

Generalized Crewther relation and a novel demonstration of the scheme independence of commensurate scale relations up to all orders*

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Abstract: In this study, we investigate in detail the generalized Crewther Relation (GCR) between the Adler function (D) and the Gross-Llewellyn Smith sum rules coefficient (C^{GLS}) using the newly proposed single-scale approach of the principle of maximum conformality (PMC). The resultant GCR is scheme-independent, with the residual scale dependence due to unknown higher-order terms highly suppressed. Thus, a precise test of QCD theory without renormalization schemes and scale ambiguities can be achieved by comparing with data. Moreover, a demonstration of the scheme independence of the commensurate scale relation up to all orders is presented. Additionally, for the first time, the Pade approximation approach has been adopted for estimating the unknown 5th-loop contributions from the known four-loop perturbative series.

Keywords: perturbative QCD, renormalization, renormalization group invariance, QCD theory

DOI: 10.1088/1674-1137/ac1934

I. INTRODUCTION

The Crewther relation [1, 2] provides a non-trivial relation for three fundamental constants, $3S = KR'$, where S is the anomalous constant of $\pi^0 \rightarrow \gamma\gamma$, K is the coefficient of the Bjorken sum rules for polarized deep-inelastic electron scattering [3], and R' is the isovector part of the cross-sectional ratio for electron-positron annihilation into hadrons [4]. In the theory of quantum chromodynamics (QCD) [5, 6], an improved Crewther relation is known as the ‘‘Generalized Crewther Relation (GCR)’’ [7-13]:

$$D^{\text{NS}}(a_s)C^{\text{BjP}}(a_s) = 1 + \Delta_{\text{csb}}^*, \quad (1)$$

or

$$D(a_s)C^{\text{GLS}}(a_s) = 1 + \Delta_{\text{csb}}, \quad (2)$$

where $a_s = \alpha_s/\pi$, D^{NS} is the non-singlet Adler function, C^{BjP} is derived from the Bjorken sum rules for polarized deep-inelastic electron scattering, D is the Adler function, and C^{GLS} is the coefficient of the Gross-Llewellyn Smith (GLS) sum rules [14]. The Δ_{csb}^* and Δ_{csb} are conformal breaking terms. For example, the Δ_{csb} -term takes the form

$$\Delta_{\text{csb}} = \frac{\beta(a_s)}{a_s} K(a_s) = - \sum_{i \geq 2} \sum_{k=1}^{i-1} K_k \beta_{i-1-k} a_s^i, \quad (3)$$

where $\beta(a_s) = -\sum_{i \geq 0} \beta_i a_s^{i+2}$ is the usual β -function, and the coefficient K_k is free of $\{\beta_i\}$ -functions.

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Received 24 June 2021; Accepted 30 July 2021; Published online 20 August 2021

* Supported by the Chongqing Graduate Research and Innovation Foundation (ydstd1912, CYB21045), the National Natural Science Foundation of China (11625520, 12047564) and the Fundamental Research Funds for the Central Universities (2020CQJQY-Z003)

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and the Bjorken sum rules have been computed up to $O(\alpha_s^3)$ -level [15-19] and $O(\alpha_s^4)$ -level [20, 21], respectively. The GCR (1) between the non-singlet Adler function and the Bjorken sum rules was discussed in Refs. [22-24]. The GCR (2) between the Adler function and the GLS sum rules was investigated up to $O(\alpha_s^3)$ -level in Ref. [10]. Using the known α_s^4 -order corrections [25, 26], we can derive a more accurate GCR (2) up to $O(\alpha_s^4)$ -level. It is well-known that a physical observable is independent of any choice of theoretical conventions, such as the renormalization scheme and renormalization scale. This property is called “renormalization group invariance” (RGI) [27-31]. For a fixed-order pQCD prediction, if the perturbative coefficient preceding α_s and the corresponding α_s -value at each order are not consistent with each other, the RGI is explicitly broken [32, 33]. Conventionally, the “guessed” typical momentum flow of the process is adopted as the optimal renormalization scale with the purpose of eliminating the large logs to improve pQCD convergence, minimize the contributions from higher-order loop diagrams, or achieve a theoretical prediction in agreement with the experimental data. Such an approach directly breaks the RGI and reduces the predictive power of pQCD. Therefore, it is important to employ a proper scale-setting approach to achieve a scale-invariant fixed-order prediction.

In the literature, the principle of maximum conformality (PMC) [34-37] was proposed to eliminate the two artificial ambiguities. The purpose of PMC is not to find an optimal renormalization scale but to fix an effective α_s for the process using the renormalization group equation (RGE). The PMC prediction satisfies all self-consistency conditions of the renormalization group [38]. Two multi-scale approaches have been suggested to achieve the goal of PMC, which are both equivalent in terms of perturbative theory [39], with a collection of their successful applications found in Ref. [40]. For the multi-scale approach, the PMC sets the scales in an order-by-order manner; the individual scales at each order reflect the varying virtuality of the amplitudes at those orders. It has been noted that the PMC multi-scale approach has two types of residual scale dependence due to unknown perturbative terms [41]. Those residual scale dependences suffer from both α_s -power suppression and exponential suppression, but their magnitudes may be large due to poor convergence of the perturbative series of either the PMC scale or the pQCD approximant [42].

Recently, the PMC single-scale approach [43] has been suggested to suppress the residual scale dependence by introducing an overall effective α_s . The argument for such an effective α_s corresponds to the overall effective momentum flow of the process, which is also independent of any choice of renormalization scale. It has been shown that by using the PMC single-scale approach and the C -scheme strong coupling constant [44], one can

achieve a strict demonstration of the scheme-invariant and scale-invariant PMC prediction up to any fixed order [45]. Moreover, the resulting renormalization scheme- and scale-independent conformal series is useful not only for achieving precise pQCD predictions but also for producing reliable predictions of the contributions of unknown higher-orders. Some of these applications have been investigated in Refs. [46-50], which are estimated by using the Padé resummation approach [51-53]. In this study, we adopt the PMC single-scale approach to deal with the GCR (2), and then, for the first time, we estimate the unknown 5th-loop contributions for GCR (2). A novel demonstration of the scheme independence of the commensurate scale relation for all orders is also presented.

