Properties of even $^{168-178}$Hf isotopes using IBM-1 and SEF

Mushtaq Abed Al-Jubbori[1;1)] Khalaf Ajaj Al-Mtiuty[2,3;2)] Khaliel. I. Saeed[1;3)] Fadhil I. Sharrad[4;4)]

[1] Department of Physics, College of Education for Pure Science, University of Mosul, 41001 Mosul, Iraq
[2] Department of Physics, College of Education for Pure Science, University of Mosul, Iraq
[3] Ministry of Education, 41001 Mosul, Iraq
[4] Department of Physics, College of Science, Kerbala University, 56001 Karbala, Iraq

Abstract: The properties of the ground and excited-state (γ- and β- bands) of $^{168-178}$Hf nuclei have been studied. The ratio $r(\frac{I+2}{I})$ and $E\gamma(I\rightarrow I-2)/I$ have been calculated as a function of the spin (I) to determine the ground-state evolution. The results indicate that these isotopes have a rotational property $SU(3)$. The energy levels for the ground-state, γ- and β- bands of $^{168-178}$Hf have been calculated using the Interacting Boson Model and Semi Empirical Formula (SEF). The parameters of the best fit to the measured data are determined. The behavior of energy and B(E2) ratios in the ground state band are examined.

Keywords: IBM, SEF, energy level, B(E2) value, Hf isotopes

PACS: 21.60.Fw, 21.10.Re, 23.20.Lv **DOI:** 10.1088/1674-1137/41/8/084103

1 Introduction

The lowlying states of the even-even Hf isotopes can be described successfully in the framework of either phenomenological or microscopic models [1–3]. The interacting boson model (IBM) is the bosonization of the shell model in terms of neutron (n) and proton (p) bosons. The nuclear structure reduces the problem of N interacting bosons by two values of angular momentum s (k=0) and d (k=2). In the IBM, there are three limiting symmetries: vibrational $U(5)$, rotational $SU(3)$ and γ-unstable $O(6)$. Furthermore, the nuclei may have transition limits, $U(5)$-$SU(3)$, $U(5)$-$O(6)$ and $SU(3)$-$O(6)$ [4, 5]. The IBM-1 cannot distinguish between protons and neutrons. The energy levels of even–even nuclei can be grouped into ground-state (GSB) β-band with $k^{\pi}=0^{+}$ and γ-band with $k^{\pi}=2^{+}$ [6]. Primary information about the properties of the nucleus can be obtained from the energy of the first excited state ($E2_1^+$), which is approximately equal to 100, 300 and 500 keV. In addition, the ratio of the second excited state to the first excited state ($R=E4_1^+/E2_1^+$) is $3<R\leqslant 3.3$,$2.4<R\leqslant 3$ and $2\leqslant R\leqslant 2.4$ for rotational, γ-soft and vibrational nuclei, respectively [7–10]. Theoretical and the experimental studies of even-even Hf isotopes have investigated the energy levels and electromagnetic transitions properties [11–14].

The aim of the present work is to study the low lying states of even-even Hf isotopes with $N=96-106$ in the frameworks of the Interacting Boson Model and Semi Empirical Formula. Furthermore, the potential energy surface $E(N,\beta,\gamma)$ will be calculated

2 Method of calculation

The IBM-1 Hamiltonian can be expressed as [1, 15, 16]:

$$\begin{aligned} H \;=\; & \varepsilon_{\rm s}(s^{\dagger}\cdot\tilde{s})+\varepsilon_{\rm d}(d^{\dagger}\cdot\tilde{d})\sum_{(L=0,2,4)}\frac{1}{2}(2L+1)^{\frac{1}{2}}C_L\left[[d^{\dagger}\times d^{\dagger}]^{(L)}\times[\tilde{d}\times\tilde{d}]^{(L)}\right]^{(0)} \\ & +\frac{1}{\sqrt{2}}v_2\left[[d^{\dagger}\times d^{\dagger}]^{(2)}\times[\tilde{d}\times\tilde{s}]^{(2)}+[d^{\dagger}\times s^{\dagger}]^{(2)}\times[\tilde{d}\times\tilde{d}]^{(2)}\right]^{(0)}+\frac{1}{2}v_0\left[[d^{\dagger}\times d^{\dagger}]^{(0)}\times[\tilde{s}\times\tilde{s}]^{(0)}+[s^{\dagger}\times s^{\dagger}]^{(0)}\times[\tilde{d}\times\tilde{d}]^{(0)}\right]^{(0)} \\ & +\frac{1}{2}u_2\left[[s^{\dagger}\times s^{\dagger}]^{(0)}\times[\tilde{s}\times\tilde{s}]^{(0)}\right]^{(0)}+\left[u_2[d^{\dagger}\times s^{\dagger}]^{(2)}\times[\tilde{d}\times\tilde{s}]^{(2)}\right]^{(0)}. \end{aligned} \quad (1)$$

