

Gupta-Bleuler triplet for massless spin- $\frac{3}{2}$ field in de Sitter space

S. Parsamehr¹⁾

Department of Physics, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran

Abstract: Physicists have been interested in quantization of spinor and vector free fields in 4-dimensional de Sitter space-time, in ambient space notation. The Gupta-Bleuler formalism has been extensively applied to the quantization of gauge invariant theories. The field equation of the massless spin- $\frac{3}{2}$ fields is gauge invariant in de Sitter space. In this paper, we study the quantization of massless spin- $\frac{3}{2}$ gauge fields in de Sitter space-time by the Gupta-Bleuler formalism. This triplet carries an indecomposable representation of the de Sitter group.

Keywords: Gupta-Bleuler triplet, quantization, de Sitter space-time, massless spin- $\frac{3}{2}$

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1 Introduction

The quantization of light was discovered in 1900 by Max Planck [1]. Quantum field theory also applies to the electromagnetic field, but some problems with negative norm and supernumerary degrees of freedom have to be solved. As discussed in electrodynamics, a vector field has four components that correspond to four sets of generation and destruction operators. The field equation and requirements for relativity would imply states with negative norm, which are unphysical. Moreover, four sets of generation and destruction operators would imply four degrees of freedom, i.e. four kinds of photons. But experimentally, only two kinds of photons, the two polarization states, are found. Gupta and Bleuler invented a procedure [2, 3] that declares just a subset of the states as physical reality. It eliminates states with negative norm and the supernumerary degrees of freedom in one shot.

De Sitter space is the solution to the Einstein equations with a positive cosmological constant and no other matter sources. It was proposed in 1917 by W. de Sitter [4]. Although this model is physically unrealistic, it introduced the idea that the real Universe might be expanding. An expansion phase which is very similar to that in the de Sitter model also plays an important role in modern theories of the inflationary universe. The astonishing result of cosmic acceleration was discovered in 1998 employing a distance indicator method similar to that used by Hubble, but using the very bright SNIa as accurate standard candles to measure the evolution of the expansion rate at large distances [5–11]. There is an extensive body of literature on quantum field theory in

de Sitter space-times [12–16].

In supergravity theories, combining general relativity and supersymmetry, the gravitino is the gauge fermion supersymmetric partner of the hypothesized graviton. It has been suggested as a candidate for dark matter [17]. The gravitino is the fermion mediating supergravity interactions, just as the photon mediates electromagnetism, and the graviton presumably mediates gravitation. Whenever supersymmetry is broken in supergravity theories, it acquires a mass which is determined by the scale at which supersymmetry is broken.

According to Wigner [18], identifying elementary particles as unitary irreducible representations (UIRs) of the de Sitter group is important. The UIRs of the de Sitter group were completely extracted by Takahashi [19]. Particles should be described by irreducible positive energy representation of the Poincaré group. In fact, they are the indecomposable building blocks of those multi-localized asymptotically stable objects in terms of which each state can be interpreted and measured in counter coincidence in the large time limit.

The field equation of massless spin- $\frac{3}{2}$ fields is gauge invariant in de Sitter space, as well as that of massless fields in Minkowski space for $s > 1$. The quantization of gauge invariant theories usually requires the Gupta-Bleuler quantization. Let us now define the Gupta-Bleuler triplet $V_g \subset V \subset V_c$ carrying the indecomposable structure for the unitary irreducible representations of the de Sitter group appearing in our study. V_c and V_g are the spaces of the gauge dependent states and the pure gauge states, respectively. In this formalism, V stands for the space of solutions with the divergencelessness condition [20, 21]. This subspace V is invariant but

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1) E-mail: sajadparsamehr@iauksh.ac.ir

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not invariantly complemented in V_c . The space V/V_g is a vector space containing the physical states, which is the Hilbert space constructed by the corresponding UIR of the de Sitter group. The structures of unphysical states V_g and V_c are obtained by employing the gauge invariant transformation and the gauge fixing field equation, respectively. The gauge invariant field equation and the gauge invariant transformation are obtained by using the Casimir operator of the de Sitter group.

In this paper, we explicitly work out some of the calculations which appear in Ref. [21]. In Section 2, we briefly recall the notation of de Sitter ambient space. The Gupta-Bleuler triplet is presented in Section 3. Finally, a brief conclusion and outlook are given in Section 4. In the appendices, some useful relations are presented.