II. GENERALIZED CREWETHER RELATION UNDER THE PMC SINGLE-SCALE APPROACH

It is helpful to define the effective charge for a physical observable [54-56], which incorporates the entire radiative correction into its definition. For example, the GLS sum rules indicate that the isospin singlet structure function $xF_3(x, Q^2)$ satisfies an unsubtracted dispersion relation [14], i.e.,

$$\frac{1}{2} \int_0^1 \frac{dx}{x} xF_3(x, Q^2) = 3C^{\text{GLS}}(a_s), \quad (4)$$

where $xF_3(x, Q^2) = xF_3^{\nu p}(x, Q^2) + xF_3^{\bar{\nu} p}(x, Q^2)$, and Q is the momentum transfer. All of the radiative QCD corrections can be defined as an effective charge $a_{F_3}(Q)$. Moreover, the Adler function of the cross-sectional ratio for electron-positron annihilation into hadrons [4] can be written as

$$D(Q^2) = -12\pi^2 Q^2 \frac{d}{dQ^2} \Pi(Q^2), \quad (5)$$

where

$$\Pi(Q^2) = -\frac{Q^2}{12\pi^2} \int_{4m_i^2}^{\infty} \frac{R_{e^+e^-}(s) ds}{s(s+Q^2)}. \quad (6)$$

The Adler function D can be defined as the effective charge $a_D(Q)$. Thus, the Adler function D and the GLS sum rules coefficient C^{GLS} can be rewritten as

$$D(a_s) = 1 + a_D(Q), \quad (7)$$

$$C^{\text{GLS}}(a_s) = 1 - a_{F_3}(Q). \quad (8)$$

The effective charges $a_D(Q)$ and $a_{F_3}(Q)$ are by definition pQCD calculable, which can be expressed in the follow-

ing perturbative form:

$$\begin{aligned}
a_S = & r_{1,0}^S a_s + (r_{2,0}^S + \beta_0 r_{2,1}^S) a_s^2 \\
& + (r_{3,0}^S + \beta_1 r_{2,1}^S + 2\beta_0 r_{3,1}^S + \beta_0^2 r_{3,2}^S) a_s^3 \\
& + (r_{4,0}^S + \beta_2 r_{2,1}^S + 2\beta_1 r_{3,1}^S + \frac{5}{2}\beta_1 \beta_0 r_{3,2}^S \\
& + 3\beta_0 r_{4,1}^S + 3\beta_0^2 r_{4,2}^S + \beta_0^3 r_{4,3}^S) a_s^4 + \mathcal{O}(a_s^5), \quad (9)
\end{aligned}$$

where $S = D$ or F_3 , respectively, $r_{i,j=0}^S$ are conformal coefficients with $r_{1,0}^S = 1$, and $r_{i,j \neq 0}^S$ are nonconformal coefficients. The β -pattern for each order is determined by RGE [35, 37]. The coefficients $r_{i,j}^{D,F_3}$ up to the four-loop level under the $\overline{\text{MS}}$ scheme can be determined from Refs. [21, 57] using the general QCD degeneracy relations [39].

After applying the PMC single-scale approach [43], all $\{\beta_i\}$ -terms are used to fix the correct α_s -value by using the RGE, up to four-loop accuracy, and we obtain the following conformal series:

$$\begin{aligned}
D(a_s)|_{\text{PMC}} = & 1 + [r_{1,0}^D a_s(\tilde{Q}_*) + r_{2,0}^D a_s^2(\tilde{Q}_*) \\
& + r_{3,0}^D a_s^3(\tilde{Q}_*) + r_{4,0}^D a_s^4(\tilde{Q}_*)], \quad (10)
\end{aligned}$$

$$\begin{aligned}
C^{\text{GLS}}(a_s)|_{\text{PMC}} = & 1 - [r_{1,0}^{F_3} a_s(\hat{Q}_*) + r_{2,0}^{F_3} a_s^2(\hat{Q}_*) \\
& + r_{3,0}^{F_3} a_s^3(\hat{Q}_*) + r_{4,0}^{F_3} a_s^4(\hat{Q}_*)], \quad (11)
\end{aligned}$$

where \tilde{Q}_* and \hat{Q}_* are in perturbative series, which can be derived from the pQCD series of a_D and a_{F_3} . \tilde{Q}_* and \hat{Q}_* correspond to the overall momentum flows of the effective charges $a_D(Q)$ and $a_{F_3}(Q)$, which are independent of any choice of renormalization scale. This property confirms that the PMC is not for choosing an optimal renormalization scale but for determining the correct momentum flow of the process. Using the four-loop pQCD series of a_D and a_{F_3} , we determine their magnitudes up to N²LL accuracy by replacing the coefficients $\hat{r}_{i,j}$ in Eqs. (8)-(11) of Ref. [43] with $r_{i,j}^D$ or $r_{i,j}^{F_3}$, respectively. We present \tilde{Q}_* and \hat{Q}_* for different accuracies in Fig. 1. As shown in Fig. 1, as the perturbative series of a_D and a_{F_3} exhibit good perturbative convergence, it is interesting to find that their magnitudes up to different accuracies, such as the leading log (LL), next-to-leading log (NLL), and next-next-leading log (N²LL), are very similar. Thus, one can treat the N²LL-accurate \tilde{Q}_* and \hat{Q}_* as their exact values.

By applying the PMC single-scale approach, one can improve the precision of $D(a_s)$ and $C^{\text{Bjp}}(a_s)$. Fig. 2 shows the PMC predictions of $D(a_s)C^{\text{GLS}}(a_s)$ up to two-loop, three-loop, and four-loop levels. Moreover, the PMC scheme-invariant and scale-invariant conformal series

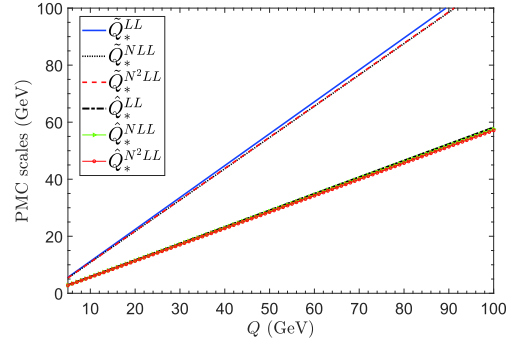


Fig. 1. (color online) PMC scales \tilde{Q}_* and \hat{Q}_* versus the momentum scale Q up to LL, NLL, and N²LL accuracies.