Received 26 March 2017

1) E-mail: mushtaq_phy8@yahoo.com
2) E-mail: kh_aj@yahoo.com
3) E-mail: saeed_khalel@yahoo.com
4) E-mail: fadhil.altaie@gmail.com

The Hamiltonian is specified by two terms of one-body interactions, ($\varepsilon_{\rm s}$ and $\varepsilon_{\rm d}$), and seven terms of two-body interactions [c_L ($L=0,\ 2,\ 4$), v_L ($L=0,\ 2$), u_L (L=0, 2)], where $\varepsilon_{\rm s}$ and $\varepsilon_{\rm d}$ are the single-boson energies, and c_L, v_L and u_L describe the two-boson interactions. However, the total number of bosons N (pairs of two nucleons) is conserved, $N=n_{\rm s}+n_{\rm d}$ [15]. Then Eq. (1) can be written in general form as [16]:

$$\hat{H}=\varepsilon\hat{n}_{\rm d}+a_0\hat{P}\cdot\hat{P}+a_1\hat{L}\cdot\hat{L}+a_2\hat{Q}\cdot\hat{Q}+a_3\hat{T}_3\cdot\hat{T}_3+a_4\hat{T}_4\cdot\hat{T}_4\ . \quad (2)$$

Here, the first term represents the boson energy operator ($\hat{n}_{\rm d}=(d^{\dagger}\cdot\tilde{d})$), and the second term represents the pairing operator interaction ($\hat{P}=1/2[(\tilde{d}\cdot\tilde{d})-(\tilde{s}\cdot\tilde{s})]$). The third term represents the $O(3)$ angular momentum contribution ($\hat{L}=\sqrt{10}[d^{\dagger}\times\tilde{d}]^1$) and the fourth term represents the quadrupole interaction of the $L=2$ d-bosons. The last two terms represent the octoupole ($r=3$) and hexadecapole ($r=4$) operator interactions ($\hat{T}_r=[d^{\dagger}\times\tilde{d}]^{(r)}$).

The quadrupole operator is given by [15, 17]:

$$\hat{Q}=\left[d^{\dagger}\times\tilde{s}+s^{\dagger}\times\tilde{d}\right]^{(2)}+\chi\left[d^{\dagger}\times\tilde{d}\right]^{(2)}, \quad (3)$$

where χ is the quadrupole structure parameter and take the values 0 and $\pm\frac{\sqrt{7}}{2}$ [15, 17].

The eigenvalues for these three limits are given by [18]:

$$\left.\begin{array}{ll} E=\varepsilon nd+\beta nd(nd+4)+2\gamma\upsilon(\upsilon+3)+2\delta L(L+1) & U(5) \\ E=\dfrac{a_2}{2}(\lambda^2+\mu^2+\lambda\mu+3(\lambda+\mu))+\left(a_1-\dfrac{3a_2}{8}\right)L(L+1) & SU(3) \\ E=a_0/4(N-\sigma)(N+\sigma+4)+a_3/2\tau(\tau+3)+(a_1-a_3/10)L(L+1) & O(6)) \end{array}\right], \quad (4)$$

where β, γ and δ give the strengths of the parameters.

By analyzing the well-known experimental energy levels of the even–even Hf isotopes, a Semi Empirical Formula (SEF) has been derived for the energy levels of the ground state band (GSB), which depend on the angular momentum (I) in the following simple form [19]:

$$E(I)=A_1[\exp(A_2I)-A_3]. \quad (5)$$

This formula contains three parameters A_1, A_2 and A_3 beside the energy levels and angular momentum (I). These parameters are determined by fitting, inserting all the experimental energy data in the positive GSB.

Furthermore, the γ-and β-bands can be calculated from the following simple form [19]:

$$E(I)=E_0+(A_1+B)[\exp(A_2I)-A_3]. \quad (6)$$

The parameters E_0 and B can be estimated from the γ-and β-bands.

The SEF is very easy, uncomplicated and is a function of angular momentum (I) only.

3 Results and discussion

The results for energy levels and the transition probabilities B(E2) values are calculated by IBM-1 and SEF, and the potential energy surface can be discussed separately as follows.

3.1 Energy levels

Even-even Hf isotopes have atomic number $Z=72$, which has ten protons less than the magic number Z=82, with neutron numbers from 96 to 104.

