymptotic behavior of the symplectic current is studied so that a second presymplectic potential current, Θ_{abc}, on $\mathscr{I}$ can be constructed, thereby setting the restriction of the symplectic current to $\mathscr{I}$. Finally, the flux density is simply Θ_{abc}, and the variation of the "conserved charge" is $\delta Q_\xi[C]=\oint_C[\delta Q_{ab}(\xi)-\xi^c\theta_{cab}+\xi^c\Theta_{cab}]$, with C denoting a cross section of $\mathscr{I}$. Once a suitable reference spacetime is chosen, the "conserved charge," Q_ξ, can be obtained that satisfies $Q_\xi[C]-Q_\xi[C']=\int_{\mathscr{B}}\Theta_{abc}$, where $\mathscr{B}$ is a patch in $\mathscr{I}$ bounded by C and C'. There are also some ambiguities in choosing $\theta_{abc},\omega_{abc}$, and Θ_{abc}, as well as an issue concerning the choice of the reference spacetime, which have been thoroughly discussed in Ref. [39]. In addition, Refs. [40, 41] nicely reviewed this formalism; it is worthwhile to read both.

In previous studies, Noether charges and currents were also considered for black holes in a more general BD with a variable $\omega(\varphi)$ and a generic potential $V(\phi)$ in both Jordan and Einstein frames [42, 43]. References [44, 45] reported that at least in GR, the BMS group is a subgroup of the so-called conformal Carroll group, whose charges have been computed. One may also add to the action terms that have no influence on the equations of motion, but that may lead to new charges, as described in Refs. [46, 47].

In this study, we applied the Wald-Zoupas formalism to BD, as will be described in the following sections. We start with a brief review of the asymptotically flat spacetime in BD in Sec. II. Then, the asymptotic structure is discussed again in Sec. III within the context of the conformal completion method. Following Refs. [48-51], the radiative modes are identified in Sec. IIIA, and we determine the infinitesimal BMS symmetries in Sec. IIIB. Section IV discusses the "conserved charges" and fluxes. The presymplectic potential current and symplectic current are computed and analyzed in Sec. IVA. Based on these computations, the fluxes and charges are obtained, as presented in the following two subsections: IVB and IVC. Finally, the flux-balance laws are applied to constrain the displacement memory (Sec. VA), spin memory (Sec. VB), and CM memory (Sec. VC) in Sec. V. Section VI presents a short summary. Some technical details have been relegated to Appendices A and B. The abstract index notation is used [52], and the speed of light is set to $c = 1$ in vacuum.

II. BRANS-DICKE THEORY

In this section, we review the asymptotically flat spacetime in the BD based on Ref. [34]. It is well known that the action of the BD is expressed as follows [35]:

$$S=\frac{1}{16\pi G_0}\int \mathrm{d}x^4\sqrt{-g}\left(\varphi R-\frac{\omega}{\varphi}\nabla_a\varphi\nabla^a\varphi\right), \tag{1}$$

where ω is a constant, G_0 is the bare gravitational constant, and the matter action is ignored. Some phenomenological aspects have been summarized in Ref. [34]. The variational principle gives rise to the following equations of motion:

$$R_{ab}-\frac{1}{2}g_{ab}R=\frac{8\pi G_0}{\varphi}\mathcal{T}_{ab}, \tag{2a}$$

$$\nabla_a\nabla^a\varphi=0, \tag{2b}$$

in which $\mathcal{T}_{ab}$ is the effective stress-energy tensor for φ, given by

$$\mathcal{T}_{ab}=\frac{1}{8\pi G_0}\left[\frac{\omega}{\varphi}\left(\nabla_a\varphi\nabla_b\varphi-\frac{1}{2}g_{ab}\nabla_c\varphi\nabla^c\varphi\right)+\nabla_a\nabla_b\varphi-g_{ab}\nabla_c\nabla^c\varphi\right]. \tag{3}$$

Equation (1) is said to be written in Jordan frame.

From a previous study [34], it is known that $\varphi=\varphi_0+O\left(r^{-1}\right)$ in an asymptotically flat spacetime. Consequently, the following conformal transformation can be applied, $\tilde{g}_{ab}=\frac{\varphi}{\varphi_0}g_{ab}$, and set $\frac{\varphi}{\varphi_0}=e^{\tilde{\varphi}}$; then, the action becomes [53]

$$S=\frac{1}{16\pi\tilde{G}}\int\sqrt{-\tilde{g}}\left(\tilde{R}-\frac{2\omega+3}{2}\tilde{\nabla}_a\tilde{\varphi}\tilde{\nabla}^a\tilde{\varphi}\right), \tag{4}$$

where $\tilde{G}=G_0/\varphi_0$. This action is written in Einstein frame. The equations of motion are given by

$$\tilde{R}_{ab}-\frac{1}{2}\tilde{g}_{ab}\tilde{R}=8\pi G_0\tilde{\mathcal{T}}_{ab}, \tag{5a}$$

$$\tilde{\nabla}_a\tilde{\nabla}^a\tilde{\varphi}=0, \tag{5b}$$

with

$$\tilde{T}_{ab} = \frac{2\omega+3}{16\pi G_0}\left(\tilde{\nabla}_a\tilde{\varphi}\tilde{\nabla}_b\tilde{\varphi} - \frac{1}{2}\tilde{g}_{ab}\tilde{\nabla}^c\tilde{\varphi}\tilde{\nabla}_c\tilde{\varphi}\right). \quad (6)$$

In Einstein frame, $\tilde{\varphi}$ is proportional to a canonical scalar field.

As discussed in Ref. [34], Eqs. (2) can be solved using the generalized Bondi-Sachs coordinates $(u, r, x^2 = \theta, x^3 = \phi)$ [54],

$$\begin{aligned} \mathrm{d}s^2 =& e^{2\beta}\frac{V}{r}\mathrm{d}u^2 - 2e^{2\beta}\mathrm{d}u\mathrm{d}r + h_{AB}(\mathrm{d}x^A \\ & - U^A\mathrm{d}u)(\mathrm{d}x^B - U^B\mathrm{d}u), \end{aligned} \quad (7)$$

with $A,B = 2,3$; β, V, U^A, and h_{AB} are six arbitrary functions. Moreover, certain boundary conditions are imposed [54]:

$$\begin{aligned} \beta =& O\left(r^{-1}\right), \quad V = -r + O\left(r^0\right), \\ U^A =& O\left(r^{-2}\right), \end{aligned} \quad (8)$$

along with the determinant condition

$$\det(h_{AB}) = r^4\left(\frac{\varphi_0}{\varphi}\right)^2\sin^2\theta. \quad (9)$$

Then, the series expansions in powers of $1/r$

$$\varphi = \varphi_0 + \frac{\varphi_1}{r} + \frac{\varphi_2}{r^2} + O\left(\frac{1}{r^3}\right), \quad (10a)$$

$$g_{uu} = -1 + \frac{2m + \varphi_1/\varphi_0}{r} + O\left(\frac{1}{r^2}\right), \quad (10b)$$

$$\begin{aligned} g_{ur} =& -1 + \frac{\varphi_1}{\varphi_0 r} + \frac{1}{r^2}\left[\frac{1}{16}\hat{c}^B_A\hat{c}^A_B + \frac{2\omega-5}{8}\left(\frac{\varphi_1}{\varphi_0}\right)^2 \right. \\ & \left. + \frac{\varphi_2}{\varphi_0}\right] + O\left(\frac{1}{r^3}\right), \end{aligned} \quad (10c)$$

$$\begin{aligned} g_{uA} =& \frac{\mathcal{D}_B\hat{c}^B_A}{2} + \frac{2}{3r}\left[N_A + \frac{1}{4}\hat{c}_{AB}\mathcal{D}_C\hat{c}^{BC} \right. \\ & \left. - \frac{\varphi_1}{12\varphi_0}\mathcal{D}_B\hat{c}^B_A\right] + O\left(\frac{1}{r^2}\right), \end{aligned} \quad (10d)$$

$$\begin{aligned} g_{AB} =& r^2\gamma_{AB} + r\left(\hat{c}_{AB} - \gamma_{AB}\frac{\varphi_1}{\varphi_0}\right) + \hat{d}_{AB} \\ & + \gamma_{AB}\left(\frac{1}{4}\hat{c}^D_C\hat{c}^C_D + \frac{\varphi_1^2}{\varphi_0^2} - \frac{\varphi_2}{\varphi_0}\right) + O\left(\frac{1}{r}\right). \end{aligned} \quad (10e)$$

Here, γ_{AB} is the metric on a unit 2-sphere, and $\mathcal{D}_A$ is its compatible covariant derivative; $\varphi_1, \varphi_2, \hat{c}_{AB}$, and $\hat{d}_{AB}$ are expansion coefficients, which are arbitrary functions of (u, x^A). The indices of $\hat{c}_{AB}$ and $\hat{d}_{AB}$ are raised by γ^{AB}, and $\gamma^{AB}\hat{c}_{AB} = \gamma^{AB}\hat{d}_{AB} = 0$. Functions m and N_A of (u, x^A) are called the Bondi mass aspect and the angular momentum aspect, respectively. Einstein's equation (2a) leads to the following evolutions of m and N_A:

$$\dot{m} = -\frac{1}{4}\mathcal{D}_A\mathcal{D}_B N^{AB} - \frac{1}{8}N_{AB}N^{AB} - \frac{2\omega+3}{4}\left(\frac{N}{\varphi_0}\right)^2, \quad (11a)$$

$$\begin{aligned} \dot{N}_A =& \mathcal{D}_A m + \frac{1}{4}(\mathcal{D}_B\mathcal{D}_A\mathcal{D}_C\hat{c}^{BC} - \mathcal{D}_B\mathcal{D}^B\mathcal{D}_C\hat{c}^C_A) \\ & - \frac{1}{16}\mathcal{D}_A(N^B_C\hat{c}^C_B) + \frac{1}{4}N^B_C\mathcal{D}_A\hat{c}^C_B + \frac{1}{4}\mathcal{D}_B(N^C_A\hat{c}^B_C \\ & - \hat{c}^C_A N^B_C) + \frac{2\omega+3}{8\varphi_0^2}(\varphi_1\mathcal{D}_A N - 3N\mathcal{D}_A\varphi_1), \end{aligned} \quad (11b)$$

where $N_{AB} = -\partial\hat{c}_{AB}/\partial u$ is the news tensor, and $N = \partial\varphi_1/\partial u$. Finally, the equation of motion (2b) for φ gives

$$\dot{\varphi}_2 = \frac{\varphi_1 N}{\varphi_0} - \frac{1}{2}\mathcal{D}^2\varphi_1, \quad (11c)$$

with $\mathcal{D}^2 = \mathcal{D}_A\mathcal{D}^A$.

As in GR, the asymptotically flat spacetime in BD also exhibits BMS symmetries. An infinitesimal BMS transformation, ξ^a, is parameterized by $\alpha(x^A)$ and $Y^A(x^B)$ defined on the unit 2-sphere. The transformation generated by α is called a supertranslation, and the one generated by Y^A is a Lorentz transformation. The action on the solution space can be easily computed, for instance, given by [34]

$$\delta_\xi\varphi_1 = fN + \frac{\psi}{2}\varphi_1 + Y^A\mathcal{D}_A\varphi_1, \quad (12a)$$

$$\delta_\xi\hat{c}_{AB} = -fN_{AB} - 2\mathcal{D}_A\mathcal{D}_B f + \gamma_{AB}\mathcal{D}^2 f + \mathcal{L}_Y\hat{c}_{AB} - \frac{\psi}{2}\hat{c}_{AB}, \quad (12b)$$

and thus,

$$\delta_\xi N_{AB} = f\dot{N}_{AB} + \mathcal{L}_Y N_{AB}, \quad (12c)$$

$$\delta_\xi N = f\dot{N} + \psi N + Y^A\mathcal{D}_A N, \quad (12d)$$

where $\psi = \mathcal{D}_A Y^A$. With these expressions, it is possible to discuss the relation between BMS symmetries and gravitational memories. It turns out that the displacement memory effect in the tensor sector is caused by the null energy fluxes, including that of scalar field φ passing through $\mathscr{I}$, which is similar to the one in GR. This memory effect is associated with the supertranslation transformation, which induces the transition among the vacua in the tensor sector. The scalar sector also contains degenerate vacua; hence, the displacement memory occurs in the scalar sector as well. This is caused by the passage of the angular momentum fluxes through $\mathscr{I}$, and a Lorentz transformation induces the transition among the vacua. The spin memory and the CM memory effects are also of interest in GR [25, 26]; they exist in the tensor sector as well. However, neither of them is present in the scalar sector.

In Ref. [34], we did not calculate the "conserved charges" of the asymptotically flat spacetime in the BD. In the present study, we computed them using the covariant phase space formulism devised by Wald and Zoupas [39]. For that purpose, we started with the asymptotic structure of BD, which will be described in the next section.

III. ASYMPTOTIC STRUCTURE AT NULL INFINITY

The asymptotic structure of spacetimes in GR has been discussed and summarized in Refs. [48, 50-52]. In this section, we follow these approaches to study the asymptotic structure at $\mathscr{I}$ in BD. In particular, we utilize the conformal completion, which brings $\mathscr{I}$ to a finite place.