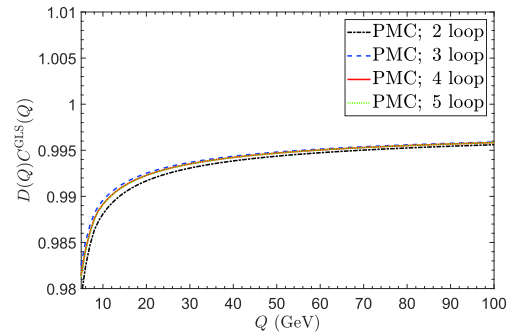


Fig. 2. (color online) $D(a_s)C^{\text{GLS}}(a_s)$ versus momentum Q under the PMC single-scale approach up to the two-loop, three-loop, and four-loop levels. The uncalculated five-loop result is predicted using the Padé approximation approach.

provides a reliable platform for predicting the uncalculated high-order terms [46]. For the first time, we also present the previously uncalculated five-loop result in Fig. 2, which is predicted using the Padé approximation approach (PAA) [51-53] and the preferable $[N/M] = [0/n-1]$ -type Padé generating function, which makes the PAA geometric series self-consistent with the PMC prediction [46]. More explicitly, the PAA has been introduced for estimating the $(n+1)$ _{th}-order coefficient for a given n _{th}-order perturbative series and for feasible conjectures on the likely high-order behavior of the series. For example, with the following conformal series

$$\rho(Q) = \sum_{i=1}^n r_{i,0} a_s^i, \quad (12)$$

for the present cases, we have $n = 4$ and $\rho(Q) = a_D|_{\text{PMC}}$ or $a_{F_3}|_{\text{PMC}}$. Its $[N/M]$ -type fractional generating function from the PAA is defined as

$$\rho^{N/M}(Q) = a_s \times \frac{b_0 + b_1 a_s + \dots + b_N a_s^N}{1 + c_1 a_s + \dots + c_M a_s^M}, \quad (13)$$

where $M \geq 1$ and $N + M = n - 1$. Then the unknown $(n+1)$ _{th}-order coefficient $r_{n+1,0}$ can be predicted using the known coefficients $\{r_{1,0}, \dots, r_{n,0}\}$ by expanding the frac-

tional generating function over a_s . That is, Eq. (13) can be expressed as

$$\rho^{N/M}(Q) = \sum_{i=1}^n r_{i,0} a_s^i + r_{n+1,0} a_s^{n+1} + \dots \quad (14)$$

We can first express all coefficients $\{b_0, \dots, b_N\}$ and $\{c_1, \dots, c_M\}$ by the known coefficients $r_{\{1, \dots, n\}, 0}$ and then obtain the coefficient $r_{n+1,0}$ over $\{b_i\}$ and $\{c_i\}$, which can be finally expressed by $\{r_{1,0}, \dots, r_{n,0}\}$. The PAA predictions of $a_{D|PMC}$ and $a_{F_3|PMC}$ are presented in Table 1 and Table 2, respectively, in which the known coefficients (“Exact”) for different orders are also presented for comparison. This suggests that with more known perturbative terms, the PAA predicted coefficients are closer to the “Exact” values.

Figure 2 shows that the two-, three-, four-, and five-

Table 1. The preferable $[0/n-1]$ -type PAA predictions of three-, four-, and five-loop coefficients of $a_{D|PMC}$ for the PMC single-scale approach. The known coefficients (“Exact”) are also presented as comparisons.

	Exact	PAA
$r_{1,0}^D$	1	–
$r_{2,0}^D$	1.84028	–
$r_{3,0}^D$	–3.03222	[0/1]: 3.38662
$r_{4,0}^D$	–23.2763	[0/2]: –17.3926
$r_{5,0}^D$	–	[0/3]: –34.1991

Table 2. The preferable $[0/n-1]$ -type PAA predictions of three-, four-, and five-loop coefficients of $a_{F_3|PMC}$ for the PMC single-scale approach. The known coefficients (“Exact”) are also presented as comparisons.

	Exact	PAA
$r_{1,0}^{F_3}$	1	–
$r_{2,0}^{F_3}$	0.840278	–
$r_{3,0}^{F_3}$	–5.71277	[0/1]: 0.706067
$r_{4,0}^{F_3}$	–16.0776	[0/2]: –10.1939
$r_{5,0}^{F_3}$	–	[0/3]: 18.2157

loop predictions have similar curves, particularly the predicted five- and four-loop curves. This is because the PMC conformal series is free of renormalon divergence [58–60], which inversely results in good pQCD convergence. The scheme-independent $D(a_s)|_{PMC}$ and $1/C^{GLS}(a_s)|_{PMC}$ have the same conformal coefficients, but as shown in Fig. 1, their PMC scales are not identical, $\tilde{Q}_* \neq \hat{Q}_*$. Thus, there is a small deviation from unity:

$$D(a_s)|_{PMC} C^{GLS}(a_s)|_{PMC} \approx 1. \quad (15)$$

This deviation, though it is very small in the large Q region, e.g., for $Q \approx 10^3$ GeV, where the ratio is approximately 0.998 for the five-loop prediction, is part of the intrinsic nature of QCD theory due to its conformal breaking property.

