Perturbative modes and black hole entropy in f(Ricci) gravity*

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Abstract: f(Ricci) gravity is a special kind of higher curvature gravity whose bulk Lagrangian density is the trace of a matrix-valued function of the Ricci tensor. It is shown that under some mild constraints, f(Ricci) gravity admits Einstein manifolds as exact vacuum solutions, and can be ghost-free and tachyon-free around maximally symmetric Einstein vacua. It is also shown that the entropy for spherically symmetric black holes in f(Ricci) gravity calculated via the Wald method and the boundary Noether charge approach are in good agreement.

Keywords: modified gravity, higher curvature gravity, black hole entropy, linear perturbation

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1 Introduction

Even after more than a hundred years of intensive studies, Einstein's general theory of relativity (GR) remains the most important theory of gravity. The success of GR as the best candidate theory for gravity stems from the fact that it agrees with most observational tests, and is the simplest model among all metric-based geometric theories of gravity. However, GR is not without its own problems. To name a few, the unavoidable development of singularities signifies the failure of GR at later times, the non-renormalizability of the canonical quantization of GR indicates its failure in the early age of the Universe, and even at the present age of the Universe GR fails to explain its accelerated expansion without resorting to the yet feature-unknown dark energy. Some of the problems of GR may be overcame by introducing higher curvature terms in the action. For instance, the renormalizability can be significantly improved and the accelerated expansion of the Universe may be explained without introducing the concept of dark energy.

A large number of higher curvature gravity models have been proposed in literature, each inheriting one feature of GR or another. To select a good alternative for GR among these models, several criteria must be considered. First, the alternative model must also be in agreement with the observational tests as is GR. Second, it must resolve at least some of the problems that GR has been confronted to. Last but not the least, the alternative model must not introduce novel problems which GR did not have.

The observational tests of GR may be subdivided in two categories. The first consists of tests on the kinematic level. Most of the well known observational tests belong to this category, including, but not limited to, the deflection of light, perihelion recession, the time delay of radar signals and the frame dragging effect, etc. These are actually tests of the metric (Schwarzschild metric in the case of light deflection, perihelion recession and time delay of radar signals, Kerr metric in the case of frame dragging effect), rather than the gravitational model, and it is a simple fact that Schwarzschild and Kerr metrics are among the universal solutions to all metric-based theories of gravity. The second category involves dynamical effects. Only a few observational tests were performed at this level. The direct observation of gravitational waves and the accelerated expansion of the Universe are in this category. However, gravitational waves have only been observed in the far field, which should be considered as a test of the weak field limit only, and it has been mentioned above that GR fails to explain the accelerated expansion of the Universe without introducing dark energy.

Among the various higher curvature modifications of GR, the Lanczos-Lovelock gravity [1, 2] and $f(R)$ gravity [3−6] are probably the two best known and intensively studied models. The Lanczos-Lovelock gravity modifies GR in such a way that the action is supplemented by a series of higher order topological densities, so that in 4-dimensions it falls back to GR without any modification. $f(R)$ gravity, on the other hand, inherits the fact that the Lagrangian density of GR consists purely of

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Ricci scalars, among which GR is the simplest choice. We can regard GR from a different angle, where its Lagrangian density contains only the metric and the Ricci tensor. From this point of view, it is natural to extend GR into a more general f (Ricci) gravity, of which our proposal of the Ricci polynomial gravity [7] is a special example.

In this paper, we extend the study of the Ricci polynomial gravity to a more general f (Ricci) gravity, where $f(x)$ is an analytic function subject to only a few mild constraints which guarantee its perturbative behavior around certain Einstein vacua. To be more specific, we analyze the perturbative spectrum of f (Ricci) gravity, identifying the ghost-free and tachyon-free conditions around all allowed Einstein vacua. We also calculate the AdS black hole entropy using the boundary Noether charge technique proposed by Majhi and Padmanabhan [8, 9], which is referred to as the MP approach (see also [10, 11] for the use of of the same method in the case of $f(R)$ and conformal gravity), since the black hole entropy plays an indispensable role in understanding the holographic properties of the black hole spacetime, and in exploring the microscopic degrees of freedom of the black hole themselves [12−18]. Our calculation shows that the black hole entropy arising from the holographic calculations agrees with the Wald geometric entropy[19].

The paper is organized as follows. In Sec.2, we introduce the basics of f (Ricci) gravity. The bulk and boundary actions, the equation of motion and the conditions under which an Einstein manifold can be a vacuum solution are given explicitly. Sec. 3 is devoted to the perturbative analysis of the model, with emphasis on the ghost-free and tachyon-free conditions around various maximally symmetric Einstein vacua. Sec. 4 goes beyond the perturbative regime and concentrates on the calculation of the entropy of spherically symmetric black holes using the Wald method and the MP approach. The paper is then concluded in Sec. 5.