2 Notation

The most symmetric vacuum solution to Einstein's equation is flat space-time. If we now add the cosmological constant as the only source of curvature in Einstein's equation, the resulting space-time is also highly symmetric and has an interesting geometrical structure. In the case of a positive cosmological constant, this is the de Sitter manifold. De Sitter space-time is best visualized as a 4-dimensional hyperboloid embedded in 5-dimensional flat Minkowski space-time:

$$X_H = \{x \in \mathbb{R}^5 \mid x \cdot x = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2}\}, \alpha, \beta = 0, 1, 2, 3, 4, \quad (1)$$

where $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$. The metric in de Sitter space-time is as follows:

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta \Big|_{x^2 = -H^{-2}} = g_{\mu\nu}^{dS} dX^\mu dX^\nu, \mu = 0, 1, 2, 3, \quad (2)$$

$$g_{\mu\nu}^{dS} = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu} \theta_{\alpha\beta} \quad (3)$$

where the de Sitter intrinsic coordinate is denoted by X^μ and the five dimensional Minkowski space-time by x^α . $\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ is the projection tensor that projects a vector in ambient space notation orthogonal to x_α . The de Sitter group has ten infinitesimal generators $L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}$ [14]. The orbital part $M_{\alpha\beta}$ is defined by:

$$M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) = -i(x_\alpha \partial_\beta^\top - x_\beta \partial_\alpha^\top), \quad (4)$$

where $\partial_\beta^\top = \theta_\beta^\alpha \partial_\alpha$ is the transverse derivative ($x \cdot \partial^\top = 0$). The form of the spinorial part $S_{\alpha\beta}$ for a spin $\frac{1}{2}$ field is [14, 22]:

$$S_{\alpha\beta} = -\frac{i}{4} [\gamma_\alpha, \gamma_\beta], \quad (5)$$

where the γ -matrices have to satisfy the basic Clifford algebra relation:

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta}, \gamma^{\alpha\dagger} = \gamma^0 \gamma^\alpha \gamma^0. \quad (6)$$

The de Sitter group has two Casimir operators:

$$Q^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta}, \alpha, \beta = 0, 1, 2, 3, 4, \quad (7)$$

$$Q^{(2)} = -W_\alpha W^\alpha, W_\alpha = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} L^{\beta\gamma} L^{\delta\eta}, \quad (8)$$

where $\epsilon_{\alpha\beta\gamma\delta\eta}$ is the usual antisymmetrical tensor. The Casimir operators are simple to manipulate in ambient space notation. Since $Q^{(1)}$ is a second order derivative operator, it is convenient to use for obtaining the field equation. The action of the Casimir operator $Q_j^{(1)}$ on a vector-spinor field $\Psi_\alpha(x)$ is [14]:

$$Q_j^{(1)} \Psi(x) = \left(-\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} + \frac{i}{2} \gamma_\alpha \gamma_\beta M^{\alpha\beta} - \frac{11}{2} \right) \Psi_\alpha(x) - 2\partial_\alpha x \cdot \Psi(x) + 2x_\alpha \partial \cdot \Psi(x) + \gamma_\alpha (\gamma \cdot \Psi(x)). \quad (9)$$

The Casimir operators commute with the generators of the group and as a consequence, they are constant on each UIR of the de Sitter group.

3 Gupta-Bleuler triplet

One can construct the Gupta-Bleuler triplet for a massless spin- $\frac{3}{2}$ field according to the method below. The gauge fixing field equation for a massless spin- $\frac{3}{2}$ field can be given in the following form [21]

$$\left(Q_{\frac{3}{2}}^{(1)} + \frac{5}{2} \right) \Psi_\alpha(x) + c \nabla_\alpha^\top \partial^\top \cdot \Psi(x) = 0, \quad (10)$$

where $\nabla_\alpha^\top = (\partial_\alpha^\top + \gamma_\alpha \not{x} - x_\alpha)$ [21], c is the gauge fixing parameter, and

$$Q_{\frac{3}{2}}^{(1)} \Psi_\alpha = Q_0^{(1)} \Psi_\alpha + \not{x} \not{\partial}^\top \Psi_\alpha + 2x_\alpha \partial^\top \cdot \Psi - \frac{11}{2} \Psi_\alpha + \gamma_\alpha \not{\Psi}, \quad (11)$$

where $Q_0^{(1)} = -\partial_\alpha^\top \partial^{\alpha\top}$ is the ‘‘scalar’’ Casimir operator. The elements of the subspace V_g are the gauge solutions [21]. This is an invariant subspace, but not a complement, of V . Putting $\Psi_\alpha^g = \nabla_\alpha^\top \psi^p$ (where p stands for a pure gauge state) into Eq. (10):

$$Q_{\frac{3}{2}}^{(1)} \nabla_\alpha^\top \psi^p + \frac{5}{2} \nabla_\alpha^\top \psi^p + c \nabla_\alpha^\top \partial^\top \cdot \nabla_\alpha^\top \psi^p = 0 \quad (12)$$