In GCR (2), the scheme dependent Δ_{csb} -term has been introduced to collect all the conformal breaking terms. This leads to explicit scheme dependence of GCR (2) under the conventional scale-setting approach due to the mismatching of α_s and its corresponding expansion coefficients. Conversely, by applying the PMC single-scale approach, we can achieve an exact scheme and scale invariant GCR at any fixed order. More explicitly, after applying the PMC single-scale approach, we obtain the following conformal series,

$$a_D(Q) = \sum_{i=1}^n a_{F_3}^i(Q_*), \quad (16)$$

where all the coefficients are exactly equal to 1, and the PMC scale Q_* is independent of the choice of renormalization scale, which can be fixed up to N²LL accuracy using the known four-loop coefficients, i.e.,

$$\ln \frac{Q_*^2}{Q^2} = T_0 + T_1 a_{F_3}(Q) + T_2 a_{F_3}^2(Q), \quad (17)$$

where

$$T_0 = r_{2,1}^{F_3} - r_{2,1}^D, \quad (18)$$

$$T_1 = 2(r_{3,1}^{F_3} - r_{3,1}^D + r_{2,0}^D r_{2,1}^D - r_{2,0}^{F_3} r_{2,1}^{F_3}) + [r_{3,2}^{F_3} - r_{3,2}^D + (r_{2,1}^D)^2 - (r_{2,1}^{F_3})^2] \beta_0, \quad (19)$$

$$T_2 = 3(r_{4,1}^{F_3} - r_{4,1}^D) - 2r_{2,1}^{F_3} r_{3,0}^{F_3} + 4r_{2,0}^D r_{3,1}^D + 3r_{2,1}^D r_{3,0}^D - 4(r_{2,0}^D)^2 r_{2,1}^D - r_{2,1}^D r_{3,0}^{F_3} + (r_{2,0}^{F_3})^2 (r_{2,1}^D + 5r_{2,1}^{F_3}) - 2r_{2,0}^{F_3} (3r_{3,1}^{F_3} - r_{3,1}^D + r_{2,0}^D r_{2,1}^D) + \{3(r_{4,2}^{F_3} - r_{4,2}^D) + 2r_{2,0}^D r_{3,2}^D + 4r_{3,1}^D r_{2,1}^D - 3r_{2,0}^D (r_{2,1}^D)^2 - 2r_{2,1}^{F_3} (3r_{3,1}^{F_3} - r_{3,1}^D + r_{2,0}^D r_{2,1}^D) + r_{2,0}^{F_3} [r_{3,2}^D - 3r_{3,2}^{F_3} + 6(r_{2,1}^{F_3})^2 - (r_{2,1}^D)^2]\} \beta_0 + \{r_{4,3}^{F_3} - r_{4,3}^D + 2r_{3,2}^D r_{2,1}^D + 2(r_{2,1}^{F_3})^3 - (r_{2,1}^D)^3 + r_{2,1}^{F_3} [r_{3,2}^D - 3r_{3,2}^{F_3} - (r_{2,1}^D)^2]\} \beta_0^2 + \frac{3}{2} [r_{3,2}^{F_3} - r_{3,2}^D + (r_{2,1}^D)^2 - (r_{2,1}^{F_3})^2] \beta_1. \quad (20)$$

The scale Q_* satisfies Eqs. (8)-(11) of Ref. [43], whose value can be determined by replacing a_s and $\hat{r}_{i,j}$ with a_{F_3} and $r_{i,j}^{DF_3}$. The $r_{i,j}^{DF_3}$ is a function of $r_{i,j}^D$ and $r_{i,j}^{F_3}$ [37]. One can derive Eqs.(18)-(20) by substituting those functions into Eqs. (9)-(11) of Ref. [43]. Eq. (16) is exactly scheme-independent and can be treated as a kind of commensurate scale relation (CSR). The CSR has been suggested in Ref. [61] for the purpose of ensuring the scheme-independence of the pQCD approximants among different renormalization schemes, with all original CSRs suggested in Ref. [61] at the NLO level. The PMC single-scale approach provides a method of extending the CSR to any order. A general demonstration of the scheme independence of the CSR (16) is presented in the next section. As a special case, taking the conformal limit that all $\{\beta_i\}$ -terms tend to zero, we have $Q_* \rightarrow Q$, and the relation (16) becomes the original Crewther relation [9]:

$$[1 + a_D(Q)][1 - a_{F_3}(Q)] \equiv 1. \quad (21)$$

By applying the PMC single-scale approach, one can obtain similar scheme-independent relations among different observables. The relation (16) not only provides a fundamental scheme-independent relation but also has phenomenologically useful consequences.

For example, the effective charge a_{F_3} can be related to the effective charge a_R of the R -ratio for the e^+e^- annihilation cross section ($R_{e^+e^-}$). The measurable R -ratio can be expressed by the perturbatively calculated Adler function, i.e.,

$$\begin{aligned} R_{e^+e^-}(s) &= \frac{1}{2\pi i} \int_{-s-i\epsilon}^{-s+i\epsilon} \frac{D(a_s(Q))}{Q^2} dQ^2 \\ &= 3 \sum_f q_f^2 (1 + a_R(Q)), \end{aligned} \quad (22)$$

where q_f is the electric charge of the active flavor. Similarly, the perturbative series of the effective charge $a_R(Q)$ can be written as

$$\begin{aligned} a_R &= r_{1,0}^R a_s + (r_{2,0}^R + \beta_0 r_{2,1}^R) a_s^2 \\ &\quad + (r_{3,0}^R + \beta_1 r_{2,1}^R + 2\beta_0 r_{3,1}^R + \beta_0^2 r_{3,2}^R) a_s^3 \\ &\quad + (r_{4,0}^R + \beta_2 r_{2,1}^R + 2\beta_1 r_{3,1}^R + \frac{5}{2}\beta_1 \beta_0 r_{3,2}^R \\ &\quad + 3\beta_0 r_{4,1}^R + 3\beta_0^2 r_{4,2}^R + \beta_0^3 r_{4,3}^R) a_s^4 + \mathcal{O}(a_s^5), \end{aligned} \quad (23)$$

where the coefficients $r_{i,j}^R$ under the $\overline{\text{MS}}$ -scheme can be derived from Refs. [20, 21, 26, 57]. Here, we will not consider the nonperturbative contributions, which may be important in small Q^2 -region [62-71], but are negligible for comparatively large s and Q^2 . After applying the PMC single-scale approach, we obtain a relation between a_R and a_{F_3} up to N³LO, i.e.,