2 The model: equation of motion and vacuum solutions

To begin with, let us write down the bulk action of f (Ricci) gravity in n dimensions:

$$I_{\rm bulk} = \frac{1}{16\pi G}\int_{\mathcal{M}} {\rm d}^n x\,\sqrt{g}\,f(R_{\mu\nu},g_{\mu\nu})$$
$$\equiv \frac{1}{16\pi G}\int_{\mathcal{M}} {\rm d}^n x\,\sqrt{g}\,g^{\mu\nu}f_{\mu\nu}, \tag{1}$$

where $f_{\mu\nu}=g_{\mu\rho}f^{\rho}_{\nu}$, $f^{\mu}_{\nu}=f(x)|_{x\to R^{\mu}_{\nu}}$ is the natural continuation of $f(x)$, an analytic function of the single real variable x, to the case of matrix-valued variable, and, of course, $g_{\mu\nu}$ is the metric of the spacetime $\mathcal{M}$ with Ricci curvature $R_{\mu\nu}$, and g represents the absolute value of $\det(g_{\mu\nu})$. When $f(x)$ is a polynomial function, the above action reduces to the Ricci polynomial gravity studied in [7]. Given a coordinate system, and assuming that at certain event in the spacetime all components of the Ricci tensor are small enough, we can understand the tensorial expression f^{μ}_{ν} in terms of the Taylor expansion of $f(x)$, i.e.

$$f^{\mu}_{\nu}=\sum_{k=0}^{\infty}\frac{1}{k!}f^{(k)}(0)(\mathcal{R}^{(k)})^{\mu}_{\nu}, \tag{2}$$

where

$$(\mathcal{R}^{(k)})^{\mu}_{\nu}\equiv R^{\mu}_{\mu_1}R^{\mu_1}_{\mu_2}\cdots R^{\mu_{k-1}}_{\nu}.$$

It is evident that the tensorial monomials $(\mathcal{R}^{(k)})^{\mu}_{\nu}$ are not all independent from expressions of the form $(\mathcal{R}^{(j_1)})^{\mu}_{\mu_1}(\mathcal{R}^{(j_2)})^{\mu_1}_{\mu_2}\cdots(\mathcal{R}^{(j_p)})^{\mu_{p-1}}_{\nu}R^{k-j_1-\cdots-j_p}$ (where $0<j_a<k$ for $a=1,\cdots p$ and $j_1+\cdots+j_p<k$) when k is large enough, because the Schouten identity

$$\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}R^{\nu_1}_{\mu_1}\cdots R^{\nu_k}_{\mu_k}=0$$

holds identically for $k>n$. This implies that the expanded form (2) of f^{μ}_{ν} can be presented in other forms. However, to keep the action simple, we prefer that each term in the expanded form of $f\equiv f(R_{\mu\nu},g_{\mu\nu})=f^{\mu}_{\mu}$ consists of a single trace rather than of a product of multiple traces. This consideration makes the analysis of our model much simpler than the cases studied in [20, 21] and in [22]. Note also that the works [20, 21] studied only the cases in three dimensions. In a generic dimension, f (Ricci) gravity was considered in [23], but the subject there was mainly frame-mapping and the calculation of holographic entanglement entropy, which does not overlap with what we are doing in this paper.

The process of calculating the first variation of the action is a little bit involved. However, the result can be arranged in the following simple form:

$$\delta I_{\rm bulk}=\frac{1}{16\pi G}\int_{\mathcal{M}}{\rm d}^n x\,\sqrt{g}\,(H_{\mu\nu}\delta g^{\mu\nu}+\nabla_\mu\mathcal{B}^\mu)$$
$$=\frac{1}{16\pi G}\left(\int_{\mathcal{M}}{\rm d}^n x\,\sqrt{g}\,H_{\mu\nu}\delta g^{\mu\nu}+\int_{\partial\mathcal{M}}{\rm d}^{n-1}x\,\sqrt{g_{(n-1)}}\,n_\mu\mathcal{B}^\mu\right), \tag{3}$$

where

$$H_{\mu\nu}\equiv-\frac{1}{2}fg_{\mu\nu}+f'_{(\mu^{\sigma}R_{\nu)\sigma}}+\frac{1}{2}\Box f'_{\mu\nu}-\nabla_\rho\nabla_{(\mu}f'^{\ \rho}_{\nu)}+\frac{1}{2}\nabla_\rho\nabla_\sigma f'^{\rho\sigma}g_{\mu\nu}, \tag{4}$$

$$\mathcal{B}^\mu=f'^{\rho\nu}\delta\Gamma^{\mu}_{\rho\nu}-f'^{\rho\mu}\delta\Gamma^{\nu}_{\nu\rho}+\nabla_\rho f'^{\mu}_{\ \sigma}\delta g^{\rho\sigma}-\frac{1}{2}\nabla^\mu f'_{\rho\sigma}\delta g^{\rho\sigma}$$
$$-\frac{1}{2}g_{\rho\sigma}\nabla_\nu f'^{\mu\nu}\delta g^{\rho\sigma}, \tag{5}$$