Here, ψ^p is a spinor field. By using the following identities [15, 21, 23]:

$$Q_{\frac{3}{2}}^{(1)} \nabla_\alpha^\top = \nabla_\alpha^\top Q_{\frac{1}{2}}^{(1)}, \partial^\top \cdot \nabla^\top \psi = - \left(Q_{\frac{1}{2}}^{(1)} + \frac{5}{2} \right) \psi, \quad (13)$$

we have

$$\nabla_\alpha^\top \left(Q_{\frac{1}{2}}^{(1)} \psi^p + \frac{5}{2} \psi^p + c \partial^\top \cdot \nabla_\alpha^\top \psi^p \right) = 0, \quad (14)$$

or

$$\left(Q_{\frac{1}{2}}^{(1)} + \frac{5}{2} \right) \psi^p + c \partial^\top \cdot \nabla_\alpha^\top \psi^p = 0, \quad (15)$$

where the second term is:

$$\begin{aligned}\partial^\top \cdot \nabla^\top \psi^p &= \partial^{\top\alpha} \nabla_\alpha^\top \psi^p = \partial^{\top\alpha} [\partial_\alpha^\top \psi^p + \gamma_\alpha \not{x} \psi^p - x_\alpha \psi^p] \\ &= -Q_0 \psi^p - \not{x} \not{\partial}^\top \psi^p.\end{aligned}\quad (16)$$

The second order Casimir operator for spin- $\frac{1}{2}$ is given by [14, 22]

$$Q_{\frac{1}{2}}^{(1)} \psi = \left(Q_0 + \not{x} \not{\partial}^\top - \frac{5}{2} \right) \psi. \quad (17)$$

Therefore, Eq. (15) can be written

$$\left(Q_{\frac{1}{2}}^{(1)} + \frac{5}{2} \right) \psi^p + c \left(-Q_0 - \not{x} \not{\partial}^\top \right) \psi^p = 0. \quad (18)$$

Otherwise, according to Equations (17) and (18) we have

$$(1-c) \left(Q_0 + \not{x} \not{\partial}^\top \right) \psi^p = (1-c) \left(Q_{\frac{1}{2}}^{(1)} + \frac{5}{2} \right) \psi^p = 0, \quad (19)$$

which is reduced to

$$(1-c) \nabla_\alpha^\top \left(Q_0 + \not{x} \not{\partial}^\top \right) \psi^p = 0, \quad (20)$$

or

$$(1-c) \nabla_\alpha^\top \left(Q_{\frac{1}{2}}^{(1)} + \frac{5}{2} \right) \psi^p = 0. \quad (21)$$

At this stage one must distinguish the two cases $c=1$ and $c \neq 1$ [21]. In the first case, $c=1$, the spinor field ψ^p is arbitrary and unlimited and the gauge state space is given by vector-spinor fields of the form $\nabla_\alpha^\top \psi^p$. In the second case, $c \neq 1$, ψ^p obeys the following field equation

$$\left(Q_{\frac{1}{2}}^{(1)} + \frac{5}{2} \right) \psi^p = 0, \quad (22)$$

which carries the representation $\Pi_{\frac{1}{2}, -\frac{1}{2}}$ which is unphysical [15].

The spinor states belong to the quotient space V_c/V . In order to characterize them, let us take the divergence of Eq. (10)

$$\partial^{\top\alpha} \left[\left(Q_{\frac{3}{2}}^{(1)} + \frac{5}{2} \right) \Psi_\alpha(x) + c \nabla_\alpha^\top \partial^\top \cdot \Psi(x) \right] = 0. \quad (23)$$

Applying the Casimir operator $Q_{\frac{3}{2}}^{(1)}$ to the vector-spinor field, we have

$$\begin{aligned}\partial^{\top\alpha} \left[Q_0 \Psi_\alpha + \not{x} \not{\partial}^\top \Psi_\alpha - \frac{11}{2} \Psi_\alpha + 2x_\alpha \partial^\top \cdot \Psi \right] \\ + \frac{5}{2} \partial^\top \cdot \Psi + c \partial^{\top\alpha} (\nabla_\alpha^\top \partial^\top \cdot \Psi) = 0.\end{aligned}\quad (24)$$