$$a_{F_3}(Q) = \sum_{i=1}^n r_{i,0} a_{F_3}^i(\bar{Q}_*), \quad (24)$$

with the first four coefficients

$$r_{1,0} = 1, \quad (25)$$

$$r_{2,0} = -1, \quad (26)$$

$$r_{3,0} = 1 + \gamma_3^S \left[\frac{99}{8} - \frac{3(\sum_f q_f)^2}{4 \sum_f q_f^2} \right], \quad (27)$$

$$\begin{aligned} r_{4,0} &= -1 + \frac{(\sum_f q_f)^2}{\sum_f q_f^2} \left(\frac{27\gamma_2^{\text{NS}}\gamma_3^S}{16} + \frac{3\gamma_3^S}{2} - \frac{3\gamma_4^S}{4} \right) \\ &\quad - \gamma_3^S \left(\frac{99}{4} + \frac{891\gamma_2^{\text{NS}}}{32} \right) + \frac{99\gamma_4^S}{8}, \end{aligned} \quad (28)$$

where $\gamma_i^{\text{S,NS}}$ are singlet and non-singlet anomalous dimensions which are unrelated to α_s -renormalization, and the effective PMC scale \bar{Q}_* can be determined up to N²LL accuracy, which reads

$$\ln \frac{\bar{Q}_*^2}{Q^2} = S_0 + S_1 a_R(\bar{Q}_*) + S_2 a_R^2(\bar{Q}_*), \quad (29)$$

where

$$S_0 = -\frac{r_{2,1}}{r_{1,0}}, \quad (30)$$

$$S_1 = \frac{\beta_0(r_{2,1}^2 - r_{1,0}r_{3,2})}{r_{1,0}^2} + \frac{2(r_{2,0}r_{2,1} - r_{1,0}r_{3,1})}{r_{1,0}^2}, \quad (31)$$

and

$$\begin{aligned} S_2 &= \frac{3\beta_1(r_{2,1}^2 - r_{1,0}r_{3,2})}{2r_{1,0}^2} \\ &\quad + \frac{4(r_{1,0}r_{2,0}r_{3,1} - r_{2,0}^2 r_{2,1}) + 3(r_{1,0}r_{2,1}r_{3,0} - r_{1,0}^2 r_{4,1})}{r_{1,0}^3} \\ &\quad + \frac{\beta_0(6r_{2,1}r_{3,1}r_{1,0} - 3r_{4,2}r_{1,0}^2 + 2r_{2,0}r_{3,2}r_{1,0} - 5r_{2,0}r_{2,1}^2)}{r_{1,0}^3} \\ &\quad + \frac{\beta_0^2(3r_{1,0}r_{3,2}r_{2,1} - 2r_{2,1}^3 - r_{1,0}^2 r_{4,3})}{r_{1,0}^3}. \end{aligned} \quad (32)$$

Experimentally, the effective charge a_R has been constrained by measuring $R_{e^+e^-}$ above the thresholds for the production of the $(c\bar{c})$ -bound state [72], i.e.

$$a_R^{\text{exp}}(\sqrt{s} = 5 \text{ GeV}) \simeq 0.08 \pm 0.03. \quad (33)$$

Substituting it into Eq. (24), we obtain

$$a_{F_3}(Q = 12.58_{-1.26}^{+1.48} \text{ GeV}) = 0.073_{-0.026}^{+0.025}. \quad (34)$$

which is consistent with the PMC prediction derived directly from Eq. (11) within errors, e.g.,

$$a_{F_3}(Q = 12.58_{-1.26}^{+1.48} \text{ GeV})|_{\text{PMC}} = 0.063_{-0.001-0.001}^{+0.002+0.001}, \quad (35)$$

where the first error is for $\Delta Q = \binom{+1.48}{-1.26}$ GeV and the second error is for $\Delta\alpha_s(M_Z) = 0.1179 \pm 0.0011$ [73]. At present, the GLS sum rules are measured at small Q^2 -values [74, 75], and an extrapolation of the data gives [76]

$$a_{F_3}^{\text{ext}}(Q = 12.25 \text{ GeV}) \simeq 0.093 \pm 0.042, \quad (36)$$

which agrees with our prediction within errors.

III. A DEMONSTRATION OF THE SCHEME INDEPENDENCE OF COMMENSURATE SCALE RELATION

In this section, we present a novel demonstration of the scheme independence of CSR for all orders by relating different pQCD approximants within the effective charge method. The effective charge a_A can be expressed as a perturbative series over another effective charge a_B ,

$$\begin{aligned} a_A = & r_{1,0}^{\text{AB}} a_B + (r_{2,0}^{\text{AB}} + \beta_0 r_{2,1}^{\text{AB}}) a_B^2 \\ & + (r_{3,0}^{\text{AB}} + \beta_1 r_{3,1}^{\text{AB}} + 2\beta_0 r_{3,2}^{\text{AB}} + \beta_0^2 r_{3,3}^{\text{AB}}) a_B^3 \\ & + (r_{4,0}^{\text{AB}} + \beta_2 r_{4,1}^{\text{AB}} + 2\beta_1 r_{4,2}^{\text{AB}} + \frac{5}{2}\beta_1 \beta_0 r_{4,3}^{\text{AB}} \\ & + 3\beta_0 r_{4,4}^{\text{AB}} + 3\beta_0^2 r_{4,2}^{\text{AB}} + \beta_0^3 r_{4,3}^{\text{AB}}) a_B^4 + \mathcal{O}(a_B^5), \end{aligned} \quad (37)$$

where a_A and a_B are the effective charges under arbitrary schemes A and B , respectively. The previously mentioned a_D and a_{F_3} are examples of effective charges. The $\{\beta_i\}$ -functions are usually calculated under the $\overline{\text{MS}}$ -scheme, and the $\{\beta_i\}$ -functions for A/B scheme can be obtained using its relation to the $\overline{\text{MS}}$ -scheme, i.e. $\beta^{A/B} = \partial\alpha_{s,A/B}/\partial\alpha_{s,\overline{\text{MS}}}\beta^{\overline{\text{MS}}}$. The effective charge a_B at any scale μ can be expanded in terms of a C -scheme coupling $\hat{a}_B(\mu)$ at the same scale [45], i.e.,