$g_{(n-1)}$ is the absolute value of the determinant of the induced metric on the spacetime boundary $\partial\mathcal{M}$, n_μ is the unit outer-pointing normal covector of $\partial\mathcal{M}$, and f'^{μ}_{ν} is defined in the spirit of f^{μ}_{ν} with $f(x)$ replaced by $f'(x)$. In Eq. (3), the term involving $n_\mu \mathcal{B}^\mu$ cannot be simply dropped by imposing the fixed boundary condition $\delta g_{\mu\nu}|_{\partial\mathcal{M}}=0$ because the first two terms on the right hand side of Eq. (5) contain derivatives of $\delta g_{\mu\nu}$ rather than $\delta g_{\mu\nu}$ itself. To get rid of this redundant term, some boundary action must be introduced. One way to achieve this is to adopt the method introduced in [24], which works for any higher curvature gravity with the Lagrangian density composed of the Riemann tensor and the metric. Schematically, one introduces two auxiliary fields, $\phi_{\mu\nu\rho\sigma}$ and $\psi_{\mu\nu\rho\sigma}$, to rearrange the bulk action into a form which is linear in the Riemann tensor,

$$I_{\rm bulk}=\int_{\mathcal{M}}\mathrm{d}^n x\,\sqrt{g}\left[f(\phi_{\mu\nu\rho\sigma},g_{\mu\nu})-\psi^{\mu\nu\rho\sigma}(\phi_{\mu\nu\rho\sigma}-R_{\mu\nu\rho\sigma})\right],\tag{6}$$

where $f(\phi_{\mu\nu\rho\sigma},g_{\mu\nu})$ is nothing else but $f(R_{\mu\nu},g_{\mu\nu})$ with all occurrences of the Riemann tensor $R_{\mu\nu\rho\sigma}$ replaced by $\phi_{\mu\nu\rho\sigma}$. Then, following a systematic variation process, one finds that the only source of boundary action comes from the variation of $R_{\mu\nu\rho\sigma}$. Finally, the boundary action can be written as

$$I_{bdry}=\frac{1}{16\pi G}\int_{\partial\mathcal{M}}\mathrm{d}^{n-1}x\,\sqrt{g_{(n-1)}}\,\mathcal{L}_{\rm B},\tag{7}$$

where the boundary Lagrangian density $\mathcal{L}_{\rm B}$ is given by

$$\mathcal{L}_{\rm B}=f'^{\mu\nu}K_{\mu\nu}-2f'^{\mu\nu}n_\nu n^\rho K_{\rho\mu}+f'^{\mu\nu}n_\mu n_\nu K,\tag{8}$$

wherein $K_{\mu\nu}=\nabla_\mu n_\nu$ is the extrinsic curvature tensor and K is the trace of $K_{\mu\nu}$.

It is common knowledge that the choice of a boundary action is not unique: two different boundary actions that differ only in some terms vanishing on the boundary will equally well make the variational problem self consistent. In our case, instead of Eq. (6), we can make the bulk action linear in the Ricci tensor rather than in the Riemann tensor by writing

$$I_{\rm bulk}=\int_{\mathcal{M}}\mathrm{d}^n x\,\sqrt{g}\left[f(\phi_{\mu\nu},g_{\mu\nu})-\psi^{\mu\nu}(\phi_{\mu\nu}-R_{\mu\nu})\right].\tag{9}$$

Then, a similar procedure yields the boundary action (7) with $\mathcal{L}_{\rm B}$ replaced by

$$\mathcal{L}_{\rm B}=n_\mu\left(f'^{\rho\mu}\Gamma^{\nu}_{\nu\rho}-f'^{\rho\nu}\Gamma^{\mu}_{\rho\nu}\right).\tag{10}$$

With the aid of either Eq. (8) or Eq. (10), we can make the variational problem of our model consistent, and hence it follows that the equation of motion of the model is simply

$$H_{\mu\nu}=0.\tag{11}$$

Recalling the form (4) of $H_{\mu\nu}$ and assuming that the equation of motion admits an Einstein metric obeying

$$R_{\mu\nu}=\chi g_{\mu\nu}\tag{12}$$

as an exact solution, where the constant χ is related to the cosmological constant Λ via $\chi=\dfrac{2\Lambda}{n-2}$, it follows from Eq. (11) that χ is a solution of the following algebraic equation:

$$\chi f'(\chi)-\frac{n}{2}f(\chi)=0.\tag{13}$$

Let us stress that this is an algebraic equation for χ and not a differential equation for $f(\chi)$, because $f(x)$ is prescribed when defining the model. However, if one happens to chose the function $f(x)$ proportional to $x^{n/2}$, then the above equation is identically satisfied for any value of χ. In the particular case of $n=4$ this equals to $f(x)\sim x^2$, and the corresponding model is simply the Ricci squared gravity with bulk Lagrangian density $\mathcal{L}\propto\mathcal{R}^{(2)}$. For a generic choice of $f(x)$, Eq. (13) is the necessary and sufficient condition in order that the Einstein manifolds (12) can be the exact solutions of our model, and the allowed values of χ may not be unique. This fact has already become evident in the earlier study [7] of the Ricci polynomial gravity.

3 Perturbative properties around Einstein vacua

In this section we analyze the perturbative properties of f (Ricci) gravity around the background Einstein metric $g_{\mu\nu}$ satisfying the equation $R_{\mu\nu}(g)=\chi g_{\mu\nu}$. The metric with fluctuation may be denoted by $\tilde{g}_{\mu\nu}=g_{\mu\nu}+h_{\mu\nu}$, where $h_{\mu\nu}$ is a small deviation from the background metric. It is customary to denote $\nabla_\nu h^{\mu\nu}=A_\mu$, where ∇_μ is the covariant derivative compatible with $g_{\mu\nu}$. We also denote the traceless part of $h_{\mu\nu}$ by $\bar{h}_{\mu\nu}=h_{\mu\nu}-\dfrac{1}{n}hg_{\mu\nu}$, where $h=g^{\mu\nu}h_{\mu\nu}$.