The asymptotically flat spacetime at $\mathscr{I}$ in BD can be defined in the following way. A spacetime (M, g_{ab}) is said to be asymptotically flat at $\mathscr{I}$ in vacuum BD, if an unphysical spacetime (M', g'_{ab}) and a conformal transformation $C: M \to C[M] \subset M'$ exist such that

1. $g'_{ab} = \Omega^2 C^* g_{ab}$ in $C[M]$, for some conformal factor Ω, where C^* is the pullback;
2. $\mathscr{I}$ is the boundary of M in M', and, on it, $\Omega = 0$ and $\nabla'_a \Omega \neq 0$;
3. the topology of $\mathscr{I}$ is $\mathbb{S}^2 \times \mathbb{R}$;
4. equations (2) are satisfied near $\mathscr{I}$.

With this definition, we can elucidate the asymptotic structure at $\mathscr{I}$ for BD. However, Eqs. (2) are very complicated because φ is not a canonical scalar field; hence, the discussion in Jordan frame would be very involved. Therefore, it is preferable to work in Einstein frame, where the equations of motion (5) are simpler, and $\tilde{\varphi}$ is a canonical scalar field modulo a factor. We are allowed to do the conformal completion in Einstein frame, because under the above conformal transformation relating (M, g_{ab}) to (M', g'_{ab}), another unphysical spacetime $(\bar{M}, \bar{g}_{ab})$ can be found with $\bar{g}_{ab} = \Omega^2 \tilde{g}_{ab}$. As a matter of fact, $\bar{g}_{ab} = \dfrac{\varphi}{\varphi_0} g'_{ab}$. In this spacetime, $\mathscr{I}$ is still the boundary of M in $\bar{M}$ with the topology being $\mathbb{S}^2 \times \mathbb{R}$, and, on it, $\Omega = 0$ and $\bar{\nabla}_a \Omega \neq 0$, given that $\bar{\nabla}_a = \nabla'_a = \partial_a$ for a scalar field. However, instead of Eqs. (2), Eqs. (5) must now hold near $\mathscr{I}$.

In the following, we will first identify the radiative modes in BD and then discuss the asymptotic symmetries.

A. Radiative modes

Consequently, in Einstein frame, we can effectively perform the conformal completion for GR with a canonical scalar field. Many results obtained in GR can be carried over directly. For example, the conformal transformation of $\tilde{\varphi}$ is $\tilde{\varphi} = \Omega \bar{\varphi}$ [48]. Then, Einstein's equation (5a) becomes [48, 52]

$$\Omega \bar{S}_{ab} + 2\bar{\nabla}_a \bar{n}_b - \bar{f} \bar{g}_{ab} = \Omega^{-1} \bar{L}_{ab}, \tag{13}$$

where $\bar{S}_{ab} = \bar{R}_{ab} - \bar{g}_{ab}\bar{R}/6$ is the Schouten tensor for $\bar{g}_{ab}$, $\bar{n}_a = \bar{\nabla}_a \Omega$, $\bar{f} = \bar{n}_a \bar{n}^a / \Omega$, and $\bar{L}_{ab}$ is given by

$$\bar{L}_{ab} = \frac{2\omega + 3}{2} \Omega^2 \left(\bar{\mathcal{T}}_{ab} - \frac{1}{6} \bar{g}_{ab} \bar{\mathcal{T}} \right), \tag{14}$$

with $\bar{\mathcal{T}}_{ab} = \bar{\varphi}^2 \bar{n}_a \bar{n}_b + 2\Omega \bar{\varphi} \bar{n}_{(a} \bar{\nabla}_{b)} \bar{\varphi} + \Omega^2 \bar{\nabla}_a \bar{\varphi} \bar{\nabla}_b \bar{\varphi}$, and $\bar{\mathcal{T}} = \bar{g}^{ab} \bar{\mathcal{T}}_{ab}$. Here and below, the index of $\bar{n}_a$ will be raised by $\bar{g}^{ab}$, i.e., $\bar{n}^a = \bar{g}^{ab} \bar{n}_b$. The scalar equation (5b) is

$$\Omega \bar{\nabla}_a \bar{\nabla}^a \bar{\varphi} + \bar{\varphi} \bar{\nabla}_a \bar{n}^a - 2\bar{f} \bar{\varphi} = 0. \tag{15}$$

Although the right hand side of Eq. (13) carries a factor of Ω^{-1}, it is vanishing on $\mathscr{I}$ because $\bar{L}_{ab}$ vanishes faster according to Eq. (14). The finiteness of Eq. (13) implies that $\bar{n}_a \bar{n}^a = 0$, i.e., $\mathscr{I}$ is null, as expected.

In addition, the conformal factor can be freely chosen. A new conformal factor, $\Omega' = \varpi \Omega$, with $\varpi > 0$, is as good as the old one. Under this type of gauge transformation, one can calculate that

$$\bar{g}'_{ab} = \varpi^2 \bar{g}_{ab}, \quad \bar{\varphi}' = \varpi^{-1} \bar{\varphi}, \tag{16a}$$

$$\bar{n}'_a = \varpi \bar{n}_a + \Omega \bar{\nabla}_a \varpi, \tag{16b}$$

$$\bar{f}' = \varpi^{-1} \bar{f} + 2\varpi^{-2} \bar{n}^a \bar{\nabla}_a \varpi + \varpi^{-3} \Omega (\bar{\nabla}^a \varpi) \bar{\nabla}_a \varpi. \tag{16c}$$

A gauge may be chosen such that $\bar{f}' \doteq 0$, which also implies that $\bar{\nabla}'_a \bar{n}'_b \doteq 0$ according to Eqs. (13) and (14). Here, the $\doteq$ symbol means to evaluate the equation on $\mathscr{I}$. This gauge is also called the *Bondi gauge* by analogy [52]. Next, we will fix such a gauge condition and drop all the prime symbols, i.e.,

$$\bar{f}\doteq 0, \quad \bar{\nabla}_a\bar{n}_b\doteq 0. \tag{17}$$

These conditions imply that, on $\mathcal{I}$, the integral curves of $\bar{n}^a$ are the affinely parameterized null geodesics, and the null congruence is free of expansion, shear, and rotation [55]. The first expression in the above equation can be rewritten as $\bar{f}=\Omega\vartheta$ for some function ϑ on $\bar{M}$; hence, $\bar{n}_a\bar{n}^a=\Omega^2\vartheta$. The second expression in Eq. (17) is equivalent to

$$\mathcal{L}_{\bar{n}}\bar{g}_{ab}\doteq 0, \tag{18}$$

that is, $\bar{n}^a$ is a null Killing vector field on $\mathcal{I}$. A further gauge transformation would maintain the Bondi gauge as long as $\mathcal{L}_{\bar{n}}\varpi\doteq 0$.

According to the above discussion, the structure of $\mathcal{I}$ is characterized by $\bar{g}_{ab}$ and $\bar{n}^a$ at the "zeroth order". However, these are spacetime quantities defined on $\bar{M}$. One may prefer the intrinsic ones to $\mathcal{I}$; hence, let γ_{ab} be the restriction of $\bar{g}_{ab}$ to $\mathcal{I}$. Note that $\bar{n}^a$ is tangent to $\mathcal{I}$; therefore, it is naturally intrinsic to $\mathcal{I}$. Then, following the terminology of Ref. [51], the zeroth-order structure of $\mathcal{I}$ is the pair $(\gamma_{ab},\bar{n}^a)$. This structure is universal, i.e., it is shared by any asymptotically flat spacetime at $\mathcal{I}$ [48]. Given that $\gamma_{ab}\bar{n}^b$ is the restriction of $\bar{g}_{ab}\bar{n}^b=\bar{\nabla}_a\Omega$ to $\mathcal{I}$, $\gamma_{ab}\bar{n}^b=0$; hence, γ_{ab} is degenerate. This is consistent with the fact that $\mathcal{I}$ is null.

The first-order structure is covariant derivative $\mathcal{D}_a$, induced on $\mathcal{I}$ by $\bar{\nabla}_a$ [51]. It satisfies

$$\mathcal{D}_a\gamma_{bc}=0, \quad \mathcal{D}_a\bar{n}^b=0. \tag{19}$$

Some of the higher-order structures require the following quantities from $\mathcal{D}_a$. The curvature tensor, $\mathcal{R}_{abc}{}^d$, can be defined for $\mathcal{D}_a$ as follows. Let v_a be a covector field on $\mathcal{I}$; then, one obtains

$$\mathcal{D}_{[a}\mathcal{D}_{b]}v_c=\frac{1}{2}\mathcal{R}_{abc}{}^d v_d. \tag{20}$$

Define $\mathcal{R}_{abcd}=\gamma_{de}\mathcal{R}_{abc}{}^e$; then, $\mathcal{R}_{ab}=\gamma^{cd}\mathcal{R}_{acbd}$, and $\mathcal{R}=\gamma^{ab}\mathcal{R}_{ab}$. Here, γ^{ab} is "inverse" to γ_{ab} such that $\gamma_{ac}\gamma_{bd}\gamma^{cd}=\gamma_{ab}$. By counting the number of algebraically independent components of $\mathcal{R}_{abc}{}^d$, one may prove that there is a tensor field $\mathcal{S}_a{}^b$, which satisfies [49]

$$\mathcal{S}_b{}^a\bar{n}^b=(\mathcal{S}_b{}^b-\mathcal{R})\bar{n}^a, \tag{21}$$

such that

$$\mathcal{R}_{abc}{}^d=\gamma_{c[a}\mathcal{S}_{b]}{}^d+\mathcal{S}_{c[a}\delta_{b]}{}^d, \tag{22}$$

with $\mathcal{S}_{ab}=\gamma_{bc}\mathcal{S}_a{}^c$. Thus, $\mathcal{R}_{abc}{}^d$ can be equivalently represented by $\mathcal{S}_a{}^b$. In fact, $\mathcal{S}_a{}^b$ is nothing but the restriction of $\bar{S}_a{}^b$ to $\mathcal{I}$.

Owing to the topology of $\mathcal{I}$, there exists a unique symmetric tensor field ρ_{ab} on $\mathcal{I}$ with the following properties [48]:

$$\rho_{ab}\bar{n}^b=0, \quad \gamma^{ab}\rho_{ab}=\mathcal{R}, \quad \mathcal{D}_{[a}\rho_{b]c}=0. \tag{23}$$

We can now introduce the second-order structure, i.e., news tensor N_{ab}, defined by

$$N_{ab}=\mathcal{S}_{ab}-\rho_{ab}. \tag{24}$$

It is transverse, i.e., $N_{ab}\bar{n}^b=0$ and traceless, i.e., $\gamma^{ab}N_{ab}=0$. Its nonvanishing nature indicates the presence of the tensor GW [34]. There also exists scalar field $\bar{\varphi}$ on $\mathcal{I}$. Its Lie-drag, $\bar{N}\equiv\mathcal{L}_{\bar{n}}\bar{\varphi}=N/\varphi_0$, along the integral curves of $\bar{n}^a$ signals the existence of the scalar GW penetrating $\mathcal{I}$; hence, $\bar{N}$ (or, equivalently N) also belongs to the second-order structure of $\mathcal{I}$.

Finally, the third-order structure can be introduced. According to [52],

$$\bar{R}_{abcd}=\bar{C}_{abcd}+\bar{g}_{a[c}\bar{S}_{d]b}-\bar{g}_{b[c}\bar{S}_{d]a}. \tag{25}$$

In the above, we have observed the roles that $\bar{S}_{ab}$ plays in the asymptotic structure. Now, consider $\bar{C}_{abcd}$. Although $\bar{L}_{ab}$ is $O(\Omega^2)$ near $\mathcal{I}$, $\bar{C}_{abcd}$ still vanishes on $\mathcal{I}$ according to Ref. [48]. Thus, the following two quantities are introduced:

$$K^{ab}=-4\Omega^{-1}\bar{C}^{acbd}\bar{n}_c\bar{n}_d, \quad {}^*K^{ab}=-4\Omega^{-1}{}^*\bar{C}^{acbd}\bar{n}_c\bar{n}_d, \tag{26}$$

where ${}^*\bar{C}^{acbd}$ is the Hodge dual [52]. Given that $K^{ab}\bar{n}_b={}^*K^{ab}\bar{n}_b=0$, they are naturally intrinsic to $\mathcal{I}$. They are symmetric and traceless, i.e., $\gamma_{ab}K^{ab}=\gamma_{ab}{}^*K^{ab}=0$. They are also dual to each other in the following sense:

$$\gamma_{ac}K^{cb}=-\bar{\epsilon}_{acd}\bar{n}^{d*}K^{cb}, \quad \gamma_{ac}{}^*K^{cb}=\bar{\epsilon}_{acd}\bar{n}^dK^{cb}, \tag{27}$$

where $\bar{\epsilon}_{abc}$ is the volume element on $\mathcal{I}$, induced from $\bar{\epsilon}_{abcd}(=4\bar{\epsilon}_{[abc}\bar{n}_{d]})$. Following the argument in Ref. [48], it can be shown that

$$\mathcal{D}_{[a}\mathcal{S}_{b]}{}^c=\frac{1}{4}\bar{\epsilon}_{abd}{}^*K^{dc}, \tag{28a}$$

$$\mathcal{D}_bK^{ab}=\frac{2(2\omega+3)}{3}\left[\bar{\varphi}\mathcal{L}_{\bar{n}}\bar{N}-2\bar{N}^2\right]\bar{n}^a, \tag{28b}$$

$$\mathcal{D}_b{}^*K^{ab}=0. \tag{28c}$$

${}^{*}K^{ab}$ is the third-order structure on $\mathscr{I}$.