$$\begin{aligned} a_B = & \hat{a}_B + C\beta_0 \hat{a}_B^2 + \left(\frac{\beta_2^B}{\beta_0} - \frac{\beta_1^2}{\beta_0^2} + \beta_0^2 C^2 + \beta_1 C \right) \hat{a}_B^3 \\ & + \left[\frac{\beta_3^B}{2\beta_0} - \frac{\beta_1^3}{2\beta_0^3} + \left(3\beta_2^B - \frac{2\beta_1^2}{\beta_0} \right) C + \frac{5}{2}\beta_0 \beta_1 C^2 \right. \\ & \left. + \beta_0^3 C^3 \right] \hat{a}_B^4 + \mathcal{O}(\hat{a}_B^5), \end{aligned} \quad (38)$$

where by choosing a suitable C , the coupling \hat{a}_B can be equivalent to a_B defined for any scheme at the same scale, i.e., $a_B = \hat{a}_B|_C$. By using the C -scheme coupling, the relation (37) becomes

$$\begin{aligned} a_A = & r_1 \hat{a}_B + (r_2 + \beta_0 r_1 C) \hat{a}_B^2 + \left[r_3 + \left(\beta_1 r_1 + 2\beta_0 r_2 \right) C \right. \\ & \left. + \beta_0^2 r_1 C^2 + r_1 \left(\frac{\beta_2^B}{\beta_0} - \frac{\beta_1^2}{\beta_0^2} \right) \right] \hat{a}_B^3 \\ & + \left[r_4 + \left(3\beta_0 r_3 + 2\beta_1 r_2 + 3\beta_2^B r_1 - \frac{2\beta_1^2 r_1}{\beta_0} \right) C \right. \\ & \left. + \left(3\beta_0^2 r_2 + \frac{5}{2}\beta_1 \beta_0 r_1 \right) C^2 + r_1 \beta_0^3 C^3 \right. \\ & \left. + r_1 \left(\frac{\beta_3^B}{2\beta_0} - \frac{\beta_1^3}{2\beta_0^3} \right) + r_2 \left(\frac{2\beta_2^B}{\beta_0} - \frac{2\beta_1^2}{\beta_0^2} \right) \right] \hat{a}_B^4 + \mathcal{O}(\hat{a}_B^5), \end{aligned} \quad (39)$$

where the coefficients r_i are

$$r_1 = r_{1,0}^{\text{AB}}, \quad (40)$$

$$r_2 = r_{2,0}^{\text{AB}} + \beta_0 r_{2,1}^{\text{AB}}, \quad (41)$$

$$r_3 = r_{3,0}^{\text{AB}} + \beta_1 r_{3,1}^{\text{AB}} + 2\beta_0 r_{3,2}^{\text{AB}} + \beta_0^2 r_{3,3}^{\text{AB}}, \quad (42)$$

$$\begin{aligned} r_4 = & r_{4,0}^{\text{AB}} + \beta_2 r_{4,1}^{\text{AB}} + 2\beta_1 r_{4,2}^{\text{AB}} + \frac{5}{2}\beta_1 \beta_0 r_{4,3}^{\text{AB}} \\ & + 3\beta_0 r_{4,4}^{\text{AB}} + 3\beta_0^2 r_{4,2}^{\text{AB}} + \beta_0^3 r_{4,3}^{\text{AB}}. \end{aligned} \quad (43)$$

Following the standard PMC single-scale approach, we obtain the following CSR:

$$a_A(Q) = \sum_{i=1}^n r_{i,0}^{\text{AB}} \hat{a}_B^i(Q_{**}), \quad (44)$$

where the effective PMC scale Q_{**} is obtained by vanishing all nonconformal terms, which can be expanded as a power series over $\hat{a}_B(Q_{**})$, i.e.,

$$\ln \frac{Q^2}{Q^{**}} = \sum_{i=0}^{n-2} \hat{S}_i \hat{a}_B^i(Q^{**}), \quad (45)$$

whose first three coefficients are

$$\hat{S}_0 = -\frac{r_{2,1}^{AB}}{r_{1,0}^{AB}} - C, \quad (46)$$

$$\hat{S}_1 = \frac{2(r_{2,0}^{AB} r_{2,1}^{AB} - r_{1,0}^{AB} r_{3,1}^{AB})}{(r_{1,0}^{AB})^2} + \frac{(r_{2,1}^{AB})^2 - r_{1,0}^{AB} r_{3,2}^{AB}}{(r_{1,0}^{AB})^2} \beta_0 + \frac{\beta_1^2}{\beta_0^3} - \frac{\beta_2^B}{\beta_0^2}, \quad (47)$$

and

$$\begin{aligned} \hat{S}_2 = & \frac{3r_{1,0}^{AB} r_{2,1}^{AB} r_{3,2}^{AB} - (r_{1,0}^{AB})^2 r_{4,3}^{AB} - 2(r_{2,1}^{AB})^3}{(r_{1,0}^{AB})^3} \beta_0^2 \\ & + \frac{3r_{1,0}^{AB} (2r_{2,1}^{AB} r_{3,1}^{AB} - r_{1,0}^{AB} r_{4,2}^{AB}) + r_{2,0}^{AB} [2r_{1,0}^{AB} r_{3,2}^{AB} - 5(r_{2,1}^{AB})^2]}{(r_{1,0}^{AB})^3} \beta_0 \\ & + \frac{3r_{1,0}^{AB} (r_{3,0}^{AB} r_{2,1}^{AB} - r_{1,0}^{AB} r_{4,1}^{AB}) + 4r_{2,0}^{AB} (r_{1,0}^{AB} r_{3,1}^{AB} - r_{2,0}^{AB} r_{2,1}^{AB})}{(r_{1,0}^{AB})^3} \\ & + \frac{3[(r_{2,1}^{AB})^2 - r_{1,0}^{AB} r_{3,2}^{AB}]}{2(r_{1,0}^{AB})^2} \beta_1 - \frac{\beta_1^3}{2\beta_0^4} + \frac{\beta_2^B \beta_1}{\beta_0^3} - \frac{\beta_3^B}{2\beta_0^2}. \end{aligned} \quad (48)$$

One may observe that only the LL coefficient \hat{S}_0 depends on the scheme parameter C , and all higher order coefficients \hat{S}_i ($i \geq 1$) are independent of C .