Up to the linear term in $h_{\mu\nu}$, the perturbed equation of motion reads

$$\begin{aligned}\delta H_{\mu\nu}=&-\left[\frac{1}{2}nf(\chi)+\chi^2f''(\chi)\right]h_{\mu\nu}\\&+\frac{1}{2}g_{\mu\nu}f'(\chi)\left[\chi h+\frac{1}{2}g^{\rho\sigma}\Delta_{\rm L}h_{\rho\sigma}\right]\\&-\frac{1}{2}\left[\chi f''(\chi)+f'(\chi)\right]\Delta_L h_{\mu\nu}\\&+f''(\chi)\left[\frac{1}{2}\nabla^\rho\nabla_{(\mu}\Delta_{\rm L}h_{\nu)\rho}+\chi\nabla^\rho\nabla_{(\mu}h_{\nu)\rho}\right]\\&-\frac{1}{2}f''(\chi)g_{\mu\nu}\left[\frac{1}{2}\nabla^\rho\nabla^\sigma\Delta_{\rm L}h_{\rho\sigma}+\chi\nabla^\rho\nabla^\sigma h_{\rho\sigma}\right]\\&-\frac{1}{2}f''(\chi)\left[\frac{1}{2}\Box\Delta_{\rm L}h_{\mu\nu}+\chi\Box h_{\mu\nu}\right]=0,\end{aligned}\tag{14}$$

where

$$\Delta_{\rm L}h_{\mu\nu}=\Box h_{\mu\nu}+\nabla_\mu\nabla_\nu h-2[\nabla_{(\mu}A_{\nu)}+R^{\rho}_{(\mu h_\nu)\rho}-R_{\mu\ \nu}^{\ \rho\ \sigma}h_{\rho\sigma}].\tag{15}$$

The operator $\Delta_{\rm L}$ is known as the Lichnerowicz operator.

We are particularly interested in the perturbative behavior around maximally symmetric vacua, because the linearized equation of motion can be further simplified using the following properties which any maximally symmetric vacuum must obey:

$$R_{\mu\nu\rho\sigma}=\frac{\chi}{n-1}[g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}],\quad R_{\mu\nu}=\chi g_{\mu\nu},\quad R=n\chi. \tag{16}$$

In such backgrounds, the action of the Lichnerowicz operator on $h_{\mu\nu}$ is simplified to

$$\Delta_{\rm L}h_{\mu\nu}=\Box h_{\mu\nu}+\nabla_\mu\nabla_\nu h-2\nabla_{(\mu}A_{\nu)}+\frac{2\chi}{n-1}g_{\mu\nu}h-\frac{2n\chi}{n-1}h_{\mu\nu}.$$

In the following, two sub-cases with $\chi=0$ and $\chi\neq 0$ are considered.

3.1 Minkowski background with $\chi=0$

The Minkowski spacetime corresponds to the choice $\chi=0$. According to Eq. (13), the Einstein manifold with $\chi=0$ is the exact vacuum solution of f (Ricci) gravity if and only if $f(0)=0$. Therefore, for the Minkowski background, the linearized equation of motion can be greatly simplified:

$$\begin{aligned}\delta H_{\mu\nu}=&-\frac{1}{4}f''(0)\left[\Box^2h_{\mu\nu}+\eta_{\mu\nu}(\Box^2h-\Box\partial^\rho A_\rho)-\Box\partial_\mu\partial_\nu h\right.\\&\left.-2\Box\partial_{(\mu}A_{\nu)}+2\partial_{(\mu}\partial_{\nu)}\partial^\rho A_\rho)\right]-\frac{1}{2}f'(0)\\&\times\left[\Box h_{\mu\nu}-\eta_{\mu\nu}(\Box h-\partial^\rho A_\rho)+\partial_\mu\partial_\nu h-2\partial_{(\mu}A_{\nu)}\right].\end{aligned}\tag{17}$$

To further analyze the perturbative spectrum of the model, we consider two different cases, $f''(0)=0$ and $f''(0)\neq 0$. When $f''(0)=0$, the linearized equation of motion becomes the same as the linearized standard Einstein equation (we assume that $f'(0)\neq 0$ in this case), which, after choosing the transverse traceless gauge, describes a massless spin-2 field. The fluctuation around the Minkowski vacuum is tachyon-free, and the ghost-free condition reads $f'(0)>0$. On the other hand, if we set $f''(0)\neq 0$ and choose the gauge fixing condition $A_\mu=\frac{1}{n}\partial_\mu h$, the linearized equation of motion becomes

$$\delta H_{\mu\nu}=-\frac{1}{4}f''(0)\left[\Box(\Box-m_1^2)\bar{h}_{\mu\nu}+(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\left(\Box-m_2^2\right)h\right],\tag{18}$$

where

$$m_1^2=-\frac{2f'(0)}{f''(0)},\quad m_2^2=\frac{2(n-2)}{n}\frac{f'(0)}{f''(0)}.$$

In this case, if $f'(0)=0$, we have $m_1^2=m_2^2=0$, and hence $\bar{h}_{\mu\nu}$ and h respectively correspond to a massless spin-2 and a massless spin-0 mode after we impose the ghost-free condition $f''(0)>0$. If, however, $f'(0)\neq 0$, then either m_1^2 or m_2^2 is negative, and the corresponding mode is exactly what is referred to as the tachyon mode.