The first indication of the symmetry shape of a nucleus is the ratio $R=E4_1^+/E2_1^+$. It is equal to 10/3 for deformed nuclei, 2.5 for γ-unstable nuclei and 2 for vibrational nuclei [1].

Table 1 shows the experimental values of $R=E4_1^+/E2_1^+$ for those nuclei. In this table, $R_{4/2}$ attains the $SU(3)$ value of $\sim$3.33 for $^{168-178}$Hf isotopes. According to the above analysis, these isotopes present features for the $SU(3)$ shape.

Table 1. Theoretical and experimental excitation energies (MeV) [20–24] for Hf Isotopes.

isotopes	Exp.		IBM	
	$E(2_1^+)$	$R=E(4_1^+)/E(2_1^+)$	$E(2_1^+)$	$R=E(4_1^+)/E(2_1^+)$
^{168}Hf	0.124100	3.1096	0.11372	3.3345
^{170}Hf	0.100800	3.1943	0.10087	3.3323
^{172}Hf	0.095220	3.2476	0.09535	3.3326
^{174}Hf	0.090985	3.2685	0.09121	3.3344
^{176}Hf	0.088349	3.2845	0.08858	3.3356
^{178}Hf	0.093180	3.2906	0.09323	3.3323

Regan et al. [25] introduced a relationship between the transition energy over spin $R=E\gamma(I\to I-2)/I$ as a function of the spin I (E-GOS), and has provided good information about the evolution that occurs in the yrast line of the nuclei. For the three limits, the relations between transition energy over spin $E\gamma(I\to I-2)$ and the spin I are given by [25]:

$$\left.\begin{array}{lll} \text{Vibrational:} & R=\dfrac{\hbar\omega}{I}\to 0 & \text{when } I\to\infty \\ \text{Rotational:} & R=\dfrac{\hbar^2}{2\vartheta}\left(4-\dfrac{2}{I}\right)\to 4\dfrac{\hbar^2}{2\vartheta} & \text{when } I\to\infty \\ \gamma\text{−soft:} & R=\dfrac{E_{2_1^+}}{4}\left(1+\dfrac{2}{I}\right)\to\dfrac{E_{2_1^+}}{4} & \text{when } I\to\infty \end{array}\right], \quad (7)$$

In the E-GOS, the vibrational nuclei curve drops quickly from the highest value ($\approx$250 keV) to (0) at

$(I=2_1^+)$ and $(I\to\infty)$, respectively. For γ-soft nuclei, the curve drops slowly from the highest value ($\approx$150 keV) at $(I=2_1^+)$ to $\frac{E_{2_1^+}}{4}$ at $I\to\infty$. For rotational nuclei, the curve increases slowly from the smallest value ($\approx$50 keV) at $(I=2_1^+)$ to $4\frac{\hbar^2}{2\vartheta}$ at $(I\to\infty)$ [25]. These theoretical limits are plotted in Fig. 1 for three typical nuclei.

A comparison of these curves with the ideal limits of vibrational, rotational and γ-soft is shown in Fig. 1. From this figure, the results show that all nuclei under study have $SU(3)$ characteristics.

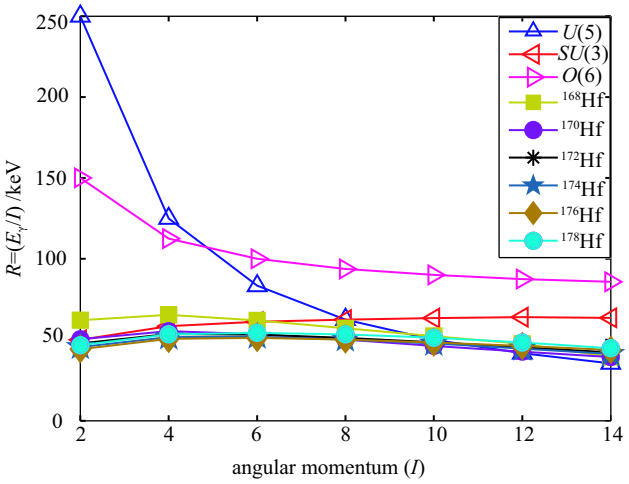

Fig. 1. (color online) E-GOS of the ground-state band for Hf isotopes [20–24].

The systematics of energy ratios $r((I+2)/I)$ of successive levels of collective bands in medium and heavy mass even-even nuclei were studied [26, 27].