Moreover, the C -scheme coupling $\hat{a}_B(Q^*)$ satisfies the following relation [45],

$$\frac{1}{\hat{a}_B(Q^{**})} + \frac{\beta_1}{\beta_0} \ln \hat{a}_B(Q^{**}) = \beta_0 \left(\ln \frac{Q^2}{\Lambda^2} + C \right). \quad (49)$$

Substituting Eq. (45) into the right hand side of this equation, we obtain

$$\begin{aligned} & \frac{1}{\hat{a}_B(Q^{**})} + \frac{\beta_1}{\beta_0} \ln \hat{a}_B(Q^{**}) \\ = & \beta_0 \left[\ln \frac{Q^2}{\Lambda^2} - \frac{r_{2,1}^{AB}}{r_{1,0}^{AB}} + \sum_{i=1}^{n-2} \hat{S}_i \hat{a}_B^i(Q^{**}) \right]. \end{aligned} \quad (50)$$

By employing this equation, we can derive a solution for $\hat{a}_B(Q^{**})$. As all the coefficients in Eq. (50) are independent of C at any fixed order, the magnitude of $\hat{a}_B(Q^{**})$ shall be exactly free of C . Together with the scheme-independent conformal coefficients and the fact that the value of C can be chosen to match any renormalization scheme, we can conclude that the CSR (44) is exactly scheme independent. This example can be extended to all orders.

IV. SUMMARY

The PMC provides a systematic approach to determine an effective α_s for a fixed-order pQCD approximant. By using the PMC single-scale approach, the determined effective α_s is scale-invariant, which is independent of any choice of renormalization scale. As all nonconformal terms have been eliminated, the resultant pQCD series is scheme independent, satisfying the requirements of RGI. Furthermore, by applying the PMC single-scale approach, we obtain a scheme-independent GCR, $D(a_s)|_{\text{PMC}} C^{\text{GLS}}(a_s)|_{\text{PMC}} \approx 1$, which provides a significant connection between the Adler function and the GLS sum rules. We have shown that their corresponding effective couplings satisfy a scheme-independent CSR, $a_D(Q) = \sum_{i=1}^n a_{F_3}^i(Q^*)$. Furthermore, we obtain a CSR that relates the effective charge a_{F_3} to the effective charge of R -ratio, $a_{F_3}(Q) = \sum_{i=1}^4 r_{i,0} a_R^i(\bar{Q}^*)$. This leads to $a_{F_3}(Q = 12.58_{-1.26}^{+1.48} \text{ GeV}) = 0.073_{-0.026}^{+0.025}$, which agrees with the extrapolated measured value within errors. A demonstration on the scheme-independence of the CSR has been presented. The scheme- and scale-independent CSRs shall provide important tests of pQCD theory.

References

- [1] R. J. Crewther, *Phys. Rev. Lett.* **28**, 1421 (1972)
- [2] S. L. Adler, C. G. Callan, Jr., D. J. Gross, and R. Jackiw, *Phys. Rev. D* **6**, 2982 (1972)
- [3] J. D. Bjorken, *Phys. Rev.* **148**, 1467 (1966)
- [4] S. L. Adler, *Phys. Rev. D* **10**, 3714 (1974)
- [5] H. D. Politzer, *Phys. Rev. Lett.* **30**, 1346 (1973)
- [6] D. J. Gross and F. Wilczek, *Phys. Rev. Lett.* **30**, 1343 (1973)
- [7] J. D. Bjorken, *Phys. Rev.* **163**, 1767 (1967)
- [8] J. D. Bjorken, *Phys. Rev. D* **1**, 1376 (1970)
- [9] D. J. Broadhurst and A. L. Kataev, *Phys. Lett. B* **315**, 179 (1993)
- [10] S. J. Brodsky, G. T. Gabadadze, A. L. Kataev *et al.*, *Phys. Lett. B* **372**, 133 (1996)
- [11] R. J. Crewther, *Phys. Lett. B* **397**, 137 (1997)
- [12] G. T. Gabadadze and A. L. Kataev, *JETP Lett.* **61**, 448 (1995)
- [13] V. M. Braun, G. P. Korchemsky, and D. Müller, *Prog. Part. Nucl. Phys.* **51**, 311 (2003)
- [14] D. J. Gross and C. H. Llewellyn Smith, *Nucl. Phys. B* **14**,