Thus, we conclude that for the Minkowski background, the ghost-free and tachyon-free condition is either $f''(0)=0$ with $f'(0)>0$, or $f''(0)>0$ with $f'(0)=0$. Note that in the special case of the Ricci polynomial gravity [7], we have fixed the coefficient in front of the first order term in the Lagrangian density to be unity, which implies $f'(0)=1$. Therefore, the latter tachyon-free condition $f''(0)\neq 0$ with $f'(0)=0$ was not seen in [7][1)].

3.2 (A)dS background

Maximally symmetric background with $\chi\neq 0$ is either de Sitter (dS) or anti-de Sitter (AdS). In this subsection we consider the perturbative spectrum around these two types of backgrounds.

In the (A)dS background, we can choose the gauge fixing condition $A_\mu=\nabla_\mu h$, and making use of Eq. (13) to change all occurrences of $f'(\chi)$ into multiples of $f(\chi)$, Eq. (14) becomes

$$\begin{aligned}\delta H_{\mu\nu}=&-\frac{1}{4}f''(\chi)\left[\Box^2h_{\mu\nu}-\nabla_{(\mu}\nabla_{\nu)}\Box h\right]\\&+\left[\frac{n}{2(n-1)}f(\chi)-\frac{1}{(n-1)^2}\chi^2f''(\chi)\right]h_{\mu\nu}\\&+\left[-\frac{n}{4}\frac{f(\chi)}{\chi}+\frac{1}{n-1}\chi f''(\chi)\right]\Box h_{\mu\nu}\\&-\frac{n+3}{4(n-1)}\chi f''(\chi)g_{\mu\nu}\Box h\\&+\left[\frac{n}{4}\frac{f(\chi)}{\chi}+\chi f''(\chi)\right]\nabla_\mu\nabla_\nu h\\&+\left[\frac{n(n-3)}{4(n-1)}f(\chi)-\frac{n-2}{(n-1)^2}\chi^2f''(\chi)\right]g_{\mu\nu}h=0.\end{aligned}\tag{19}$$

We again proceed by considering two choices, $f''(\chi)=0$ and $f''(\chi)\neq 0$.

When $f''(\chi)=0$, Eq. (19) can be simplified to

$$\begin{aligned}&\frac{n}{2(n-1)}f(\chi)h_{\mu\nu}-\frac{n}{4}\frac{f(\chi)}{\chi}\Box h_{\mu\nu}\\&\quad+\frac{n}{4}\frac{f(\chi)}{\chi}\nabla_\mu\nabla_\nu h+\frac{n(n-3)}{4(n-1)}f(\chi)hg_{\mu\nu}=0.\end{aligned}\tag{20}$$

Taking the trace by contraction with $g^{\mu\nu}$, we get

$$\frac{n(n-1)}{4}f(\chi)h=0,\tag{21}$$

i.e. $h=0$. Thus, Eq. (19) becomes

$$\delta H_{\mu\nu}=-\frac{n}{4}\frac{f(\chi)}{\chi}\left[\Box-\frac{2\chi}{n-1}\right]\bar{h}_{\mu\nu}=0,\tag{22}$$

from which we can say that $\bar{h}_{\mu\nu}$ is a massless spin-2 mode in the (A)dS background if we impose the ghost-free condition $\frac{f(\chi)}{\chi}>0$.

When $f''(\chi)\neq 0$, we can separate Eq. (19) into the

1) The terminology "ghost" in [7] should be the tachyon.

traceless and trace parts, i.e.

$$\delta H_{\mu\nu} = \delta H^{(1)}_{\mu\nu} + \delta H^{(2)}_{\mu\nu} = 0, \tag{23}$$

where

$$\delta H^{(1)}_{\mu\nu} = -\frac{1}{4} f''(\chi)\left(\Box - \frac{2\chi}{n-1} - m_1^2\right)\left(\Box - \frac{2\chi}{n-1} - m_2^2\right)\bar{h}_{\mu\nu}, \tag{24}$$

with

$$m_1^2 = 0, \quad m_2^2 = -n\frac{f(\chi)}{\chi f''(\chi)}, \tag{25}$$

and

$$\delta H^{(2)}_{\mu\nu} = -\frac{1}{4} f''(\chi)\left[\frac{1}{n} g_{\mu\nu}\Box - \nabla_{(\mu}\nabla_{\nu)}\right]\left[\Box + \left(\frac{nf(\chi)}{\chi f''(\chi)} + 4\chi\right)\right]h - \frac{1}{4}\chi f''(\chi) g_{\mu\nu}\left\{\Box - \left[\frac{(n-2)f(\chi)}{\chi f''(\chi)} - \frac{4\chi}{n}\right]\right\}h. \tag{26}$$