For a given band of each spin I, the following ratios were constructed to define the symmetry for the excited band of even-even nuclei [9, 19, 26, 27]:

$$r\left(\frac{I+2}{I}\right)=\left[R\left(\frac{I+2}{I}\right)_{\rm exp.}-\frac{(I+2)}{I}\right]\times\frac{I(I+1)}{2(I+2)}, \qquad (8)$$

where $R\left(\frac{I+2}{I}\right)_{\rm exp.}$ is the experimental energy ratio values between the I+2 and I states. The ratios $r(I+2/I)$ with $I=2,4,6,\cdots$ have been studied. These ratios show distinctly different behavior in the vibrational, rotational, and γ-unstable limits. The ratio r should be close to zero and one for vibrational and rotational nuclei, respectively, while it should have intermediate values between zero and one for γ-unstable nuclei. In Eq. (8), the value of energy ratios (r) have changed between 0.1 and 1 for yrast bands of even-even nuclei [9, 19, 26, 27]:

$$\left.\begin{array}{ll} 0.1\leqslant r\leqslant 0.35 & \text{for vibrational nuclei} \\ 0.4\leqslant r\leqslant 0.6 & \text{for transitional nuclei} \\ 0.6\leqslant r\leqslant 1.0 & \text{for rotational nuclei} \end{array}\right]. \qquad (9)$$

The relation between $r((I+2)/I)$ and I for the ground state bands of the $^{168-178}$Hf isotopes are plotted in Fig. 2. This plot is used to distinguish between different kinds of collective behavior of vibrational $U(5)$, rotational $SU(3)$ and γ-unstable $O(6)$ nuclei. From this figure, the ratios $r((I+2)/I)$ start with a value very close to one and then constantly decrease with I to value $\geqslant$0.6, so this confirms that the isotopes have a rotational limit.

The SEF and IBM are used to calculate the energy levels of the ground states (GSB), γ- and β-bands with special MATLAB software and the PHINT code [28], respectively. The number of bosons and the values of the parameters which give the best fitting between theoretical and the experimental energy levels [20–24] are presented in Table 2 for all nuclei under study.

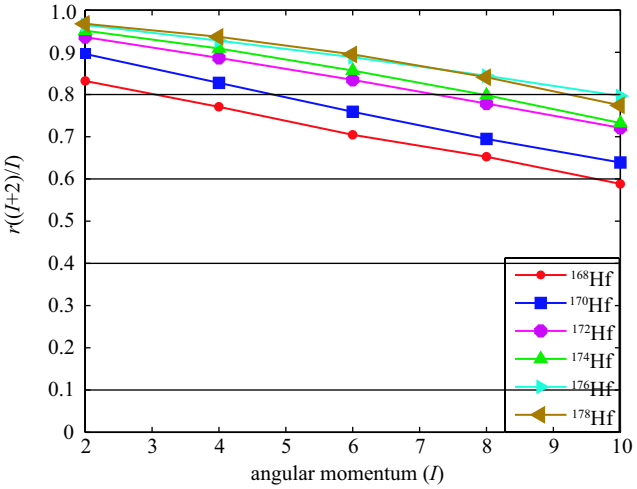

Fig. 2. (color online) The ratio $r((I+2)/I)$ as a function of I for $^{168-178}$Hf Isotopes [20–24].

Table 2. IBM-1 and SEF parameters used in the calculations (in units of MeV), except N, CHI, A_2 and A_3 for Hf isotopes.

isotopes	N	IBM			SEF		
		QQ	ELL	CHI	A_1	A_2	A_3
^{168}Hf	12	−0.0210	0.030	−1.333	2.1601	0.0635	1.0965
^{170}Hf	13	0.0229	0.0250	−1.333	1.4521	0.076	1.1158
^{172}Hf	14	−0.0212	0.0238	−1.333	1.2522	0.0847	1.1355
^{174}Hf	15	−0.0186	0.0234	−1.333	1.2069	0.0855	1.1398
^{176}Hf	16	−0.0245	0.0203	−1.333	1.0504	0.0933	1.1529
^{178}Hf	15	−0.0249	0.0217	−1.333	1.1848	0.0898	1.1501

($ELL=2a_1$ and $QQ=2a_2$) [18].

Table 3. SEF parameters of the β- and γ- bands in MeV for Hf isotopes.

isotopes	β-band		$\gamma-$band	
	E_0	B	E_0	B
^{168}Hf	1.0415	−0.969	0.7893	−0.0791
^{170}Hf	1.0423	−0.9747	0.8934	−0.06
^{172}Hf	0.9312	−0.6067	0.9831	−0.0984
^{174}Hf	0.8744	−0.2978	1.1544	0.016
^{176}Hf	1.2045	−0.2879	1.2794	−0.0763
^{178}Hf	1.2743	−0.4303	1.0948	−0.0885

The SEF parameters of the β- and γ- bands are presented in Table 3 for Hf isotopes.