The gauge transformations of the above structures are given by [50]

$$\gamma'_{ab} = \varpi^2 \gamma_{ab}, \quad \bar{n}'^a = \varpi^{-1} \bar{n}^a, \tag{29a}$$

$$\mathscr{D}'_a v_b = \mathscr{D}_a v_b - 2\varpi^{-1} v_{(a} \mathscr{D}_{b)} \varpi + \varpi^{-1} \gamma_{ab} \varpi^c v_c, \tag{29b}$$

$$N'_{ab} = N_{ab}, \quad \bar{N}' = \varpi^{-2} \bar{N}, \tag{29c}$$

$$K'^{ab} = \varpi^{-5} K^{ab}, \quad {}^{*}K'^{ab} = \varpi^{-5}\, {}^{*}K^{ab}, \tag{29d}$$

where ϖ^a is the restriction of $\bar{\nabla}^a \varpi$ to $\mathscr{I}$. Now, consider a special gauge transformation with $\varpi = 1$ on $\mathscr{I}$. Then, the first-order structure, $\mathscr{D}_a$, changes according to

$$\mathscr{D}'_a v_b = \mathscr{D}_a v_b + \kappa \gamma_{ab} \bar{n}^c v_c, \tag{30}$$

where $\bar{\nabla}^a \varpi \doteq \kappa \bar{n}^a$ for a function κ on $\mathscr{I}$, but the remaining structures stay the same. Therefore, although the zeroth-order structure does not change, i.e., $(\gamma'_{ab}, \bar{n}'^a) = (\gamma_{ab}, \bar{n}^a)$, covariant derivatives $\mathscr{D}'_a$ and $\mathscr{D}_a$ can be different. This suggests the introduction of the concept of an equivalence class $\{\mathscr{D}_a\}$, which is the set of covariant derivatives associated with each other via Eq. (30) [49]. The radiative degrees of freedom are encoded in $\{\mathscr{D}_a\}$. Now, let $\bar{g}_{ab}$ and $\bar{g}'_{ab}$ be two metric fields in unphysical spacetime $\bar{M}$, and their covariant derivatives be $\bar{\nabla}_a$ and $\bar{\nabla}'_a$, respectively. Further, let $\{\mathscr{D}_a\}$ and $\{\mathscr{D}'_a\}$ be two equivalence classes of the induced covariant derivatives from $\bar{\nabla}_a$ and $\bar{\nabla}'_a$, respectively. Their difference is completely characterized by a symmetric tensor field σ_{ab} with $\sigma_{ab}\bar{n}^b = 0$ and $\gamma^{ab}\sigma_{ab} = 0$. If one introduces a covector field, ℓ_a, on $\mathscr{I}$ such that $\bar{n}^a \ell_a = 1$, it can be shown that σ_{ab} is the traceless part of the following tensor [49]:

$$\Sigma_{ab} = (\mathscr{D}'_a - \mathscr{D}_a)\ell_b, \tag{31}$$

where $\mathscr{D}'_a$ and $\mathscr{D}_a$ are two representatives of $\{\mathscr{D}'_a\}$ and $\{\mathscr{D}_a\}$, respectively. One can easily verify that σ_{ab} has two independent components, and they represent the radiative degrees of freedom in the tensor sector. In fact, by replacing v_b with ℓ_b, substituting Eq. (22) into Eq. (20), and contracting both sides of the result by $\bar{n}^b$, the following is obtained:

$$N_{ab} = -2\mathcal{L}_{\bar{n}} \sigma_{ab}. \tag{32}$$

Here, to derive this relation, a trivial derivative, $\mathring{\mathscr{D}}_a$, with $\mathring{\mathscr{D}}_a \ell_b = 0$, is applied, setting $\Sigma_{ab} = (\mathscr{D}_a - \mathring{\mathscr{D}}_a)\ell_b = \mathscr{D}_a \ell_b$. In this sense, σ_{ab} is the shear of a null congruence with tangent vector fields $\ell^a = \bar{g}^{ab}\ell_b$ on $\mathscr{I}$.

The metric solution presented in the previous section is actually in the Bondi gauge. To demonstrate this, first, the solution is transformed to the one in Einstein frame, and then, a conformal transformation is performed with $\Omega = 1/r$. In the coordinates $\{u, \Omega, \theta, \phi\}$, the metric is

$$\begin{aligned} \mathrm{d}\bar{s}^2 = &\left[-\Omega^2 + 2\Omega^3 m + O\left(\Omega^4\right)\right]\mathrm{d}u^2 + 2\left[1 + O\left(\Omega^2\right)\right]\mathrm{d}u\mathrm{d}\Omega \\ &+ \left[\Omega^2 \mathscr{D}_B \hat{c}^B_A + O\left(\Omega^3\right)\right]\mathrm{d}u\mathrm{d}x^A + \Bigg[\gamma_{AB} + \Omega \hat{c}_{AB} \\ &+ \Omega^2 \left(\hat{d}_{AB} + \frac{\varphi_1}{\varphi_0}\hat{c}_{AB} + \frac{\gamma_{AB}}{4}\hat{c}^D_C \hat{c}^C_D\right) + O\left(\Omega^3\right)\Bigg]\mathrm{d}x^A \mathrm{d}x^B, \end{aligned} \tag{33}$$

and the scalar field is

$$\bar{\varphi} = \frac{\varphi_1}{\varphi_0} + \Omega\left(\frac{\varphi_2}{\varphi_0} - \frac{\varphi_1^2}{2\varphi_0^2}\right) + O\left(\Omega^2\right). \tag{34}$$

With Eq. (33), one can verify the validity of the Bondi gauge condition (17). Note also that $\bar{n}_a = \bar{\nabla}_a \Omega$; hence, $\bar{n}^a = (\partial_u)^a$. Finally, by setting $\ell_a = (\mathrm{d}u)_a + O(\Omega)$, we obtain $\sigma_{AB} = \hat{c}_{AB}/2$ and $N_{AB} = -\partial_u \hat{c}_{AB}$.

B. BMS generators

As discussed in Ref. [56], an infinitesimal asymptotic symmetry, ξ^a, induces the following variation in $\tilde{g}_{ab}$:

$$\begin{aligned} \Omega^2 \delta_\xi \tilde{g}_{ab} &= \Omega^2 \mathcal{L}_\xi \tilde{g}_{ab} \\ &= \mathcal{L}_\xi \bar{g}_{ab} - 2\bar{K}\bar{g}_{ab} = 2\Omega \bar{X}_{ab}, \end{aligned} \tag{35}$$

for a smooth scalar field $\bar{K} = \xi^a \bar{n}_a / \Omega$ [57] and a smooth tensor field $\bar{X}_{ab}$ in $\bar{M}$. The well-posedness of this expression requires that $\xi^a \bar{n}_a \doteq 0$ for ξ^a to be tangent to $\mathscr{I}$. This equation can be rewritten as

$$\mathcal{L}_\xi \bar{g}_{ab} = 2(\bar{K}\bar{g}_{ab} + \Omega \bar{X}_{ab}). \tag{36}$$

By examining $(\mathcal{L}_\xi \mathcal{L}_{\bar{n}} - \mathcal{L}_{\bar{n}} \mathcal{L}_\xi - \mathcal{L}_{[\xi,\bar{n}]})\bar{g}_{ab} = 0$ with the conformal Einstein's equation (13), the following is obtained:

$$\begin{aligned} &-\bar{\nabla}_a \bar{\nabla}_b \bar{K} + 4\bar{n}_{(a}\bar{X}_{b)} + 2\Omega \bar{\nabla}_{(a}\bar{X}_{b)} - \bar{g}_{ab}\bar{n}_c \bar{X}^c \\ &-\frac{1}{2}\mathcal{L}_\xi(\bar{S}_{ab} - \Omega^{-2}\bar{L}_{ab}) - \mathcal{L}_{\bar{n}}\bar{X}_{ab} = 0, \end{aligned} \tag{37}$$

where $\bar{X}_a = \Omega^{-1}\bar{X}_{ab}\bar{n}^b$ and $\bar{X} = \bar{g}^{ab}\bar{X}_{ab}$. Again, the well-posedness of Eq. (37) leads to the fact that $\bar{X}_{ab}$ is transverse to $\bar{n}^a$ so that $\bar{X}_a$ is finite on $\mathscr{I}$. The action of ξ^a on $\bar{n}^a$ can be easily calculated as

$$\mathcal{L}_\xi \bar{n}^a = -\bar{K}\bar{n}^a + \Omega\bar{\nabla}^a\bar{K} - 2\Omega^2\bar{X}^a. \tag{38}$$

Contracting both sides by $\bar{n}_a$ gives

$$\mathcal{L}_{\bar{n}}\bar{K} = \frac{1}{2}(\mathcal{L}_\xi \bar{f} - \bar{K}\bar{f}), \tag{39}$$

without imposing the Bondi gauge condition. What about the action of ξ^a on $\bar{\varphi}$? First, one can perform $\delta_\xi\tilde{\varphi} = \mathcal{L}_\xi\tilde{\varphi} = \Omega(\mathcal{L}_\xi\bar{\varphi} + \bar{K}\bar{\varphi})$. Second, according to the definition of the asymptotic symmetry in Ref. [34], the transformed "physical" $\tilde{\varphi}$ is allowed to decay as $1/r \sim \Omega$. Therefore, we have that

$$\delta_\xi\bar{\varphi} = \mathcal{L}_\xi\bar{\varphi} + \bar{K}\bar{\varphi}, \tag{40}$$

which actually agrees with the transformation property of φ_1 in Ref. [34]. Indeed, $\bar{\varphi} \doteq \varphi_1/\varphi_0$.

Now, we know how a BMS generator acts on $\bar{g}_{ab}$ and $\bar{n}^a$ in unphysical spacetime $\bar{M}$ according to Eqs. (36) and (38). By restricting these equations to $\mathscr{I}$, we obtain [48]

$$\mathcal{L}_\xi\gamma_{ab} = 2\bar{K}\gamma_{ab}, \quad \mathcal{L}_\xi\bar{n}^a = -\bar{K}\bar{n}^a. \tag{41}$$

Eq. (39) implies that $\mathcal{L}_\xi\bar{K} = 0$ on $\mathscr{I}$ in the Bondi gauge. Therefore, ξ^a is a conformal Killing vector field on $\mathscr{I}$. Among the BMS generators, there are infinitesimal supertranslations given by [56]

$$\xi^a = \alpha\bar{n}^a - \Omega\bar{\nabla}^a\alpha + \Omega^2 u^a, \tag{42}$$

where α is a smooth function, and u^a is a smooth vector field on $\bar{M}$. Moreover, α should satisfy $\mathcal{L}_{\bar{n}}\alpha = \Omega\varsigma_\alpha$ for some smooth function ς_α on $\bar{M}$. One can show that

$$\bar{K} = \Omega(\alpha\vartheta - \varsigma_\alpha + \varrho), \tag{43a}$$

$$\bar{X}_{ab} = -\bar{\nabla}_a\bar{\nabla}_b\alpha - \frac{1}{2}(\alpha\vartheta - 2\varsigma_\alpha + 2\varrho)\bar{g}_{ab} - \frac{\alpha}{2}(\bar{S}_{ab} - \Omega^{-2}\bar{L}_{ab}) + 2\bar{n}_{(a}u_{b)} + \Omega\bar{\nabla}_{(a}u_{b)}, \tag{43b}$$

$$\bar{X}_a = \frac{1}{2}\bar{\nabla}_a(\alpha\vartheta - 2\varsigma_\alpha + \varrho) - \frac{1}{2}(\bar{S}_{ab} - \Omega^{-2}\bar{L}_{ab})\bar{\nabla}^b\alpha + \frac{1}{2}\bar{n}^b\bar{\nabla}_b u_a + \frac{\Omega}{4}[3\vartheta u_a + (\bar{S}_{ab} - \Omega^{-2}\bar{L}_{ab})u^b], \tag{43c}$$

$$\bar{X} = -\bar{\nabla}^2\alpha - 2(\alpha\vartheta - 2\varsigma_\alpha + \varrho) - \alpha\left(\frac{\bar{R}}{3} - \Omega^{-2}\bar{L}\right) + \Omega\bar{\nabla}_a u^a, \tag{43d}$$

where $\varrho = u^a\bar{n}_a$ and $u_a = \bar{g}_{ab}u^b$. On $\mathscr{I}$, $\xi^a \doteq \alpha\bar{n}^a$ and $\mathcal{L}_{\bar{n}}\alpha \doteq 0$. Given that $\bar{K} \doteq 0$, $\alpha\bar{n}^a$ is a Killing vector field.