By the supplementary identities [Appendix A]

$$\partial^{\top\alpha} (Q_0 \Psi_\alpha) = (Q_0 - 6) \partial^\top \cdot \Psi, \quad (25)$$

$$\partial^{\top\alpha} (\not{x} \not{\partial}^\top \Psi_\alpha) = (1 + \not{x} \not{\partial}^\top) \partial^\top \cdot \Psi, \quad (26)$$

$$c \partial^{\top\alpha} (\gamma_\alpha \not{x} \not{\partial}^\top \cdot \Psi) = 4c \partial^\top \cdot \Psi - c \not{x} \not{\partial}^\top \partial^\top \cdot \Psi, \quad (27)$$

$$c \partial^{\top\alpha} (x_\alpha \partial^\top \cdot \Psi) = -4c \partial^\top \cdot \Psi, \quad (28)$$

one can write Eq. (23) in the form

$$\begin{aligned}Q_0 \partial^\top \cdot \Psi - 6 \partial^\top \cdot \Psi + \partial^\top \cdot \Psi + \not{x} \not{\partial}^\top \partial^\top \cdot \Psi - 3 \partial^\top \cdot \Psi + 8 \partial^\top \cdot \Psi \\ - c Q_0 \partial^\top \cdot \Psi + 4c \partial^\top \cdot \Psi - c \not{x} \not{\partial}^\top \partial^\top \cdot \Psi - 4c \partial^\top \cdot \Psi = 0 \\ (1-c) Q_0 \partial^\top \cdot \Psi + (1-c) \not{x} \not{\partial}^\top \partial^\top \cdot \Psi = 0 \\ (1-c) \left(Q_0 + \not{x} \not{\partial}^\top \right) \partial^\top \cdot \Psi = 0\end{aligned}$$

which can be written in terms of the Casimir operator as

$$(1-c) \left(Q_{\frac{1}{2}}^{(1)} + \frac{5}{2} \right) \partial^\top \cdot \Psi = 0$$

Again one must distinguish between $c=1$ and $c \neq 1$ [21]. For $c=1$, $\partial^\top \cdot \Psi \equiv \psi^s$, s stands for spinor, and ψ is an arbitrary spinor field. For $c \neq 1$, it again corresponds to a massless spinor field associated with the representation $\Pi_{\frac{1}{2}, -\frac{1}{2}}$, which is unphysical [15]. So, ψ^s satisfies the following eigenvalue equation:

$$\left(Q_{\frac{1}{2}}^{(1)} + \frac{5}{2} \right) \psi^s = 0. \quad (29)$$

Finally, the quotient spaces V/V_g are the physical states $\Pi_{\frac{3}{2}, \frac{3}{2}}^\pm$ which are associated with the UIR of the discrete series representation of the de Sitter group [15, 20, 21]

4 Conclusions

Supergravity, the supersymmetric generalization of Einstein's theory of gravity, is a fundamental tool in current research in theoretical physics. There has recently been renewed interest in supergravity in de Sitter space-time. The motivation for our study of massless spin- $\frac{3}{2}$ fields in de Sitter space-time originates in supergravity. The Gupta-Bleuler scheme is required to quantize the gauge invariant theory and we have indicated how this technique can be used to find the Gupta-Bleuler triplet.

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Appendix A

Supplementary identities

Here, we prove the supplementary identities Eqs. (25), (26), (27) and (28), respectively:

$$\begin{aligned}
 \partial^{\top\alpha}(Q_0\Psi_\alpha) &= -\partial^{\top\alpha}\partial_\beta^\top\partial^{\top\beta}\Psi_\alpha = -[\partial_\beta^\top\partial^{\top\alpha} - x^\alpha\partial_\beta^\top + x_\beta\partial^{\top\alpha}]\partial^{\top\beta}\Psi_\alpha \\
 &= -\partial_\beta^\top\partial^{\top\alpha}\partial^{\top\beta}\Psi_\alpha + x^\alpha\partial_\beta^\top\partial^{\top\beta}\Psi_\alpha - x_\beta\partial^{\top\alpha}\partial^{\top\beta}\Psi_\alpha \\
 &= -\partial_\beta^\top(\partial^{\top\beta}\partial^{\top\alpha} + x^\beta\partial^{\top\alpha} - x^\alpha\partial^{\top\beta})\Psi_\alpha + x^\alpha\partial_\beta^\top\partial^{\top\beta}\Psi_\alpha - x_\beta\partial^{\top\alpha}\partial^{\top\beta}\Psi_\alpha \\
 &= -\partial_\beta^\top\partial^{\top\beta}\partial^{\top\alpha} - \partial_\beta^\top x^\beta\partial^{\top\alpha} + \partial_\beta^\top x^\alpha\partial^{\top\beta}\Psi_\alpha + x^\alpha\partial_\beta^\top\partial^{\top\beta}\Psi_\alpha - x_\beta\partial^{\top\alpha}\partial^{\top\beta}\Psi_\alpha \\
 &= Q_0\partial^{\top\beta}\Psi_\alpha - 4\partial^{\top\beta}\Psi_\alpha + (\delta_\beta^\alpha + x_\beta x^\alpha)\partial^{\top\beta}\Psi_\alpha + x^\alpha\partial_\beta^\top\partial^{\top\beta}\Psi_\alpha - x_\beta\partial_\alpha^\top\partial^{\top\beta}\Psi_\alpha - x^\alpha Q_0\Psi_\alpha \\
 &= Q_0\partial^{\top\beta}\Psi_\alpha - 3\partial^{\top\beta}\Psi_\alpha - 2x^\alpha Q_0\Psi_\alpha - x_\beta(x^\beta\partial_\alpha^\top - x^\alpha\partial_\beta^\top + \partial^{\top\alpha}\partial^{\top\beta})\Psi_\alpha \\
 &= Q_0\partial^{\top\beta}\Psi_\alpha - 3\partial^{\top\beta}\Psi_\alpha - 2x^\alpha Q_0\Psi_\alpha + \partial^{\top\beta}\Psi_\alpha + x^\alpha x\cdot\partial^{\top\beta}\Psi_\alpha - x\cdot\partial^{\top\beta}\Psi_\alpha \\
 &= Q_0\partial^{\top\beta}\Psi_\alpha - 2\partial^{\top\beta}\Psi_\alpha - 2(Q_0x\cdot\Psi + 2\partial^{\top\alpha}\Psi_\alpha + 4x\cdot\Psi) = Q_0\partial^{\top\beta}\Psi_\alpha - 6\partial^{\top\beta}\Psi_\alpha.
 \end{aligned}$$

$$\begin{aligned}
 \partial^{\top\alpha}(\not{x}\not{\theta}^\top\Psi_\alpha) &= \partial^{\top\alpha}(\gamma_\rho x^\rho \gamma_\lambda \partial^{\top\lambda}\Psi_\alpha) = \gamma_\rho \gamma_\lambda \partial^{\top\alpha}(x^\rho \partial^{\top\lambda}\Psi_\alpha) \\
 &= \gamma_\rho \gamma_\lambda [(\partial^{\top\alpha} x^\rho \partial^{\top\lambda}\Psi_\alpha) + x^\rho \partial^{\top\alpha} \partial^{\top\lambda}\Psi_\alpha] \\
 &= \gamma_\rho \gamma_\lambda [(\eta^{\alpha\rho} + x^\alpha x^\rho) \partial^{\top\lambda}\Psi_\alpha + x^\rho \partial^{\top\alpha} \partial^{\top\lambda}\Psi_\alpha] \\
 &= \gamma^\alpha \gamma_\lambda \partial^{\top\lambda}\Psi_\alpha + x^\alpha \not{x} \gamma_\lambda \partial^{\top\lambda}\Psi_\alpha + \gamma_\rho \gamma_\lambda (x^\rho \partial^{\top\alpha} \partial^{\top\lambda}\Psi_\alpha) \\
 &= (2\delta_\lambda^\alpha - \gamma_\lambda \gamma^\alpha) \partial^{\top\lambda}\Psi_\alpha + x^\alpha \not{x} \not{\theta}^\top \Psi_\alpha + \gamma_\rho \gamma_\lambda (x^\rho (\partial^{\top\lambda} \partial^{\top\alpha} + x^\lambda \partial^{\top\alpha} - x^\alpha \partial^{\top\lambda}) \Psi_\alpha) \\
 &= 2\partial^{\top\alpha}\Psi_\alpha + x^\alpha \not{x} \not{\theta}^\top \Psi_\alpha + \not{x} \gamma_\lambda \partial^{\top\lambda} \partial^{\top\alpha}\Psi_\alpha + \gamma_\rho \gamma_\lambda x^\rho \partial^{\top\alpha} \partial^{\top\lambda}\Psi_\alpha - \not{x} x^\alpha \not{\theta}^\top \Psi_\alpha \\
 &= 2\partial^{\top\alpha}\Psi_\alpha + \not{x} \not{\theta}^\top \partial^{\top\alpha}\Psi_\alpha + \not{x} \not{\theta}^\top \partial^{\top\alpha}\Psi_\alpha = \partial^{\top\alpha}\Psi_\alpha + \not{x} \not{\theta}^\top \partial^{\top\alpha}\Psi_\alpha.
 \end{aligned}$$