Taking the trace of the above equation, we get

$$\delta H^{(2)} = -\frac{n}{4}\chi f''(\chi)\left(\Box - m_3^2\right)h = 0, \tag{27}$$

where

$$m_3^2 = \frac{(n-2)f(\chi)}{\chi f''(\chi)} - \frac{4\chi}{n}. \tag{28}$$

Now, because $\bar{h}_{\mu\nu}$ and h are essentially independent, Eq. (23) implies both $\delta H^{(1)}_{\mu\nu} = 0$ and $\delta H^{(2)}_{\mu\nu} = 0$. It follows from the first of these two conditions that $\bar{h}_{\mu\nu}$ consists of two traceless spin-2 modes, one of which is massless and the other is either massive or a tachyon mode. The condition that $\bar{h}_{\mu\nu}$ does not contain a tachyon mode is $m_2^2 \geqslant 0$, or explicitly

$$\frac{f(\chi)}{\chi f''(\chi)} \leqslant 0. \tag{29}$$

The second condition, $\delta H^{(2)}_{\mu\nu} = 0$, looks more complicated. According to Eq. (27), the last term in Eq. (26) can be dropped. Then the remaining terms in Eq. (26) imply that h is subject to some extra constraints in addition to the wave-like Eq (27), unless the following condition is satisfied,

$$m_3^2 = \frac{(n-2)f(\chi)}{\chi f''(\chi)} - \frac{4\chi}{n} = -\left(\frac{nf(\chi)}{\chi f''(\chi)} + 4\chi\right), \tag{30}$$

in which case Eq. (26) can be rewritten in a completely factorized form, i.e.

$$\delta H^{(2)}_{\mu\nu} = -\frac{1}{4} f''(\chi)\left[\frac{1}{n} g_{\mu\nu}\Box - \nabla_{(\mu}\nabla_{\nu)} + \chi g_{\mu\nu}\right]\left(\Box - m_3^2\right)h. \tag{31}$$

Eq. (30) can be simplified into

$$\frac{f(\chi)}{f''(\chi)} = -\frac{2\chi^2}{n}. \tag{32}$$

Now, since the condition that h does not correspond to a scalar tachyon is

$$m_3^2 = \frac{(n-2)f(\chi)}{\chi f''(\chi)} - \frac{4\chi}{n} \geqslant 0,$$

which gives

$$\chi \leqslant \frac{n(n-2)}{4}\frac{f(\chi)}{\chi f''(\chi)}, \tag{33}$$

we have from Eq. (29) that $\chi < 0$ and

$$\frac{f(\chi)}{f''(\chi)} > 0. \tag{34}$$

However, Eq. (32) requires $\frac{f(\chi)}{f''(\chi)} < 0$. Therefore, we conclude that for $f''(\chi) \neq 0$, the model cannot be tachyon-free around maximally symmetric Einstein vacua with $\chi \neq 0$. In other words, f (Ricci) gravity can only be ghost-free and tachyon-free around maximally symmetric Einstein vacua when $f''(\chi) = 0$ and $\frac{f(\chi)}{\chi} > 0$. Let us recall that the condition $f''(\chi) = 0$ is identical to the tachyon-free condition presented in [7] for the special case of the Ricci polynomial gravity.

4 Black hole entropy

It is clear from Sec. 2 that provided the condition (13) is satisfied, an Einstein manifold obeying Eq. (12) is a vacuum solution of our model. In this section, we are particularly interested in the spherically symmetric black hole solutions and concentrate on the calculation of black hole entropy for such solutions. Unlike the standard GR, the commonly acknowledged form of the black hole entropy associated with higher curvature gravity is not the Bekenstein-Hawking entropy, but rather the Wald geometric entropy. We will show that the Wald entropy is identical to the holographic entropy calculated using the MP approach.

For any higher curvature gravity with bulk action

$$I_{\rm bulk} = \int {\rm d}^n x \sqrt{g}\, L(g_{\mu\nu}, R_{\mu\nu\rho\sigma}),$$

the Wald entropy associated with a spherically symmetric black hole solution of the form (here we take a coordinate system $x^\mu = (x^0, x^1, \cdots x^{n-1}) = (t, r, x^2, \cdots x^{n-1})$)

$${\rm d}s^2 = -\tilde{f}(r){\rm d}t^2 + \frac{1}{\tilde{f}(r)}{\rm d}r^2 + r^2\Omega_{ij}{\rm d}x^i{\rm d}x^j, \quad (i,j = 2,\cdots,n-1) \tag{35}$$

can be evaluated via the following formula:

$$S_{\rm Wald} = -2\pi\int {\rm d}^{n-2}x \sqrt{g_{(n-2)}}\frac{\delta L}{\delta R_{\alpha\beta\gamma\delta}}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}, \tag{36}$$

where Ω_{ij} is the metric on a $(n-2)$-dimensional unit sphere and $g_{(n-2)} = |\det(r_h^2\Omega_{ij})|$, where r_h is the radius of the black hole event horizon, and $\epsilon_{\alpha\beta}$ is given via $\epsilon_{01} = -\epsilon_{10} = 1$. Note that the integration in the above formula is over the compact spacial section of the horizon hypersurface, a sphere with radius r_h.