For a generic BMS generator ξ^a, let us directly consider its restriction to $\mathscr{I}$. It satisfies the following conditions [48]:

$$\bar{n}^a\xi_a = 0,$$
$$\mathcal{D}_{(a}\xi_{b)} = \bar{K}\gamma_{ab},$$
$$\mathcal{L}_{\bar{n}}\xi_a = 0, \tag{44}$$

with $\xi_a = \bar{g}_{ab}\xi^b$. The first expression is because ξ^a is tangent to $\mathscr{I}$. The second and third are basically Eqs. (41). Conversely, if a covector field, ξ_a, satisfies Eqs. (44), one can find a BMS generator, ξ^a, that satisfies Eqs. (41) and $\xi_a = \gamma_{ab}\xi^b$. Owing to the degeneracy of γ_{ab}, ξ^a is not unique; one can add to it an arbitrary supertranslation generator $\alpha\bar{n}^a$ without modifying ξ_a. If ξ^a and ξ'^a are deemed equivalent, as long as they only differ by a supertranslation, the solutions to Eqs. (44) belong to an equivalence class. The set of such equivalence classes is isomorphic to the Lorentz algebra owing to the topology of $\mathscr{I}$. Given that this set is also the quotient algebra of the BMS algebra modulo the supertranslation algebra, one verifies that the BMS algebra is, indeed, the semi-direct sum of the supertranslation algebra and the Lorentz algebra.

Once a foliation of $\mathscr{I}$ is chosen, ξ^a can be uniquely decomposed. This foliation can be obtained by starting with a reference leaf C_0, a cross section, at some retarded time u_0 and then Lie-dragging it along the integral curves of $\bar{n}^a$ to an arbitrary C. One can further let the normal to C be ℓ_a; then, ξ^a is decomposed according to

$$\xi^a \doteq \left(\alpha + \frac{u}{2}\mathcal{D}\cdot Y\right)\bar{n}^a + Y^a, \tag{45}$$

where $\mathcal{D}\cdot Y = \gamma^{ab}\mathcal{D}_a Y_b = 2\bar{K}$ on $\mathscr{I}$. Here, component Y^a is tangent to C, generating the infinitesimal Lorentz transformation and leaving C invariant, but $\alpha\bar{n}^a$, i.e., an infinitesimal supertranslation, induces a one-parameter group of diffeomorphisms that changes the foliation for a general α. It can be demonstrated that $\mathcal{L}_Y\gamma_{ab} = 2\bar{K}\gamma_{ab}$ but $\mathcal{L}_Y\bar{n}^a = 0$, which means that Y^a itself is not a BMS generator. This explains the presence of the term proportional to u, which, along with Y^a, is a genuine BMS generator. One should also know how a BMS generator transforms $\mathcal{D}_a$ to calculate the flux and the "conserved charge". To this end, it should first be noted that for any ξ^a and v_a,

$$(\mathcal{L}_\xi\mathcal{D}_a - \mathcal{D}_a\mathcal{L}_\xi)v_b = (\xi^d\mathcal{R}_{dab}{}^c - \mathcal{D}_a\mathcal{D}_b\xi^c)v_c. \tag{46}$$

Consequently, for a supertranslation $\xi^a \doteq \alpha\bar{n}^a$, the following useful result can be obtained:

$$\begin{aligned}\delta_{\alpha\bar{n}}\Sigma_{ab} =&(\mathcal{L}_{\alpha\bar{n}}\mathcal{D}_a-\mathcal{D}_a\mathcal{L}_{\alpha\bar{n}})\ell_b\\ =&-\mathcal{D}_a\mathcal{D}_b\alpha-\frac{\alpha}{2}N_{ab}+\kappa'\gamma_{ab},\end{aligned}\quad(47)$$

where Eq. (22) has been used, and κ' is a function on $\mathcal{I}$ that is irrelevant for the coming discussion. Then, the Lorentz transformation also transforms $\mathcal{D}_a$ as follows [58]:

$$\begin{aligned}\delta_Y\Sigma_{ab} =&(\mathcal{L}_\xi\mathcal{D}_a-\mathcal{D}_a\mathcal{L}_\xi)\ell_b\\ =&-\frac{u}{2}\mathcal{D}_a\mathcal{D}_b(\mathcal{D}\cdot Y)-\frac{1}{2}\sigma_{ab}\mathcal{D}\cdot Y\\ &+\mathcal{L}_Y\sigma_{ab}-\frac{u}{4}(\mathcal{D}\cdot Y)N_{ab}-\ell_{(a}\mathcal{D}_{b)}(\mathcal{D}\cdot Y)\\ &+\frac{1}{4}\gamma_{ab}\ell\mathcal{D}\cdot Y+\frac{1}{2}\gamma_{ab}Y^c\mathcal{D}_c\ell,\end{aligned}\quad(48)$$

where $\ell=\gamma^{ab}\mathcal{D}_a\ell_b$. The traceless parts of Eqs. (47) and (48) are $\delta_{\alpha\bar{n}}\sigma_{ab}$ and $\delta_Y\sigma_{ab}$, respectively. In the end, ξ^a induces the variation of $\bar{g}_{ab}$ according to Eq. (36); hence, its connection, $\bar{\Gamma}^c{}_{ab}$, also changes, which is given by

$$\begin{aligned}(\mathcal{L}_\xi\bar{\nabla}_a-\bar{\nabla}_a\mathcal{L}_\xi)v_b=&-v_c\delta_\xi\bar{\Gamma}^c_{ab}=v_c[\bar{n}^c\bar{X}_{ab}-2\bar{n}_{(a}\bar{X}_{b)}{}^c\\ &-2\delta^c_{(a}\bar{\nabla}_{b)}\bar{K}+\bar{g}_{ab}\bar{\nabla}^c\bar{K}+O(\Omega)].\end{aligned}\quad(49)$$

Now, take the restriction to $\mathcal{I}$, and set $v_a=\ell_a$, thereby obtaining

$$\bar{X}_{ab}=(\mathcal{L}_\xi\mathcal{D}_a-\mathcal{D}_a\mathcal{L}_\xi)\ell_b+2\ell_{(a}\mathcal{D}_{b)}\bar{K}-\gamma_{ab}\ell_c\bar{\nabla}^c\bar{K},\quad(50)$$

where $\bar{\nabla}^c$ is not replaced by $\mathcal{D}_a$ in the last term, because this term is useless in the following calculation.

IV. “CONSERVED CHARGES” AND FLUXES

A. (Pre)symplectic currents

Following Ref. [39], we can start with the variation of the action described by Eq. (1),

$$\begin{aligned}\delta S=&\frac{1}{16\pi G_0}\int d^4\sqrt{-g}(E_{ab}\delta g^{ab}+E_\varphi\delta\varphi)\\ &+\int d^4x\sqrt{-g}\nabla_a\theta^a,\end{aligned}\quad(51)$$

where E_{ab} is Einstein's equation, which takes a different form to that of Eq. (2a) but is equivalent, and $E_\varphi=R+\frac{2\omega}{\varphi}\nabla_a\nabla^a\varphi-\frac{\omega}{\varphi^2}\nabla_a\varphi\nabla^a\varphi$. The last term above is a surface term, where θ^a, or its Hodge dual, is the so-called presymplectic potential current, given by

$$\begin{aligned}\theta_{abc}(\delta g,\delta\varphi)=&\frac{1}{16\pi G_0}\epsilon_{dabc}\Big[\varphi g^{de}g^{fh}(\nabla_f\delta g_{eh}-\nabla_e\delta g_{fh})\\ &+g^{de}g^{fh}(\delta g_{fh}\nabla_e\varphi-\delta g_{eh}\nabla_f\varphi)-\frac{2\omega}{\varphi}\delta\varphi\nabla^d\varphi\Big].\end{aligned}\quad(52)$$

With θ_{abc}, the symplectic current is given by

$$\begin{aligned}\omega_{abc}=&\delta\theta_{abc}(\delta' g,\delta'\varphi)-\delta'\theta_{abc}(\delta g,\delta\varphi)\\ =&\frac{1}{16\pi G_0}\epsilon_{dabc}\varphi w^d+\frac{1}{16\pi G_0}\epsilon_{dabc}\\ &\times\Big[2g^{d[e}g^{f]h}(\delta\varphi\nabla_f\delta' g_{eh}-\delta' g_{eh}\nabla_f\delta\varphi)\\ &+(g^{dp}g^{eq}g^{fh}+g^{de}g^{fp}g^{qh})\delta g_{pq}\delta' g_{eh}\nabla_f\varphi\\ &+\frac{1}{2}g^{fh}\delta' g_{fh}\delta g^{de}\nabla_e\varphi-\frac{2\omega}{\varphi}\delta'\varphi\Big(\delta g^{de}\nabla_e\varphi+\nabla^d\delta\varphi\\ &+\frac{1}{2}g^{ef}\delta g_{ef}\delta'\varphi\nabla^d\varphi\Big)-\langle\delta\leftrightarrow\delta'\rangle\Big],\end{aligned}\quad(53)$$

where $\langle\delta\leftrightarrow\delta'\rangle$ represents the terms obtained by switching δ and δ' within the remaining terms in the square brackets, and w^a has been calculated in Ref. [39] for GR, i.e,

$$\begin{aligned}w^a=&(g^{a[e}g^{d]c}g^{bf}+g^{ae}g^{b[f}g^{c]d}+g^{a[d}g^{b]c}g^{ef})\\ &\times(\delta' g_{bc}\nabla_d\delta g_{ef}-\delta g_{bc}\nabla_d\delta' g_{ef}).\end{aligned}\quad(54)$$

However, the above results were computed in Jordan frame, where Eqs. (2) are complicated, and the calculation of the “conserved charges” and fluxes is very likely to be involved.

To resolve the complication, all quantities in Eqs. (52) and (53) should be replaced by the corresponding ones in Einstein frame. One may also directly calculate the presymplectic potential current and the symplectic current using the action described in Eq. (4) in Einstein frame, which are

$$\begin{aligned}\tilde{\theta}_{abc}=&\frac{1}{16\pi\tilde{G}}\tilde{\epsilon}_{dabc}[\tilde{g}^{de}\tilde{g}^{fh}(\tilde{\nabla}_f\delta\tilde{g}_{eh}-\tilde{\nabla}_e\delta\tilde{g}_{fh})\\ &-(2\omega+3)\delta\tilde{\varphi}\tilde{\nabla}^d\tilde{\varphi}],\end{aligned}\quad(55a)$$

$$\begin{aligned}\tilde{\omega}_{abc}=&\frac{1}{16\pi\tilde{G}}\tilde{\epsilon}_{dabc}\tilde{w}^d-\frac{2\omega+3}{16\pi\tilde{G}}\tilde{\epsilon}_{dabc}\Big[\delta'\tilde{\varphi}\tilde{\nabla}^d\delta\tilde{\varphi}\\ &+\delta'\tilde{\varphi}\delta\tilde{g}^{de}\tilde{\nabla}_e\tilde{\varphi}+\frac{1}{2}\tilde{g}^{ef}\delta\tilde{g}_{ef}\delta'\tilde{\varphi}\tilde{\nabla}^d\tilde{\varphi}-\langle\delta\leftrightarrow\delta'\rangle\Big],\end{aligned}\quad(55b)$$

where $\tilde{w}^a$ takes similar forms to w^a in Eq. (54), with all g's and ∇'s replaced by $\tilde{g}$ and $\tilde{\nabla}$, respectively. However, a careful examination reveals that these two methods give distinct presymplectic potential currents,

$$\theta_{abc}(\delta g,\delta\varphi)=\tilde{\theta}_{abc}(\delta\tilde{g},\delta\tilde{\varphi})+\Delta_{abc}, \tag{56a}$$

$$\Delta_{abc}\equiv\frac{3}{16\pi\tilde{G}}\tilde{\epsilon}_{dabc}\left(\delta\tilde{g}^{de}\tilde{\nabla}_e\tilde{\varphi}+\frac{1}{2}\tilde{g}^{ef}\delta\tilde{g}_{ef}\tilde{\nabla}^d\tilde{\varphi}+\tilde{\nabla}^d\delta\tilde{\varphi}\right). \tag{56b}$$

Nevertheless, the symplectic currents are the same, i.e., $\omega_{abc}(\delta g,\delta\varphi)=\tilde{\omega}_{abc}(\delta\tilde{g},\delta\tilde{\varphi})$. Although Δ_{abc} is nonvanishing in general, it is closed, i.e., $\tilde{\nabla}_{[a}\Delta_{bcd]}=0$, if Eqs. (5) and their linear perturbations are satisfied. Indeed, the following can be obtained:

$$\tilde{\epsilon}^{abcd}\tilde{\nabla}_a\Delta_{bcd}=\frac{9}{8\pi\tilde{G}}\Bigg[\frac{1}{2}\tilde{g}^{ab}\delta\tilde{g}_{ab}\tilde{\nabla}^2\tilde{\varphi}+\Bigg(\delta\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\varphi}+\tilde{\nabla}_a\delta\tilde{g}^{ab}\tilde{\nabla}_b\tilde{\varphi}+\frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_c\delta\tilde{g}_{ab}\tilde{\nabla}^c\tilde{\varphi}+\tilde{\nabla}^2\delta\tilde{\varphi}\Bigg)\Bigg], \tag{57}$$

where the terms in the round brackets represent the linearized scalar field equation. Therefore, $\Delta_{abc}=3\tilde{\nabla}_{[a}\mathcal{Y}_{bc]}$ locally for some 2-form $\mathcal{Y}_{ab}$, locally constructed out of $\tilde{g}_{ab},\tilde{\varphi}$, and their variations [59]. According to Ref. [39], there is always an ambiguity in choosing θ_{abc}. Given that we work in Einstein frame, we ignore the difference, Δ_{abc}.