$$\begin{aligned}
 c\partial^{\top\alpha}(\gamma_\alpha x^\rho \gamma_\rho \partial^{\top\alpha}\Psi) &= c\gamma_\alpha \gamma_\rho \partial^{\top\alpha}(x^\rho \partial^{\top\alpha}\Psi) \\
 &= c\gamma_\alpha \gamma_\rho [(\eta^{\rho\alpha} + x^\alpha x^\rho) \partial^{\top\alpha}\Psi + x^\rho \partial^{\top\alpha} \partial^{\top\alpha}\Psi] = 5c\partial^{\top\alpha}\Psi + c\not{x}\not{\theta}^\top \Psi + c\gamma_\alpha \not{x} \partial^{\top\alpha}\Psi \\
 &= 4c\partial^{\top\alpha}\Psi + c(2x_\alpha - \not{x} \gamma_\alpha) \partial^{\top\alpha}\Psi \\
 &= 4c\partial^{\top\alpha}\Psi - c\not{x} \not{\theta}^\top \partial^{\top\alpha}\Psi.
 \end{aligned}$$

$$-c\partial^{\top\alpha}(x_\alpha \partial^{\top\alpha}\Psi) = -c[(\partial^{\top\alpha} x_\alpha) \partial^{\top\alpha}\Psi + x_\alpha \partial^{\top\alpha} \partial^{\top\alpha}\Psi] = -4c\partial^{\top\alpha}\Psi.$$

We have used the following conditions in the calculation : $x\cdot\Psi=0$, $\gamma\cdot\Psi=0$ and $x\cdot\partial^{\top\alpha}=0$ [14].

Appendix B

Some useful relations

In this appendix, some useful relations are given which are used in this paper:

$$\begin{aligned}
 [\partial_\alpha^\top, \partial_\beta^\top] &= x_\beta \partial_\alpha^\top - x_\alpha \partial_\beta^\top, & [\partial_\alpha^\top, x_\beta] &= \theta_{\alpha\beta}, \\
 [x_\alpha, \not{\theta}^\top] &= -\gamma_\alpha^\top, & [\gamma_\alpha^\top, \partial_\alpha^\top] &= -4\not{x}, \\
 [Q_0, \not{x}] &= -4\not{x} - 2\not{\theta}^\top, & [x_\alpha, Q_0] &= 2\partial_\alpha^\top + 4x_\alpha, \\
 [\not{x}, \not{\theta}^\top] &= 4 - 2\not{\theta}^\top \not{x}, & \gamma_\alpha^\top &= \gamma_\alpha + x_\alpha x\cdot\gamma, \\
 [\not{x}, \partial_\alpha^\top] &= -\gamma_\alpha^\top, & [\not{x}, \gamma_\alpha^\top] &= 2x_\alpha - 2\gamma_\alpha \not{x}, \\
 [\partial_\alpha^\top, \not{\theta}^\top] &= \not{x} \partial_\alpha^\top - x_\alpha \not{\theta}^\top, & [\gamma_\alpha^\top, \partial_\alpha^\top] &= -4\not{x}, \\
 [\partial_\alpha^\top, Q_0] &= -6\partial_\alpha^\top - 2(Q_0 + 4)x_\alpha, & [\not{\theta}^\top, \gamma_\alpha^\top] &= -2\gamma_\alpha^\top \not{\theta}^\top + 2\partial_\alpha^\top + \gamma_\alpha^\top \not{x} + 4x_\alpha, \\
 [Q_0, \gamma_\alpha^\top] &= -8x_\alpha \not{x} - 2\not{x} \partial_\alpha^\top - 2\gamma_\alpha^\top - 2x_\alpha \not{\theta}^\top, & \partial_\beta^\top x^\alpha &= (\delta_\beta^\alpha + x_\beta x^\alpha).
 \end{aligned}$$

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