In our case, $L=\dfrac{1}{16\pi G}f(R_{\mu\nu},g_{\mu\nu})$. Therefore, the Wald entropy (36) becomes

$$\begin{aligned}S_{\rm Wald}&=-\frac{1}{8G}\int \mathrm{d}^{n-2}x\,\sqrt{g_{(n-2)}}\frac{\delta f}{\delta R_{\alpha\beta\gamma\delta}}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\\&=-\frac{1}{8G}\int \mathrm{d}^{n-2}x\,\sqrt{g_{(n-2)}}f'^{\rho\sigma}g^{\xi\eta}\frac{\delta R_{\rho\xi\sigma\eta}}{\delta R_{\alpha\beta\gamma\delta}}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}.\end{aligned}\quad(37)$$

Using the identity [25]

$$\begin{aligned}\frac{\delta R_{a_1b_1c_1d_1}}{\delta R_{abcd}}=\frac{1}{12}\Big(&\delta^{ab}_{a_1b_1}\delta^{cd}_{c_1d_1}-\frac{1}{2}\delta^{ac}_{a_1b_1}\delta^{db}_{c_1d_1}-\frac{1}{2}\delta^{ad}_{a_1b_1}\delta^{bc}_{c_1d_1}\\&+\delta^{cd}_{a_1b_1}\delta^{ab}_{c_1d_1}-\frac{1}{2}\delta^{db}_{a_1b_1}\delta^{ac}_{c_1d_1}-\frac{1}{2}\delta^{bc}_{a_1b_1}\delta^{ad}_{c_1d_1}\Big)\end{aligned}$$

and the explicit form of $\epsilon_{\alpha\beta}$, we can reduce the expression for the Wald entropy into the final result

$$S_{\rm Wald}=\frac{A}{8G}\left(f'^{0}_{\ 0}+f'^{1}_{\ 1}\right)_{r\to r_h},\quad(38)$$

where

$$A=\int \mathrm{d}^{n-2}x\,\sqrt{g_{(n-2)}}|_{r=r_h}\quad(39)$$

is the area of the event horizon. It is trivial to verify that for the standard GR, the above result reduces to the Bekenstein-Hawking entropy $S=\dfrac{A}{4G}$.

Let us now consider the black hole entropy from a holographic point of view. There are several methods to pursue the holographic calculation of black hole entropy. The earliest attempt was made by Brown and Henneaux [12], who successfully calculated the asymptotic Virasoro symmetry for three dimensional AdS spacetime by assuming some mild boundary conditions. Their work was further extended by Strominger [13, 14] and Carlip [15, 16] to the case of black hole spacetimes, and using the Cardy formula [17], they were able to calculate the entropy of various black holes. In this approach, the bulk action must be used. Alternatively, the MP approach makes use only of the boundary action, and the process of obtaining the black hole entropy is much simpler. Therefore, we adopt the MP approach in the following calculation.

The MP approach is applicable to any metric-based geometric theory of gravity. The key ingredient in this approach is the boundary Noether charge associated with the asymptotic diffeomorphism $x^\mu\to x^\mu+\xi^\mu(x)$, where the spacetime boundary $\partial\mathcal{M}$ is taken to be the near-horizon hypersurface. For the generic boundary action of the form (7), the boundary Noether charge can be evaluated via

$$Q[\xi]=\frac{1}{2}\int \sqrt{h}\,\mathrm{d}\Sigma_{\mu\nu}J^{\mu\nu},\quad(40)$$

where

$$\mathrm{d}\Sigma_{\mu\nu}=\mathrm{d}^{n-2}x(n_\mu m_\nu-m_\mu n_\nu)$$

is the area element of the constant-time slice Σ of the near-horizon hypersurface $\partial\mathcal{M}$, n_μ and m_μ are respectively the unit outer-pointing spacelike normal covector and the unit future pointing timelike normal covector of Σ, h is the determinant of the induced metric on Σ (which reduces to $g_{(n-2)}$ described above in the near-horizon limit), and

$$J^{\mu\nu}[\xi]=\frac{1}{16\pi G}\mathcal{L}_B[\xi^\mu n^\nu-\xi^\nu n^\mu]\quad(41)$$

is known as the Noether potential [8–11]. Note that n_μ is identical to the unit normal covector that appeared in the boundary action. In the following, we take the boundary Lagrangian density (8) as the working example, and leave it to the reader to check that the alternative boundary Lagrangian density (10) works equally well.

Using the local Rindler coordinate $\rho=r-r_h$, the metric (35) can be rewritten as

$$\mathrm{d}s^2=-\tilde f(r_h+\rho)\mathrm{d}t^2+\frac{1}{\tilde f(r_h+\rho)}\mathrm{d}\rho^2+r^2\Omega_{ij}\mathrm{d}x^i\mathrm{d}x^j.\quad(42)$$

In this spacetime, the unit normal vectors of the black hole event horizon can be chosen as

$$\begin{aligned}n^\mu&=\left(0,\sqrt{\tilde f(r)},0,\cdots,0\right),\\ m^\mu&=\left(\frac{1}{\sqrt{\tilde f(r)}},0,0,\cdots,0\right),\end{aligned}\quad(43)$$

and the generator ξ^μ of the boundary diffeomorphism is

$$\xi^t=T-\frac{\rho}{\tilde f(r_H+\rho)}\partial_t T,\quad \xi^\rho=-\rho\partial_t T,\quad(44)$$

where $T=T(t,\rho)$ is an arbitrary function. We can expand T in terms of a set of basis functions

$$T=\sum a_m T_m,\quad(45)$$

where the basis functions T_m satisfy the Diff(S^1) algebra. A standard choice is

$$T_m=\frac{1}{\alpha}\exp\left[im(\alpha t+g(\rho)+p\cdot x)\right],\quad(46)$$

where α is a constant, p is an integer, and $g(\rho)$ is a regular function on the horizon. Clearly, we must have $a^*_m=a_{-m}$ in order to make T real.