Now, let us choose an arbitrary, closed, embedded 3-dimensional hypersurface, Σ, without boundary. The presymplectic form, Ξ_Σ, is given by the following integral:

$$\Xi_\Sigma(\delta\tilde{g},\delta\tilde{\varphi};\delta'\tilde{g},\delta'\tilde{\varphi})=\int_\Sigma\tilde{\omega}_{abc}(\delta\tilde{g},\delta\tilde{\varphi};\delta'\tilde{g},\delta'\tilde{\varphi}). \tag{58}$$

Let us suppose now that $\delta'\tilde{g}_{ab}$ and $\delta'\tilde{\varphi}$ are induced by a vector field ξ^a, i.e., $\delta'\tilde{g}_{ab}=\mathcal{L}_\xi\tilde{g}_{ab}$ and $\delta'\tilde{\varphi}=\mathcal{L}_\xi\tilde{\varphi}$. Furthermore, if the equations of motion (5) are satisfied by $\tilde{g}_{ab}$ and $\tilde{\varphi}$, and the linearized equations of motion are also satisfied by $\delta\tilde{g}_{ab}$ and $\delta\tilde{\varphi}$, then the above integral defines the variation of a Hamiltonian, or a charge Q_ξ, conjugate to ξ^a,

$$\not{\delta}Q_\xi[\Sigma]=\int_\Sigma\tilde{\omega}_{abc}(\delta\tilde{g},\delta\tilde{\varphi};\mathcal{L}_\xi\tilde{g},\mathcal{L}_\xi\tilde{\varphi}). \tag{59}$$

The integral above can be rewritten as the one over a 2-dimensional surface $\partial\Sigma$ [39],

$$\not{\delta}Q_\xi[\partial\Sigma]=\int_{\partial\Sigma}[\delta\tilde{Q}_{ab}-\xi^c\tilde{\theta}_{cab}(\delta\tilde{g},\delta\tilde{\varphi})], \tag{60}$$

and, therefore, we now take $\not{\delta}Q_\xi$ as a function of $\partial\Sigma$, instead of Σ. In the above expression, the Noether charge 2-form, $\tilde{Q}_{ab}$, is

$$\tilde{Q}_{ab}=-\frac{1}{16\pi\tilde{G}}\tilde{\epsilon}_{abcd}\tilde{\nabla}^c\xi^d, \tag{61}$$

which takes exactly the same form as that in GR [60]. Here, the $\not{\delta}$ symbol means that a function Q_ξ might not exist. The sufficient and necessary condition for the existence of Q_ξ on Σ is that, for all $(\delta\tilde{g}_{ab},\delta\tilde{\varphi})$ and $(\delta'\tilde{g}_{ab},\delta'\tilde{\varphi})$ satisfying the linearized equations of motion [39],

$$\int_{\partial\Sigma}\xi^c\tilde{\omega}_{cab}(\delta\tilde{g},\delta\tilde{\varphi};\delta'\tilde{g},\delta'\tilde{\varphi})=0. \tag{62}$$

When this condition is violated, for example, when $\partial\Sigma$ is a cross section of $\mathscr{I}$, there is a prescription to find a "conserved charge" Q_ξ conjugate to ξ^a, which will be discussed in the next two subsections. Before that, the behaviors of $\tilde{\theta}_{abc}$, $\tilde{\omega}_{abc}$, and $\tilde{Q}_{ab}$ must be analyzed near $\mathscr{I}$.

Equations (55) and (61) are the most important ones for calculating the "conserved charges" at $\mathscr{I}$. Given that $\mathscr{I}$ in the physical spacetime is not at a finite place, it is probable that these equations blow up at $\mathscr{I}$. Thus, it is necessary to check whether they are finite at $\mathscr{I}$ or not and to know the behaviors of $\delta\tilde{g}_{ab}$ and $\delta\tilde{\varphi}$. For that purpose, the field variation should not change the conformal factor, i.e., $\delta\Omega=0$. Simultaneously, $\mathscr{I}$ is a universal structure for any asymptotically flat spacetime [48]. Consequently, the unphysical metric, $\bar{g}_{ab}$, should remain the same at $\mathscr{I}$,

$$\delta\bar{g}_{ab}=\Omega^2\delta\tilde{g}_{ab}\doteq 0, \tag{63}$$

which implies that a smooth tensor field τ_{ab} exists such that

$$\delta\bar{g}_{ab}=\Omega\tau_{ab},\quad \delta\tilde{g}_{ab}=\Omega^{-1}\tau_{ab}. \tag{64}$$

By applying a method similar to the one applied to obtain Eq. (50), it can be shown that $\tau_{ab}\doteq 2\delta\Sigma_{ab}$. As discussed in Sec. III, the Bondi gauge condition described by Eq. (17) is used for simplicity. This condition should be preserved under field variation $\delta\bar{g}_{ab}$; a smooth covector field τ_a exists such that

$$\tau_{ab}\bar{n}^b=\Omega\tau_a. \tag{65}$$

Finally, there are no requirements for $\delta\tilde{\varphi}$; hence, $\delta\tilde{\varphi}=\Omega\delta\bar{\varphi}$.

Now, it is straightforward to reexpress Eqs. (55) and (61) in the unphysical spacetime $(\bar{M},\bar{g}_{ab})$. First, the presymplectic potential current is

$$\tilde{\theta}_{abc}=\frac{1}{16\pi\tilde{G}_0}\bar{\epsilon}_{abcd}\Big\{\Omega^{-1}\Big[\bar{\nabla}_e\tau^{de}-\bar{\nabla}^d\tau-3\tau^d-(2\omega+3)\chi\bar{\varphi}\bar{n}^d\Big]-(2\omega+3)\chi\bar{\nabla}^d\bar{\varphi}\Big\}, \tag{66}$$

where $\tau = \bar{g}^{ab}\tau_{ab}$ and $\chi = \delta\bar{\varphi}$. The presence of the Ω^{-1} factor inside the curly brackets formally indicates the blowing up of this expression at $\mathscr{I}$, but it is actually finite. To demonstrate this, let us start with Einstein's equation, i.e., Eq. (13), in the unphysical spacetime without imposing the Bondi gauge explicitly, and then, let us vary it:

$$\delta\bar{S}_{ab} \doteq 4\bar{n}_{(a}\tau_{b)} - \bar{n}^c\bar{\nabla}_c\tau_{ab} - \bar{g}_{ab}\bar{n}^c\tau_c + (2\omega+3)\bar{n}_a\bar{n}_b\chi\bar{\varphi}. \quad (67)$$

Simultaneously, by definition, the variation of $\bar{S}_{ab}$ is [39]

$$\delta\bar{S}_{ab} \doteq -\bar{n}_{(a}\bar{\nabla}_{b)}\tau - \bar{n}^c\bar{\nabla}_c\tau_{ab} + \bar{n}_{(a}\bar{\nabla}^c\tau_{b)c} + \bar{n}_{(a}\tau_{b)} - \frac{1}{3}\bar{g}_{ab}(\bar{n}^c\tau_c - \bar{n}^c\bar{\nabla}_c\tau). \quad (68)$$

Comparing these two expressions, we obtain the following:

$$\bar{\nabla}^b\tau_{ab} - \bar{\nabla}_a\tau - 3\tau_a - (2\omega+3)\chi\bar{\varphi}\bar{n}_a \doteq 0, \quad (69a)$$

$$\bar{n}^a\bar{\nabla}_a\tau + 2\bar{n}^a\tau_a \doteq 0. \quad (69b)$$

Because of Eq. (69a), the presymplectic potential 3-form (66) is finite at $\mathscr{I}$. Then, the Noether charge 2-form is

$$\tilde{Q}_{ab}(\xi) = -\frac{1}{16\pi\tilde{G}}\bar{\epsilon}_{abcd}\bar{\nabla}^c(\Omega^{-2}\xi^d), \quad (70)$$

which takes the same form as the integrand of Eq. (7) in Ref. [56], as expected. Again, this 2-form seems to diverge at $\mathscr{I}$, in an even worse manner than Eq. (66), but it is also finite there, as proved in Appendix A. Finally, after some tedious algebraic manipulations, the symplectic current 3-form is given by

$$\tilde{\omega}_{abc} = -\frac{1}{32\pi\tilde{G}}\bar{\epsilon}_{abc}(\tau'^{de}\delta N_{de} - \tau^{de}\delta N'_{de}) + \frac{2\omega+3}{16\pi\tilde{G}}\bar{\epsilon}_{abc}(\chi'\delta\bar{N} - \chi\delta\bar{N}'), \quad (71)$$

where τ'_{ab} is defined for $\delta'\bar{g}_{ab}$, and $\chi' = \delta'\bar{\varphi}$. Given that $\tau_{ab} = 2\delta\Sigma_{ab}$, the form of this symplectic current suggests that σ_{ab} and N_{ab} are canonically conjugate to each other, and so are $\bar{\varphi}$ and $\bar{N}$. From the above equation, one may choose a presymplectic potential current, given by

$$\tilde{\Theta}_{abc}(\delta\tilde{g},\delta\tilde{\varphi}) = -\frac{1}{32\pi\tilde{G}}\bar{\epsilon}_{abc}\tau^{de}N_{de} + \frac{2\omega+3}{16\pi\tilde{G}}\bar{\epsilon}_{abc}\chi\bar{N}, \quad (72)$$

such that the pullback of $\tilde{\omega}_{abc}$ to $\mathscr{I}$ is $\delta\tilde{\Theta}_{abc}(\delta'\tilde{g},\delta'\tilde{\varphi}) - \delta'\tilde{\Theta}_{abc}(\delta\tilde{g},\delta\tilde{\varphi})$. There is also an ambiguity in $\tilde{\Theta}_{abc}$, but this is the only one, according to the argument of Ref. [39]. Note that $\tilde{\Theta}_{abc}$ enables the computation of the flux, as will be discussed below.

B. Fluxes

Once $\tilde{\Theta}_{abc}$ is determined, a flux through a patch $\mathscr{B}$, i.e., a subset of $\mathscr{I}$, can be obtained as follows:

$$F_{\xi,\mathscr{B}} = \int_{\mathscr{B}}\tilde{\Theta}_{abc}(\mathcal{L}_\xi\tilde{g},\mathcal{L}_\xi\tilde{\varphi}) = -\frac{1}{16\pi\tilde{G}}\int_{\mathscr{B}}\bar{\epsilon}_{abc}\Big\{N_{de}[(\mathcal{L}_\xi\mathcal{D}_p - \mathcal{D}_p\mathcal{L}_\xi)\ell_q + 2\ell_{(p}\mathcal{D}_{q)}\bar{K}]\gamma^{dp}\gamma^{eq} - (2\omega+3)(\mathcal{L}_\xi\bar{\varphi} + \bar{K}\bar{\varphi})\bar{N}\Big\}, \quad (73)$$

where τ_{ab} and χ in Eq. (72) are given by $2\bar{X}_{ab}$ [as in Eqs. (35) and (50)] and $\mathcal{L}_\xi\bar{\varphi} + \bar{K}\bar{\varphi}$ [refer to Eq. (40)], respectively. This should be compared with Eq. (4.14) in Ref. [50], which does not contain the term with $\bar{\varphi}$. Let us suppose that $\mathscr{B}$ is bounded by two cross sections, C_1 and C_2, with the latter in the future of the former, leading to

$$F_{\xi,\mathscr{B}} = -(Q_\xi[C_2] - Q_\xi[C_1]). \quad (74)$$

This expresses the conservation of the charge and is also called the flux-balance law. The overall negative sign above indicates that as the GW escapes from $\mathscr{I}$, the charge of the spacetime decreases.

If $\mathscr{B}$ is replaced by $\mathscr{I}$ in Eq. (73) and the resultant integral is finite, $\mathcal{H}_\xi \equiv F_{\xi,\mathscr{I}}$ is the Hamiltonian generator on the radiative phase space on $\mathscr{I}$ associated with ξ^a [41]. Using the transformations expressed by Eqs. (40), (47), and (48), the Hamiltonian generators for supertranslation $\alpha\bar{n}^a$ and the Lorentz generator parameterized by Y^a can be obtained as

$$\mathcal{H}_\alpha = \frac{1}{16\pi\tilde{G}}\int_{\mathscr{I}}\bar{\epsilon}_{abc}\Big[N_{de}\Big(\mathcal{D}_p\mathcal{D}_q\alpha + \frac{\alpha}{2}N_{pq}\Big)\gamma^{dp}\gamma^{eq} + \alpha(2\omega+3)\bar{N}^2\Big], \quad (75a)$$

$$\mathcal{H}_Y = \frac{1}{16\pi\tilde{G}}\int_{\mathscr{I}}\bar{\epsilon}_{abc}\Big\{N_{de}\Big[\frac{u}{2}\mathcal{D}_p\mathcal{D}_q(\mathcal{D}\cdot Y) + \frac{1}{2}\sigma_{pq}\mathcal{D}\cdot Y - \mathcal{L}_Y\sigma_{pq} + \frac{u}{4}N_{pq}\mathcal{D}\cdot Y\Big]\gamma^{dp}\gamma^{eq} + (2\omega+3)\bar{N}\Big[\frac{1}{2}(u\bar{N} + \bar{\varphi})\mathcal{D}\cdot Y + \mathcal{L}_Y\bar{\varphi}\Big]\Big\}, \quad (75b)$$

respectively. In GR, the linear term in N_{ab} in Eq. (75a) gives the soft charge and the quadratic one in N_{ab} the hard charge [58, 61]. Likewise, the linear terms in N_{ab} and $\bar{N}$ determine the soft fluxes, and the quadratic ones in N_{ab} and $\bar{N}$ denote the hard fluxes [62]. Using the results presented in Sec. II, the Hamiltonian generators can be explicitly calculated:

$$\mathcal{H}_\alpha = \frac{\varphi_0}{16\pi G_0}\int \alpha\Big[\mathcal{D}_A\mathcal{D}_B N^{AB} + \frac{1}{2}N_A^B N_B^A + (2\omega+3)\left(\frac{N}{\varphi_0}\right)^2\Big]\mathrm{d}u\mathrm{d}^2\mathbf{\Omega}, \tag{76a}$$

$$\mathcal{H}_Y = \mathcal{H}_{\alpha'} + \frac{\varphi_0}{32\pi G_0}\int Y^A\Big[\frac{1}{2}(\hat{c}_B^C\mathcal{D}_A N_C^B - N_B^C\mathcal{D}_A\hat{c}_C^B) + \mathcal{D}^B(N_B^C\hat{c}_{AC} - \hat{c}_B^C N_{AC}) + \frac{2\omega+3}{\varphi_0^2}(N\mathcal{D}_A\varphi_1 - \varphi_1\mathcal{D}_A N)\Big]\mathrm{d}u\mathrm{d}^2\mathbf{\Omega}, \tag{76b}$$

where $\alpha' = \frac{u}{2}\mathcal{D}_A Y^A$, and $\mathrm{d}^2\mathbf{\Omega} = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$; integration by parts has been applied. These results are consistent with those in Ref. [36].