- 337 (1969)
- [15] S. G. Gorishnii, A. L. Kataev, and S. A. Larin, *Phys. Lett. B* **259**, 144 (1991)
- [16] L. R. Surguladze and M. A. Samuel, *Phys. Rev. Lett.* **66**, 560 (1991)
- [17] K. G. Chetyrkin, J. H. Kuhn, and A. Kwiatkowski, *Phys. Rep.* **277**, 189 (1996)
- [18] K. G. Chetyrkin, *Phys. Lett. B* **391**, 402 (1997)
- [19] S. A. Larin and J. A. M. Vermaseren, *Phys. Lett. B* **259**, 345 (1991)
- [20] P. A. Baikov, K. G. Chetyrkin, and J. H. Kuhn, *Phys. Rev. Lett.* **101**, 012002 (2008)
- [21] P. A. Baikov, K. G. Chetyrkin, and J. H. Kuhn, *Phys. Rev. Lett.* **104**, 132004 (2010)
- [22] G. Cvetič and A. L. Kataev, *Phys. Rev. D* **94**, 014006 (2016)
- [23] A. V. Garkusha, A. L. Kataev, and V. S. Molokoedov, *JHEP* **1802**, 161 (2018)
- [24] J. M. Shen, X. G. Wu, Y. Ma, and S. J. Brodsky, *Phys. Lett. B* **770**, 494 (2017)
- [25] P. A. Baikov, K. G. Chetyrkin, and J. H. Kuhn, *Nucl. Phys. Proc. Suppl.* **205-206**, 237 (2010)
- [26] P. A. Baikov, K. G. Chetyrkin, J. H. Kuhn *et al.*, *Phys. Lett. B* **714**, 62 (2012)
- [27] A. Petermann, *Helv. Phys. Acta* **26**, 499 (1953)
- [28] M. Gell-Mann and F. E. Low, *Phys. Rev.* **95**, 1300 (1954)
- [29] C. G. Callan, Jr., *Phys. Rev. D* **2**, 1541 (1970)
- [30] K. Symanzik, *Commun. Math. Phys.* **18**, 227 (1970)
- [31] A. Peterman, *Phys. Rept.* **53**, 157 (1979)
- [32] X. G. Wu, S. J. Brodsky, and M. Mojaza, *Prog. Part. Nucl. Phys.* **72**, 44 (2013)
- [33] X. G. Wu, Y. Ma, S. Q. Wang *et al.*, *Rep. Prog. Phys.* **78**, 126201 (2015)
- [34] S. J. Brodsky and X. G. Wu, *Phys. Rev. D* **85**, 034038 (2012)
- [35] M. Mojaza, S. J. Brodsky, and X. G. Wu, *Phys. Rev. Lett.* **110**, 192001 (2013)
- [36] S. J. Brodsky and X. G. Wu, *Phys. Rev. Lett.* **109**, 042002 (2012)
- [37] S. J. Brodsky, M. Mojaza, and X. G. Wu, *Phys. Rev. D* **89**, 014027 (2014)
- [38] S. J. Brodsky and X. G. Wu, *Phys. Rev. D* **86**, 054018 (2012)
- [39] H. Y. Bi, X. G. Wu, Y. Ma *et al.*, *Phys. Lett. B* **748**, 13 (2015)
- [40] X. G. Wu, S. Q. Wang, and S. J. Brodsky, *Front. Phys.* **11**, 111201 (2016)
- [41] X. C. Zheng, X. G. Wu, S. Q. Wang *et al.*, *JHEP* **1310**, 117 (2013)
- [42] X. G. Wu, J. M. Shen, B. L. Du *et al.*, *Prog. Part. Nucl. Phys.* **108**, 103706 (2019)
- [43] J. M. Shen, X. G. Wu, B. L. Du *et al.*, *Phys. Rev. D* **95**, 094006 (2017)
- [44] D. Boito, M. Jamin, and R. Miravittlas, *Phys. Rev. Lett.* **117**, 152001 (2016)
- [45] X. G. Wu, J. M. Shen, B. L. Du *et al.*, *Phys. Rev. D* **97**, 094030 (2018)
- [46] B. L. Du, X. G. Wu, J. M. Shen *et al.*, *Eur. Phys. J. C* **79**, 182 (2019)
- [47] Q. Yu, X. G. Wu, J. Zeng *et al.*, *Eur. Phys. J. C* **80**, 362 (2020)
- [48] Q. Yu, X. G. Wu, S. Q. Wang *et al.*, *Chin. Phys. C* **43**, 093102 (2019)
- [49] X. D. Huang, X. G. Wu, J. Zeng *et al.*, *Phys. Rev. D* **101**, 114024 (2020)
- [50] H. M. Yu, W. L. Sang, X. D. Huang *et al.*, *JHEP* **01**, 131 (2021)
- [51] J. L. Basdevant, *Fortsch. Phys.* **20**, 283 (1972)
- [52] M. A. Samuel, G. Li, and E. Steinfelds, *Phys. Lett. B* **323**, 188 (1994)
- [53] M. A. Samuel, J. R. Ellis, and M. Karliner, *Phys. Rev. Lett.* **74**, 4380 (1995)
- [54] G. Grunberg, *Phys. Lett. B* **95**, 70 (1980)
- [55] G. Grunberg, *Phys. Rev. D* **29**, 2315 (1984)
- [56] A. Dhar and V. Gupta, *Phys. Rev. D* **29**, 2822 (1984)
- [57] P. A. Baikov, K. G. Chetyrkin, J. H. Kuhn *et al.*, *JHEP* **1207**, 017 (2012)
- [58] M. Beneke and V. M. Braun, *Phys. Lett. B* **348**, 513 (1995)
- [59] M. Neubert, *Phys. Rev. D* **51**, 5924 (1995)
- [60] M. Beneke, *Phys. Rep.* **317**, 1 (1999)
- [61] S. J. Brodsky and H. J. Lu, *Phys. Rev. D* **51**, 3652 (1995)
- [62] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Nucl. Phys. B* **147**, 385 (1979)
- [63] R. L. Jaffe and M. Soldate, *Phys. Rev. D* **26**, 49 (1982)
- [64] E. V. Shuryak and A. I. Vainshtein, *Nucl. Phys. B* **199**, 451 (1982)
- [65] E. V. Shuryak and A. I. Vainshtein, *Nucl. Phys. B* **201**, 141 (1982)
- [66] X. D. Ji and P. Unrau, *Phys. Lett. B* **333**, 228 (1994)
- [67] V. M. Braun and A. V. Kolesnichenko, *Nucl. Phys. B* **283**, 723 (1987)
- [68] I. I. Balitsky, V. M. Braun, and A. V. Kolesnichenko, *Phys. Lett. B* **242**, 245 (1990)
- [69] G. G. Ross and R. G. Roberts, *Phys. Lett. B* **322**, 425 (1994)
- [70] E. Stein, P. Gornicki, L. Mankiewicz *et al.*, *Phys. Lett. B* **343**, 369 (1995)
- [71] E. Stein, P. Gornicki, L. Mankiewicz *et al.*, *Phys. Lett. B* **353**, 107 (1995)
- [72] A. C. Mattingly and P. M. Stevenson, *Phys. Rev. D* **49**, 437 (1994)
- [73] P. A. Zyla *et al.* (Particle Data Group), *PTEP* **2020**, 083C01 (2020)
- [74] W. C. Leung *et al.*, *Phys. Lett. B* **317**, 655 (1993)
- [75] P. Z. Quintas *et al.*, *Phys. Rev. Lett.* **71**, 1307 (1993)
- [76] A. L. Kataev and A. V. Sidorov, *Phys. Lett. B* **331**, 179 (1994)