With the above prescription, the boundary Noether charge can be expressed as

$$Q[\xi]=\frac{1}{16\pi G}\int \mathrm{d}^{n-2}x\,\sqrt{g_{(n-2)}}\left[f'^t_{\ t}+f'^\rho_{\ \rho}\right]\Big|_{\rho\to0}\left(\kappa T-\frac{1}{2}\partial_t T\right)\quad(47)$$

in the near horizon limit $\rho\to0$. The commutator between the two charges is then given by

$$[Q[\xi_1],Q[\xi_2]]=\frac{1}{16\pi G}\int \mathrm{d}^{n-2}x\sqrt{g_{(n-2)}}\left[f'^{t}_{t}+f'^{\rho}_{\rho}\right]\Big|_{\rho\to 0}\times\left[\kappa(T_1\partial_t T_2-T_2\partial_t T_1)-\frac{1}{2}\left(T_1\partial_t^2 T_2-T_2\partial_t^2 T_1\right)+\frac{1}{4\kappa}\left(\partial_t T_1\partial_t^2 T_2-\partial_t T_2\partial_t^2 T_1\right)\right],\tag{48}$$

where $\kappa=\frac{\tilde{f}'(r_h)}{2}$. Denoting by Q_m the mode corresponding to T_m, we have

$$Q_m=\frac{1}{16\pi G}\left[f'^{t}_{t}+f'^{\rho}_{\rho}\right]\Big|_{\rho\to 0}\frac{\kappa A}{\alpha}\delta_{m,0},\tag{49}$$

$$[Q_m,Q_n]=-\mathrm{i}(m-n)Q_{m+n}-\mathrm{i}m^3\frac{C}{12}\delta_{m+n,0},\tag{50}$$

where A is given in Eq. (39), and

$$C=\frac{3}{8\pi G}\left[f'^{t}_{t}+f'^{\rho}_{\rho}\right]\Big|_{\rho\to 0}\left(\frac{\alpha A}{\kappa}\right).\tag{51}$$

Note that the commutator (50) is actually the famous Virasoro algebra with the generator Q_0 shifted by a constant, where the central charge C is given by Eq. (51). This observation implies that the well-known Cardy formula is applicable in the present case. Finally, using the Cardy formula, the entropy of the black hole is evaluated to be

$$S=2\pi\sqrt{\frac{CQ_0}{6}}=\frac{A}{8G}\left[f'^{t}_{t}+f'^{\rho}_{\rho}\right]\Big|_{\rho\to 0}.\tag{52}$$

This result is in exact agreement with the Wald entropy (38).

Before concluding, it may be interesting to make a comparison between the black hole entropy for f (Ricci) gravity and $f(R)$ gravity for the same $f(x)$ and the same black hole metric. To be more specific, we consider the Tangherlini-(A)dS black hole solution

$$\mathrm{d}s^2=-\left(1-\frac{2M}{r^{n-3}}-\frac{\chi r^2}{n-1}\right)\mathrm{d}t^2+\left(1-\frac{2M}{r^{n-3}}-\frac{\chi r^2}{n-1}\right)^{-1}\mathrm{d}r^2+r^2\Omega_{ij}\mathrm{d}x^i\mathrm{d}x^j,\tag{53}$$

which obviously obeys (12). For this solution, the corresponding entropy in f (Ricci) gravity, i.e. Eq. (38) or (52), reduces to

$$S_{f(\mathrm{Ricci})}=\frac{A}{4G}f'(\chi).\tag{54}$$

In comparison, the same black hole solution in $f(R)$ gravity has the entropy [11]

$$S_{f(R)}=\frac{A}{4G}f'(R)=\frac{A}{4G}f'(n\chi).\tag{55}$$

5 Conclusions

f (Ricci) gravity is an extension of the Ricci polynomial gravity proposed in [7]. Besides similar results as in [7], we actually obtain several results beyond the Ricci polynomial gravity. First, our analysis of the Einstein metric solutions, perturbative modes and surface terms are so generic that they allow to study more complicated models such as $f(x)=\mathrm{e}^x$, which can not be written as a polynomial function, and to study the total effects brought by the infinite higher curvature terms. Second, we distinguish manifestly in this paper between the two similar but different physical concepts, "tachyon" and "ghost", while the "ghost-free" conditions in [7] are in fact "tachyon-free" conditions. Third, we find all tachyon-free and ghost-free conditions for the perturbative modes in (A)dS space, some of which were not found in [7], and the present results look more compact and elegant. Finally, beyond the perturbative regime, we calculated the entropy of spherically symmetric black hole solutions using the Wald entropy formula and the MP approach, and the results are in good agreement. A similar analysis was performed for the BTZ black hole solutions for f (Ricci) gravity in three-dimensions [26]. We expect that f (Ricci) gravity may become a competitive gravitational model in future studies.

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