C. “Conserved charges”

Now, we can calculate the “conserved charges”. According to the decomposition expressed in Eq. (45), any BMS generator ξ^a contains a component tangent to a cross section C, and a component transverse to C, once a foliation of $\mathcal{I}$ is prescribed. The “conserved charges” for different components will be calculated in different ways. Thus, $\xi^a = \xi_1^a + \xi_a^2$ can be re-written using [63]

$$\xi_1^a \doteq \frac{u-u_0}{2}\psi(\partial_u)^a + Y^A(\partial_A)^a, \tag{77a}$$

$$\xi_2^a \doteq \left(\alpha + \frac{u_0}{2}\psi\right)(\partial_u)^a, \tag{77b}$$

where u_0 labels a reference cross section C_0 such that ξ_1^a is tangent to C_0 at $u = u_0$. These expressions imply that ξ_1^a is an infinitesimal Lorentz transformation, and ξ_2^a is a supertranslation generator. Thus, the charges on C_0 will be determined.

For the Lorentz generator ξ_1^a, the “conserved charge” on C_0 is given by [39]

$$Q_{\xi_1}[C_0] = \oint_{C_0}\tilde{Q}_{ab}(\xi_1), \tag{78}$$

with the requirement that $\tilde{\nabla}_a\xi^a = O\left(\Omega^2\right)$ [63]. This requirement is satisfied by ξ^a, obtained in Ref. [34]. To calculate this, we employ the asymptotic solutions presented in Sec. II to obtain

$$Q_{\xi_1}[C_0] = \frac{1}{16\pi\tilde{G}}\oint_{C_0} Y^A\Big[2N_A + \frac{1}{16}\mathcal{D}_A(\hat{c}_{BC}\hat{c}^{BC}) + \frac{2\omega+3}{4}\frac{\varphi_1\mathcal{D}_A\varphi_1}{\varphi_0^2}\Big]\mathrm{d}^2\mathbf{\Omega}. \tag{79}$$

For supertranslation generator ξ_2^a, the “conserved charge” satisfies [39]

$$\delta Q_{\xi_2}[C_0] = \oint_{C_0}[\tilde{Q}_{ab}(\alpha\bar{n}) - \alpha\bar{n}^c\tilde{\theta}_{cab} + \alpha\bar{n}^c\tilde{\Theta}_{cab}]. \tag{80}$$

Unfortunately, it is very difficult to calculate this expression directly. Instead, one can take the advantage of Eq. (74). Now, let $\xi^a = \alpha\bar{n}^a$; hence, the flux for this generator is

$$F_{\alpha\bar{n},\mathcal{B}} = -\frac{1}{16\pi\tilde{G}}\int_{\mathcal{B}}\bar{\epsilon}_{abc}\Big[\gamma^{df}\gamma^{eh}N_{fh}\Big(\mathcal{L}_{\alpha\bar{n}}\mathcal{D}_f - \mathcal{D}_f\mathcal{L}_{\alpha\bar{n}}\Big)\ell_h + (2\omega+3)\alpha\bar{N}^2\Big], \tag{81}$$

according to Eq. (73). Next, we apply the Stokes' theorem to obtain the “conserved charge” for $\xi^a = \alpha\bar{n}^a$ [50]:

$$Q_\alpha[C] = \frac{1}{8\pi\tilde{G}}\oint_C P^d\ell_d\bar{n}^c\bar{\epsilon}_{cab}, \tag{82}$$

with

$$P^a = \frac{\alpha}{4}K^{ab}\ell_b + N_{cd}\gamma^{bd}\gamma^{c[e}\bar{n}^{a]}(\alpha\mathcal{D}_e\ell_b + \ell_e\mathcal{D}_b\alpha) - \frac{2\omega+3}{6}\alpha\bar{n}^a\bar{\varphi}\bar{N}. \tag{83}$$

By setting $\bar{N} = 0$ in Eqs. (81) and (83), the GR's results are recovered [48, 50]. Now, using the results presented in Sec. II, the “conserved charge” conjugate to ξ_2^a is obtained as follows:

$$Q_{\xi_2}[C_0] = \frac{1}{8\pi\tilde{G}}\oint_{C_0}\left(2\alpha m - u_0 Y^A\mathcal{D}_A m\right)\mathrm{d}^2\mathbf{\Omega}. \tag{84}$$

where α was replaced by $\alpha + u_0\psi/2$ in Eq. (82).

The total “conserved charge” is the sum of Eqs. (79) and (84):

$$Q_\xi[C] = \frac{\varphi_0}{8\pi G_0}\oint_C\Big[2\alpha m - u\mathcal{L}_Y m + Y^A N_A + \frac{1}{32}\mathcal{L}_Y(\hat{c}_B^A\hat{c}_A^B) + \frac{2\omega+3}{8}\frac{\varphi_1\mathcal{L}_Y\varphi_1}{\varphi_0^2}\Big]\mathrm{d}^2\mathbf{\Omega}, \tag{85}$$

which is evaluated at some arbitrary C and is consistent with Eq. (3.5) in Ref. [63]. Note that, in the above computation, we implicitly assume that the all charges of the Minkowski spacetime vanish, given that any constant can always be added to Q_ξ without breaking Eq. (59). This imposes a nontrivial condition [39]

$$\int_{\partial\Sigma}\Big\{\eta^c\tilde{\theta}_{cab}(\mathcal{L}_\xi\tilde{g},\mathcal{L}_\xi\tilde{\varphi})-\xi^c\tilde{\theta}_{cab}(\mathcal{L}_\eta\tilde{g},\mathcal{L}_\eta\tilde{\varphi}) + \tilde{\mathcal{L}}\tilde{\epsilon}_{abcd}\eta^c\xi^d-\tilde{Q}_{ab}[\mathcal{L}_\eta\xi]\Big\}=0, \tag{86}$$

where η^a is also a BMS generator, and $\tilde{\mathcal{L}}$ is the Lagrange density in Eq. (4). In addition, $\tilde{g}_{ab}=\eta_{ab}$ and $\tilde{\varphi}=\tilde{\varphi}_0$, which are implicitly included in this expression. The demonstration that this condition is satisfied is presented in Appendix B.

Now, let us work out the "conserved charges" for some specific BMS generators. First, consider a generic supertranslation generator $\alpha\bar{n}^a$ with $\mathcal{L}_{\bar{n}}\alpha=0$. The "conserved charge" is called the supermomentum, which is given by

$$\mathcal{P}_\alpha[C]=\frac{\varphi_0}{4\pi G_0}\oint_C \alpha m \mathrm{d}^2\mathbf{\Omega}. \tag{87}$$

Among these supermomenta, four of them are special and are obtained by replacing α by $l=0,1$ spherical harmonics. They constitute the Bondi 4-momentum P^a. In particular, zeroth component P^0 is the Bondi mass

$$M=\frac{\varphi_0}{4\pi G_0}\oint_C m\mathrm{d}^2\mathbf{\Omega}, \tag{88}$$

which justifies the name of m. In some literature, "supermomenta" do not include P^a [63-65]. Second, switch off α, and write Y^A as follows [63]:

$$Y^A=\mathcal{D}^A\mu+\epsilon^{AB}\mathcal{D}_B\upsilon, \tag{89}$$

where ϵ^{AB} is the totally antisymmetric tensor on the unit 2-sphere, and μ and υ are linear combinations of $l=1$ spherical harmonics, satisfying $(\mathcal{D}^2+2)\mu=(\mathcal{D}^2+2)\upsilon=0$; μ is the electric part, and υ the magnetic part of Y^A. The electric part generates the Lorentz boost, whose charge is

$$\mathcal{K}_\mu[C]=-\frac{\varphi_0}{8\pi G_0}\oint_C\mu\left(\mathcal{D}^AN_A+2um-\frac{\hat{c}^B_A\hat{c}^A_B}{16}-\frac{2\omega+3}{8}\frac{\varphi_1^2}{\varphi_0^2}\right)\mathrm{d}^2\mathbf{\Omega}, \tag{90}$$

and the magnetic part generates the rotation with the following charge:

$$\mathcal{J}_\upsilon[C]=-\frac{\varphi_0}{8\pi G_0}\oint_C\upsilon\epsilon^{AB}\mathcal{D}_AN_B\mathrm{d}^2\mathbf{\Omega}, \tag{91}$$

which explains why N_A is called the angular momentum aspect; $\mathcal{K}_\mu$ and $\mathcal{J}_\upsilon$ are called the CM and the spin charges, respectively. Given that there are three linearly independent $l=1$ spherical harmonics, there are both three linearly independent boost and rotation charges. In total, there are six, which is consistent with the fact that the Lorentz algebra is six dimensional. Note also that the scalar field only contributes to boost charge $\mathcal{K}_\mu$. A remark regarding the forms of the spin and CM charges is in order. There are different conventions in defining what is called the Bondi angular momentum aspect [66, 67]. Thus, the spin and CM charges and the relevant fluxes take different forms. These differences are summarized in Ref. [68] in GR.

V. MEMORIES

As discussed in Ref. [34], GWs in both tensor and scalar sectors induce the displacement memory effects. In that study, the focus was on the relation between the memory effects and the asymptotic symmetries that induce the vacuum transitions. Here, we reanalyze the memory effects, with a focus on the constraints on memories imposed by the flux-balance laws. We consider not only the displacement memory but also the spin and the CM memory effects [25, 26].

We also consider the memory effects between vacuum states in the tensor and scalar sectors. Following Ref. [34], a vacuum state in the scalar sector is simply given by $N=\dot{\varphi}_1=0$. However, a vacuum state in the tensor sector is determined not only by $N_{AB}=-\partial_u\hat{c}_{AB}=0$ but also by the vanishing of the Newman-Penrose variables [69] Ψ_4,Ψ_3 and $\Psi_2-\bar{\Psi}_2$ at leading orders in $1/r$ [70]. This definition agrees with the one in GR and also with the requirement that $N_{ab}={}^*K^{ab}=0$ [50]. Now, $\hat{c}_{AB}$ can be written as follows [63]:

$$\hat{c}_{AB}=\left(\mathcal{D}_A\mathcal{D}_B-\frac{1}{2}\gamma_{AB}\mathcal{D}^2\right)\Phi+\epsilon_{C(A}\mathcal{D}_{B)}\mathcal{D}^C\Upsilon, \tag{92}$$

where Φ is the electric part, and Υ the magnetic part. In vacuum, $\Upsilon=0$.

A. Displacement memory effects

Let us start with the displacement memory effect in the tensor sector. First, we rewrite the flux-balance law associated with supertranslation $\alpha\bar{n}^a$ in the generalized Bondi-Sachs coordinates:

$$F_{\alpha\bar{n},\mathcal{B}}=\frac{\varphi_0}{16\pi G_0}\int_{\mathcal{B}}\alpha\left[\mathcal{D}^A\mathcal{D}^BN_{AB}+\frac{N_{AB}N^{AB}}{2} +(2\omega+3)\left(\frac{N}{\varphi_0}\right)^2\right]\mathrm{d}u\mathrm{d}^2\mathbf{\Omega}=-\Delta\mathcal{P}_\alpha, \tag{93}$$

where $\Delta\mathcal{P}_\alpha=\mathcal{P}_\alpha[C_2]-\mathcal{P}_\alpha[C_1]$ for simplicity. We can calculate the retarded time integral of the soft flux above and then rearrange the expression to obtain

$$\oint_C \alpha\mathcal{D}^2(\mathcal{D}^2+2)\Delta\Phi \mathrm{d}^2\mathbf{\Omega} = \frac{32\pi G_0}{\varphi_0}(\mathcal{E}_\alpha + \Delta\mathcal{P}_\alpha), \tag{94}$$

where $\mathcal{E}_\alpha$ is $F_{\alpha\bar{n},\mathcal{B}}$ without the first term in the square brackets; Eq. (92) has been used. Therefore, $\Delta\Phi$ fully captures the displacement memory in the tensor sector, and it is completely constrained by the above equation. It is often stated that $\mathcal{E}_\alpha$ causes the null memory, whereas $\Delta\mathcal{P}_\alpha$ causes the ordinary memory [71].

Now, let us consider the displacement memory effect in the scalar sector. Following the above argument, one may want to consider the flux-balance law for Lorentz generator Y^A, given by

$$\begin{aligned} F_{Y,\mathcal{B}} =& \Delta\mathcal{P}_{\alpha'} - \frac{\varphi_0}{16\pi G_0}\int_{\mathcal{B}}\Big[Y^A J_A + \upsilon\epsilon_{AB}N^{CA}\hat{c}^B_C\Big]\mathrm{d}u\mathrm{d}^2\mathbf{\Omega} \\ &+ \frac{\varphi_0}{64\pi G_0}\oint_C \mu\Delta\left(\frac{2\omega+3}{\varphi_0^2}\varphi_1^2 + \frac{\hat{c}^B_A\hat{c}^A_B}{2}\right)\mathrm{d}^2\mathbf{\Omega} \\ =& -\Delta\mathcal{K}_\mu - \Delta\mathcal{J}_\upsilon, \end{aligned} \tag{95}$$

where $\Delta\mathcal{P}_{\alpha'}$ is the integral of Eq. (93), with α replaced by $\alpha' = u\mathcal{D}^2\mu/2 = -u\mu$; $\Delta\mathcal{K}_\mu$ and $\Delta\mathcal{J}_\upsilon$ are defined similarly to $\Delta\mathcal{P}_\alpha$, and

$$J_A = \frac{1}{2}N^B_C\mathcal{D}_A\hat{c}^C_B - \frac{2\omega+3}{\varphi_0^2}N\mathcal{D}_A\varphi_1. \tag{96}$$

Note that $\Delta\mathcal{P}_{\alpha'}$ is not a flux, given that α' depends on u. In fact, the magnetic part could be set such that $\upsilon = 0$, and the expression could be rearranged to obtain

$$\oint \mu\Delta\left[\frac{2\omega+3}{\varphi_0^2}\varphi_1^2 + \frac{\hat{c}^B_A\hat{c}^A_B}{2}\right]\mathrm{d}^2\mathbf{\Omega} = -\frac{16\pi G_0}{\varphi_0}(\Delta\mathcal{K}_\mu + \Delta\mathcal{P}_{\alpha'} + \mathcal{J}_\mu), \tag{97}$$

where $\mathcal{J}_\mu$ is given by

$$\mathcal{J}_\mu = \frac{\varphi_0}{16\pi G_0}\int_{\mathcal{B}}\mu\mathcal{D}^A J_A \mathrm{d}u\mathrm{d}^2\mathbf{\Omega}. \tag{98}$$

It may seem that this is a constraint equation on $\Delta\varphi_1^2$, but that is not the case, according to Eq. (90). In fact, the left hand side is canceled by the terms in $\Delta\mathcal{K}_\mu$. Nevertheless, Eq. (95) is useful for CM memory.

In fact, the equation of motion gives a constraint on $\Delta\varphi_1^2$, which is

$$\begin{aligned} \Delta\varphi_1^2 =& \frac{16\varphi_0^2}{2\omega+3}\left\{\frac{1}{32}\Delta(\hat{c}^B_A\hat{c}^A_B) + \mathcal{D}^{-2}\mathcal{D}^A\Delta N_A \right. \\ &\left. - \int_{u_i}^{u_f}\mathrm{d}u\left[m + \frac{1}{2}\mathcal{D}^{-2}\mathcal{D}^A J_A\right]\right\}, \end{aligned} \tag{99}$$

where $\mathcal{D}^{-2}$ is the inverse operator of $\mathcal{D}^2$ and is explicitly given in Ref. [34]. These results suggest that $\Delta\varphi_1$ is a persistent variable [72], as stated in Ref. [36].

B. Spin memory effect

Spin memory effect exists only in the tensor sector, as it depends on the leading order term in g_{uA} [25, 34]. To determine the constraint on the spin memory effect from the flux-balance law, one needs to consider the extended BMS algebra, which includes all Y^A satisfying the conformal relation $\mathcal{L}_Y\gamma_{AB} = \gamma_{AB}\mathcal{D}\cdot Y$. These Y^A may not be globally smooth on the unit 2-sphere [54, 63, 73]. However, in Sec. IV, we assumed Y^A are smooth vector fields; hence, the fluxes and charges calculated there cannot be directly used here. Fortunately, there is a simple remedy. We can still use the fluxes and charges defined above, examine the flux-balance law, find the discrepancy, and fix it. It turns out that, without modifying the definition of the charge, one may want to add the following correction to the flux associated with Y^A [63]:

$$\begin{aligned} \mathcal{F}_{Y,\mathcal{B}} =& \frac{\varphi_0}{32\pi G_0}\int_{\mathcal{B}} Y^A\mathcal{D}^B(\mathcal{D}_A\mathcal{D}_C\hat{c}^C_B - \mathcal{D}_B\mathcal{D}_C\hat{c}^C_A)\mathrm{d}u\mathrm{d}^2\mathbf{\Omega} \\ =& \frac{\varphi_0}{64\pi G_0}\int_{\mathcal{B}}\epsilon_{AB}Y^A\mathcal{D}^B\mathcal{D}^2(\mathcal{D}^2+2)\Upsilon\mathrm{d}u\mathrm{d}^2\mathbf{\Omega}. \end{aligned} \tag{100}$$

This correction vanishes when Y^A is smooth on the unit 2-sphere. Now, adding this term to the right hand side of the first line in Eq. (95), and setting $\mu = 0$, we obtain the constraint on the spin memory, measured by $\Delta\mathcal{R} = \int\mathrm{d}u\Upsilon$ [63]:

$$\oint_C \upsilon\mathcal{D}^2\mathcal{D}^2(\mathcal{D}^2+2)\Delta\mathcal{R}\mathrm{d}^2\mathbf{\Omega} = -\frac{32\pi G_0}{\varphi_0}(\Delta\mathcal{J}_\upsilon + \mathcal{Q}_\upsilon + \bar{\mathcal{J}}_\upsilon), \tag{101}$$

where

$$\mathcal{Q}_\upsilon = -\frac{\varphi_0}{16\pi G_0}\int_{\mathcal{B}}\upsilon\epsilon_{AB}N^{AC}\hat{c}^B_C\mathrm{d}u\mathrm{d}^2\mathbf{\Omega}, \tag{102a}$$

$$\bar{\mathcal{J}}_\upsilon = \frac{\varphi_0}{16\pi G_0}\int_{\mathcal{B}}\upsilon\epsilon^{AB}\mathcal{D}_A J_B\mathrm{d}u\mathrm{d}^2\mathbf{\Omega}. \tag{102b}$$

Note that, here, υ is not necessarily a linear combination of $l = 1$ spherical harmonics.

C. Center-of-mass memory effect

Now, consider the CM memory. Given that $\mathcal{D}^2(\mathcal{D}^2+2)$ in Eq. (94) is linear, one may define $\Phi = \Phi_n + \Phi_o$ such that

$$\oint_C \alpha\mathscr{D}^2(\mathscr{D}^2+2)\Delta\Phi_n\mathrm{d}^2\mathbf{\Omega}=\frac{32\pi G_0}{\varphi_0}\mathcal{E}_\alpha, \tag{103a}$$

$$\oint_C \alpha\mathscr{D}^2(\mathscr{D}^2+2)\Delta\Phi_o\mathrm{d}^2\mathbf{\Omega}=\frac{32\pi G_0}{\varphi_0}\Delta\mathcal{P}_\alpha. \tag{103b}$$

Then, the CM memory effect is determined by [26, 36]

$$\Delta\mathcal{K}=\int_{u_i}^{u_f}u\partial_u\Phi_o\mathrm{d}u.$$

It appears in $\Delta\mathcal{P}_{\alpha'}$, i.e.,

$$\Delta\mathcal{P}_{\alpha'}=-\frac{\varphi_0}{64\pi G_0}\oint_C\mu\mathscr{D}^2\mathscr{D}^2(\mathscr{D}^2+2)\Delta\mathcal{K}\mathrm{d}^2\mathbf{\Omega}. \tag{105}$$

Thus, Eq. (95) can be rewritten to yield

$$\oint_C\mu\mathscr{D}^2\mathscr{D}^2(\mathscr{D}^2+2)\Delta\mathcal{K}\mathrm{d}^2\mathbf{\Omega}=\frac{64\pi G_0}{\varphi_0}(\mathcal{J}_\mu-\Delta\mathcal{K}'_\mu), \tag{106}$$

where $\Delta\mathcal{K}'_\mu$ is not the change in any charge, given by

$$\Delta\mathcal{K}'_\mu=-\frac{\varphi_0}{8\pi G_0}\oint_C\mu\Delta(\mathscr{D}^AN_A+2um)\mathrm{d}^2\mathbf{\Omega}. \tag{107}$$

Therefore, the CM memory is constrained by Eq. (106), as long as μ is not simply a linear combination of $l=1$ spherical harmonics.

VI. CONCLUSION

In this study, we analyzed the asymptotic structure and BMS symmetries in an isolated system in the BD using the covariant conformal completion method. The results thus obtained are independent of the coordinate system used. There are four different orders of the asymptotic structure, as in GR. The zeroth-order structure $(\gamma_{ab},\bar{n}^a)$ is universal, and the first-order structure $\{\mathscr{D}_a\}$ characterizes the differences among spacetimes. The second-order structure $(N_{ab},\bar{N})$ constitutes the radiative degrees of freedom, and the third-order structure, ${}^*K^{ab}$, contains the full gauge covariant information in $\{\mathscr{D}_a\}$ [51]. The BMS symmetries also include the supertranslations and the Lorentz transformations, and their actions on the asymptotic structure are discussed. Based on these, the "conserved charges" and fluxes were computed according to the Wald-Zoupas formulism. If one switches off the scalar field, the GR's results are reproduced. The scalar field only contributes to the CM charge, but it appears in all fluxes. Finally, the flux-balance laws are used to constrain various memory effects. Among them, the displacement memory effect in the scalar sector cannot be restricted by the flux-balance laws, but the equation of motion constrains it partially. Memory effects in the tensor sector are well constrained by the flux-balance laws, as in GR.

APPENDIX A: FINITENESS OF THE NOETHER CHARGE

In this section, the finiteness of Eq. (70) is demonstrated. According to Stokes' theorem,

$$\int_C\tilde{Q}_{ab}=\int_{S_0}\tilde{Q}_{ab}+\int_{\Sigma'}\mathrm{d}_a\tilde{Q}_{bc}, \tag{A1}$$

where S_0 is a finite topological 2-sphere in the physical spacetime, and Σ' is a 3-dimensional hypersurface joining S_0 to C. If the last two integrals are both finite, then the first is also finite, and so is integrand $\tilde{Q}_{ab}$.

The integrand of the third integral can be contracted with $\bar{\epsilon}^{abcd}$, and one can, thus, examine [56]

$$\begin{aligned}\bar\nabla^b\bar\nabla_{[a}(\Omega^{-2}\xi_{b]})=&\Omega^{-2}\bar R_{ab}\xi^b+\bar\nabla_a\bar\nabla_b(\Omega^{-2}\xi^b)\\&-\bar\nabla^b\bar\nabla_{(a}(\Omega^{-2}\xi_{b)}),\end{aligned} \tag{A2}$$

with $\bar{R}_{ab}$ being the Ricci tensor of $\bar{g}_{ab}$. Using the property described by Eq. (35), one can reexpress the last two terms in the above equation as follows:

$$\begin{aligned}\bar\nabla_a\bar\nabla_b(\Omega^{-2}\xi^b)-\bar\nabla^b\bar\nabla_{(a}(\Omega^{-2}\xi_{b)})=&\Omega^{-1}(3X_a+\bar\nabla_aX-\bar\nabla^bX_{ab})\\&+\Omega^{-3}\xi^b(2\bar\nabla_a\bar n_b+\bar g_{ab}\bar\nabla_c\bar n^c\\&-3\Omega^{-1}\bar g_{ab}\bar n_c\bar n^c).\end{aligned} \tag{A3}$$

Then, according to Eq. (13), we obtain

$$\begin{aligned}&\Omega\bar R_{ab}+2\bar\nabla_a\bar n_b+\bar g_{ab}\bar\nabla_c\bar n^c-2\Omega^{-1}\bar g_{ab}\bar n_c\bar n^c\\&-\frac{1}{2}\Omega^{-1}\bar g_{ab}\bar L=\Omega^{-1}\bar L_{ab},\end{aligned} \tag{A4}$$

with $\bar{L}=\bar{g}^{ab}\bar{L}_{ab}$. Therefore, Eq. (A2) becomes

$$\begin{aligned}\bar\nabla^b\bar\nabla_{[a}(\Omega^{-2}\xi_{b]})=&\Omega^{-1}(3X_a+\bar\nabla_aX-\bar\nabla^bX_{ab})\\&+\Omega^{-4}\xi^b\left(\bar L_{ab}+\frac{1}{2}\bar g_{ab}\bar L\right).\end{aligned} \tag{A5}$$

Now, substituting this in the definition of $\bar{L}_{ab}$ expressed by Eq. (14), we obtain

$$\begin{aligned}\bar\nabla^b\bar\nabla_{[a}(\Omega^{-2}\xi_{b]})=&\Omega^{-1}\Big[3X_a+\bar\nabla_aX-\bar\nabla^bX_{ab}\\&+\frac{2\omega+3}{2}(\bar K\bar\varphi^2+\bar\varphi\mathcal{L}_\xi\bar\varphi)\bar n_a\Big]\\&+\bar K\bar\varphi\bar\nabla_a\bar\varphi+\bar\nabla_a\bar\varphi\mathcal{L}_\xi\bar\varphi.\end{aligned} \tag{A6}$$

Again, Eq. (A6) on $\mathcal{I}$ seems to diverge as well, which is not true. To understand this, let us use Eq. (37) to calculate

$$\begin{aligned}\mathcal{L}_\xi \bar{R}_{ab} =& -2\bar\nabla_a\bar\nabla_b\bar{K} + 2\bar{n}_{(a}(X_{b)} - \bar\nabla_{b)}X + \bar\nabla^c X_{b)c}) \\ &+ 4X_{c(a}\bar\nabla_{b)}\bar{n}^c - X_{ab}\bar\nabla_c\bar{n}^c - X\bar\nabla_a\bar{n}_b - 2\mathcal{L}_{\bar n}X_{ab} \\ &+ \Omega(2\bar\nabla_c\bar\nabla_{(a}X^c_{b)} + 2\bar\nabla_{(a}X_{b)} - \bar\nabla_c\bar\nabla^c X_{ab} - \bar\nabla_a\bar\nabla_b X),\end{aligned} \tag{A7a}$$

$$\begin{aligned}\mathcal{L}_\xi \bar{R} =& -2\bar\nabla_a\bar\nabla^a\bar{K} + 4\bar{n}^a X_a - 2X_{ab}\bar\nabla^a\bar{n}^b \\ &- 2X\bar\nabla_a\bar{n}^a - 4\mathcal{L}_{\bar n}X - 2\bar{K}\bar{R} + 2\Omega(\bar\nabla_a\bar\nabla_b X^{ab} \\ &- \bar\nabla_a\bar\nabla^a X + 2\bar\nabla_a X^a - X^{ab}\bar{R}_{ab}).\end{aligned} \tag{A7b}$$

Knowing $\mathcal{L}_\xi(\Omega^{-2}\bar{L}_{ab})$, which can be demonstrated to be $O(\Omega)$, Eq. (37) leads to

$$\begin{aligned}&\bar{n}_{(a}\left[3X_{b)} + \bar\nabla_{b)} - \bar\nabla^c X_{b)c} + \frac{2\omega+3}{2}\bar\varphi(\mathcal{L}_\xi\bar\varphi + \bar{K}\bar\varphi)\bar{n}_{b)}\right] \\ &- \frac{1}{6}\bar{g}_{ab}(\bar\nabla_c\bar\nabla^c\bar{K} + 4\bar{n}^c X_c + 2\mathcal{L}_{\bar n}X) \doteq 0.\end{aligned} \tag{A8}$$

It turns out that $\bar\nabla_a\bar\nabla^a\bar{K} = 0$. Therefore, we obtain

$$3X_a + \bar\nabla_a X - \bar\nabla^b X_{ab} + \frac{2\omega+3}{2}\bar\varphi(\mathcal{L}_\xi\bar\varphi + \bar{K}\bar\varphi)\bar{n}_a \doteq 0, \tag{A9a}$$

$$2\bar{n}^a X_a + \mathcal{L}_{\bar n}X \doteq 0. \tag{A9b}$$

This implies that Eq. (A6) is, indeed, finite, and so is Eq. (A1). Therefore, Eq. (70) is finite on $\mathcal{I}$.

APPENDIX B: VERIFY CONDITION (86)

Equation (86) should hold in BD. In the Minkowski spacetime, any BMS generator is a sum of a supertranslation and a Killing vector field. If either η^a or ξ^a are a Killing vector field, Eq. (86) is satisfied [39]. As in GR, we only have to check if

$$\int_{\partial\Sigma}[\eta^c\tilde\theta_{cab}(\mathcal{L}_\xi\tilde{g}, \mathcal{L}_\xi\tilde\varphi) - \xi^c\tilde\theta_{cab}(\mathcal{L}_\eta\tilde{g}, \mathcal{L}_\eta\tilde\varphi)] = 0, \tag{B1}$$

where $\xi^a \doteq \alpha\bar{n}^a$ and $\eta^a \doteq \beta\bar{n}^a$ are two supertranslation generators, with $\mathcal{L}_{\bar n}\alpha = \mathcal{L}_{\bar n}\beta = 0$. Let us calculate the first term in the square brackets above, which is

$$\eta^c\tilde\theta_{cab} = \frac{1}{16\pi\tilde{G}}\beta F(\alpha)\bar{n}^c\bar\epsilon_{cab}, \tag{B2}$$

with function $F(\alpha)$ given by

$$\begin{aligned}F(\alpha) =& -\bar{n}_a\Omega^{-1}[\bar\nabla_b\lambda^{ab} - \bar\nabla^a\lambda - 3\lambda^a - (2\omega+3)\chi\bar\varphi\bar{n}^a] \\ &+ (2\omega+3)\chi\bar{N} \doteq -\bar\nabla_a\bar\nabla_b\lambda^{ab} + \bar\nabla^2\lambda \\ &+ 3\bar\nabla_a\lambda^a + (2\omega+3)\chi\bar{N},\end{aligned} \tag{B3}$$

where $\lambda_{ab} = \Omega\mathcal{L}_\xi\tilde{g}_{ab} = \Omega^{-1}(\mathcal{L}_\xi\bar{g}_{ab} - 2\bar{K}\bar{g}_{ab})$, $\lambda = \bar{g}^{ab}\lambda_{ab}$, and $\lambda_a = \lambda_{ab}\bar{n}^b/\Omega$. The first three terms add up to a quantity proportional to the so-called "flux" defined by Eq. (19) in Ref. [56], in which the gauge condition $\tilde\nabla_a\xi^a = 0$ was imposed. As discussed in that study, their flux can also be calculated using Eq. (20), which is gauge invariant. In the current case, we can also rewrite the above expression as follows:

$$F(\alpha) \doteq -\bar\nabla_a\bar\nabla_b\lambda^{ab} + 3\bar\nabla_a\lambda^a + \frac{3}{4}\bar\nabla^2\lambda + \frac{1}{24}\bar{R}\lambda + (2\omega+3)\chi\bar{N}. \tag{B4}$$

Now, we must calculate $F(\alpha)$ with $\xi^a = \alpha\bar{n}^a - \Omega\bar\nabla^a\alpha$ [56], thereby obtaining

$$\bar{K} = \Omega(\alpha\vartheta - \sigma_\alpha), \tag{B5}$$

$$\lambda_{ab} = -2\bar\nabla_a\bar\nabla_b\alpha - (\alpha\vartheta - 2\sigma_\alpha)\bar{g}_{ab} - \alpha(\bar{S}_{ab} - \Omega^{-2}\bar{L}_{ab}), \tag{B6}$$

$$\lambda_a = \bar\nabla_a(\alpha\vartheta - 2\sigma_\alpha) - (\bar{S}_{ab} - \Omega^{-2}\bar{L}_{ab})\bar\nabla^b\alpha, \tag{B7}$$

$$\lambda = -2\bar\nabla^2\alpha - 4(\alpha\vartheta - 2\sigma_\alpha) - \alpha\left(\frac{\bar{R}}{3} - \Omega^{-2}\bar{L}\right). \tag{B8}$$

With these, we obtain

$$\begin{aligned}F(\alpha) \doteq& -\frac{\alpha(2\omega+3)}{3}\left[2\bar{N}^2 - \bar\varphi\mathcal{L}^2_{\bar n}\bar\varphi\right] - \frac{1}{4}\left(\bar\nabla^2 - \frac{\bar{R}}{6}\right) \\ &\times\left(\bar\nabla^2\alpha + 2\alpha\vartheta - 4\sigma_\alpha + \frac{\alpha\bar{R}}{6}\right),\end{aligned} \tag{B9}$$

where $\mathcal{L}^2_{\bar n}\bar\varphi = \mathcal{L}_{\bar n}\bar{N}$. It can be easily verified that

$$\bar\nabla^2\alpha \doteq \mathcal{D}^2\alpha + 2\sigma_\alpha. \tag{B10}$$

To calculate $\bar\nabla^2\bar\nabla^2\alpha$, one needs

$$\begin{aligned}\mathcal{L}_{\bar n}\bar\nabla^2\alpha =& 2\mathcal{L}_{\bar n}\sigma_\alpha + \Omega\Big(\bar\nabla^2\sigma_\alpha + 2\vartheta\sigma_\alpha - \frac{\bar{R}\sigma_\alpha}{3} \\ &+ \frac{1}{6}\bar\nabla_a\alpha\bar\nabla^a\bar{R} - \vartheta\bar\nabla^2\alpha + \bar{S}^{ab}\bar\nabla_a\bar\nabla_b\alpha\Big) + O\left(\Omega^2\right),\end{aligned} \tag{B11}$$

where $O(\Omega^n)$ denotes a finite term at $\mathcal{I}$ multiplied by Ω^n.

Hence, we obtain

$$\begin{aligned}\bar{\nabla}^2\bar{\nabla}^2\alpha =& \mathcal{D}^2\mathcal{D}^2\alpha+4\bar{\nabla}^2\sigma_\alpha+4\vartheta\sigma_\alpha-\frac{2}{3}\bar{R}\sigma_\alpha\\ &+\frac{1}{3}\bar{\nabla}_a\alpha\bar{\nabla}^a\bar{R}-2\vartheta\bar{\nabla}^2\alpha+2\bar{S}^{ab}\bar{\nabla}_a\bar{\nabla}_b\alpha+O(\Omega).\end{aligned} \tag{B12}$$

To proceed further, we need $\mathcal{L}_{\bar{n}}\vartheta$ and $\mathcal{L}_{\bar{n}}\bar{R}$. Then, Eq. (A7b) becomes useful, setting $\xi^a=\bar{n}^a$. Thus, we obtain

$$X^{\bar{n}}_{ab}=-\frac{\vartheta}{2}\bar{g}_{ab}-\frac{1}{2}(\bar{S}_{ab}-\Omega^{-2}\bar{L}_{ab}),\quad X^{\bar{n}}_a=\frac{\bar{\nabla}_a\vartheta}{2}, \tag{B13}$$

$$X^{\bar{n}}=-2\vartheta-\frac{\bar{R}}{6}+\frac{\Omega^{-2}\bar{L}}{2},\quad \bar{K}^{\bar{n}}=\Omega\vartheta. \tag{B14}$$

Given that $\bar{\nabla}^2\bar{K}^{\bar{n}}=0$, we obtain

$$\bar{\nabla}^2\bar{K}^{\bar{n}}=2\mathcal{L}_{\bar{n}}\vartheta+\Omega\left(\bar{\nabla}^2\vartheta+2\vartheta^2-\frac{\vartheta\bar{R}}{6}\right)+O\left(\Omega^2\right), \tag{B15}$$

and, therefore,

$$\mathcal{L}_{\bar{n}}\vartheta=-\Omega\left(\frac{1}{2}\bar{\nabla}^2\vartheta+\vartheta^2-\frac{\vartheta\bar{R}}{12}\right)+O\left(\Omega^2\right). \tag{B16}$$

This implies that

$$\bar{\nabla}^2\vartheta=-\vartheta^2+\frac{\vartheta\bar{R}}{12}+\frac{1}{2}\mathcal{D}^2\vartheta+O(\Omega). \tag{B17}$$

Equation (A7b) gives

$$\begin{aligned}\mathcal{L}_{\bar{n}}\bar{R}=&\Omega\Big\{-\frac{\vartheta\bar{R}}{2}+\frac{3}{2}\bar{S}^{ab}\bar{S}_{ab}+(2\omega+3)\\ &\times\left[4\bar{N}^2+\bar{\varphi}\mathcal{L}^2_{\bar{n}}\bar{\varphi}\right]\Big\}+O\left(\Omega^2\right),\end{aligned} \tag{B18}$$

and thus,

$$\bar{\nabla}^2\bar{R}=-\vartheta\bar{R}+3\bar{S}^{ab}\bar{S}_{ab}+2(2\omega+3)\left[4\bar{N}^2+\bar{\varphi}\mathcal{L}^2_{\bar{n}}\bar{\varphi}\right]+O(\Omega). \tag{B19}$$

Finally, the "flux" is given by

$$\begin{aligned}F(\alpha)=&\frac{1}{2}\Big(\mathcal{D}^2\mathcal{D}^2\alpha+2\mathcal{D}^2\alpha+2N^{ab}\mathcal{D}_a\mathcal{D}_b\alpha\\ &+\frac{\alpha}{2}N^{ab}N_{ab}\Big)+\frac{2(2\omega+3)}{3}\alpha\mathcal{L}_{\bar{n}}(\bar{\varphi}\bar{N}).\end{aligned} \tag{B20}$$

In the Minkowski spacetime, $N_{ab}=0$ and $\bar{N}=0$; therefore,

$$F(\alpha)=\frac{1}{2}\left(\mathcal{D}^2\mathcal{D}^2\alpha+2\mathcal{D}^2\alpha\right), \tag{B21}$$

which implies that Eq. (86) is satisfied. $F(\alpha)$ is exactly the same as that in GR, up to a certain factor [74].

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