The calculated GSB, β- and γ- band and the experimental data [20–24] of low lying states for even-even Hf isotopes are plotted in Fig. 3, and the calculated results are in good agreement with the experimental data for all isotopes.

Levels with "()" in GBS, β and γ-band correspond to cases for which the spin and/or parity of the corresponding states are not well established experimentally. Furthermore, the calculations of SEF are better than those of IBM.

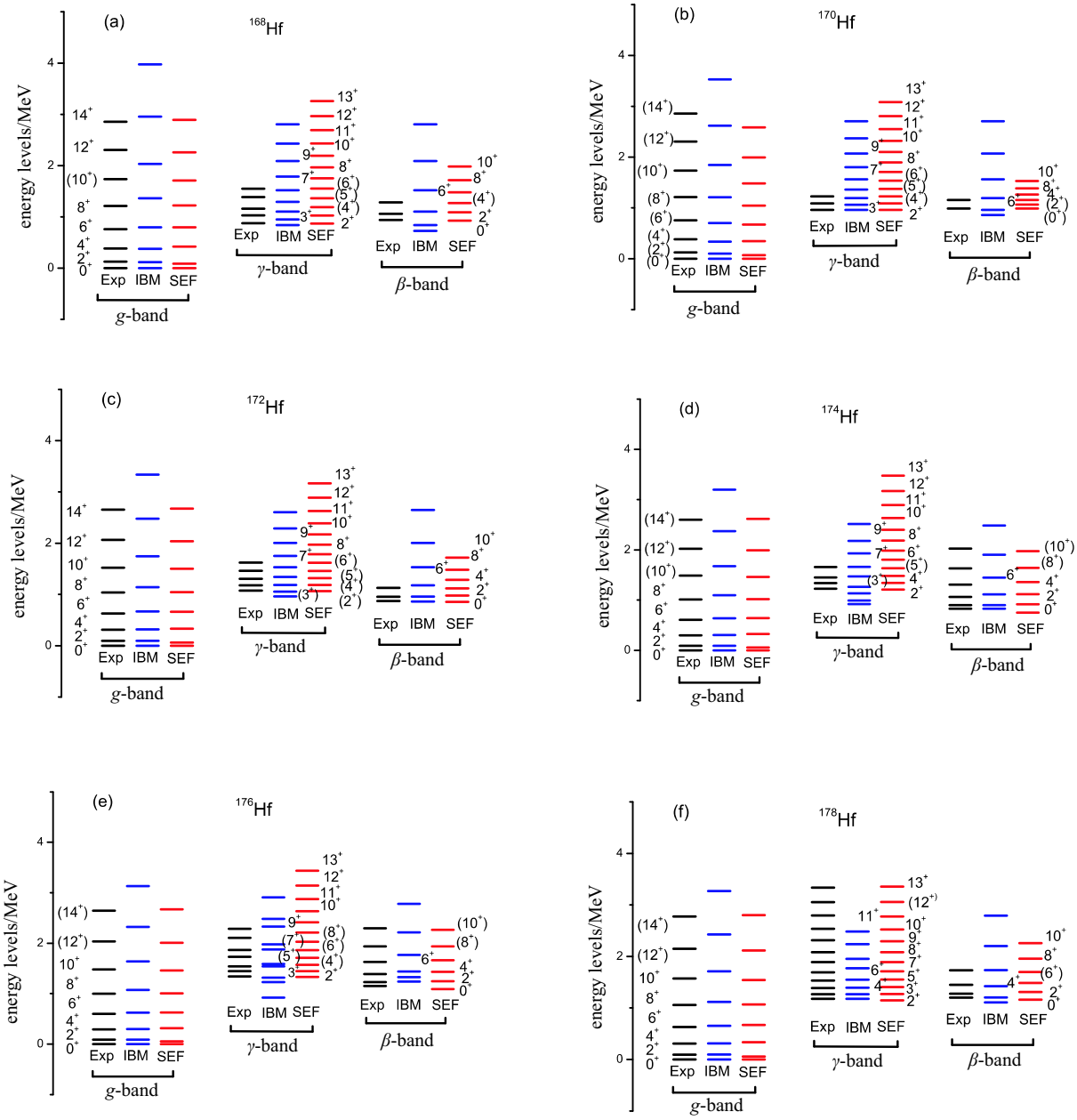

Fig. 3. (color online) Comparison between the calculated IBM-1, SEF and the experimental data [20–24] for Hf isotopes.

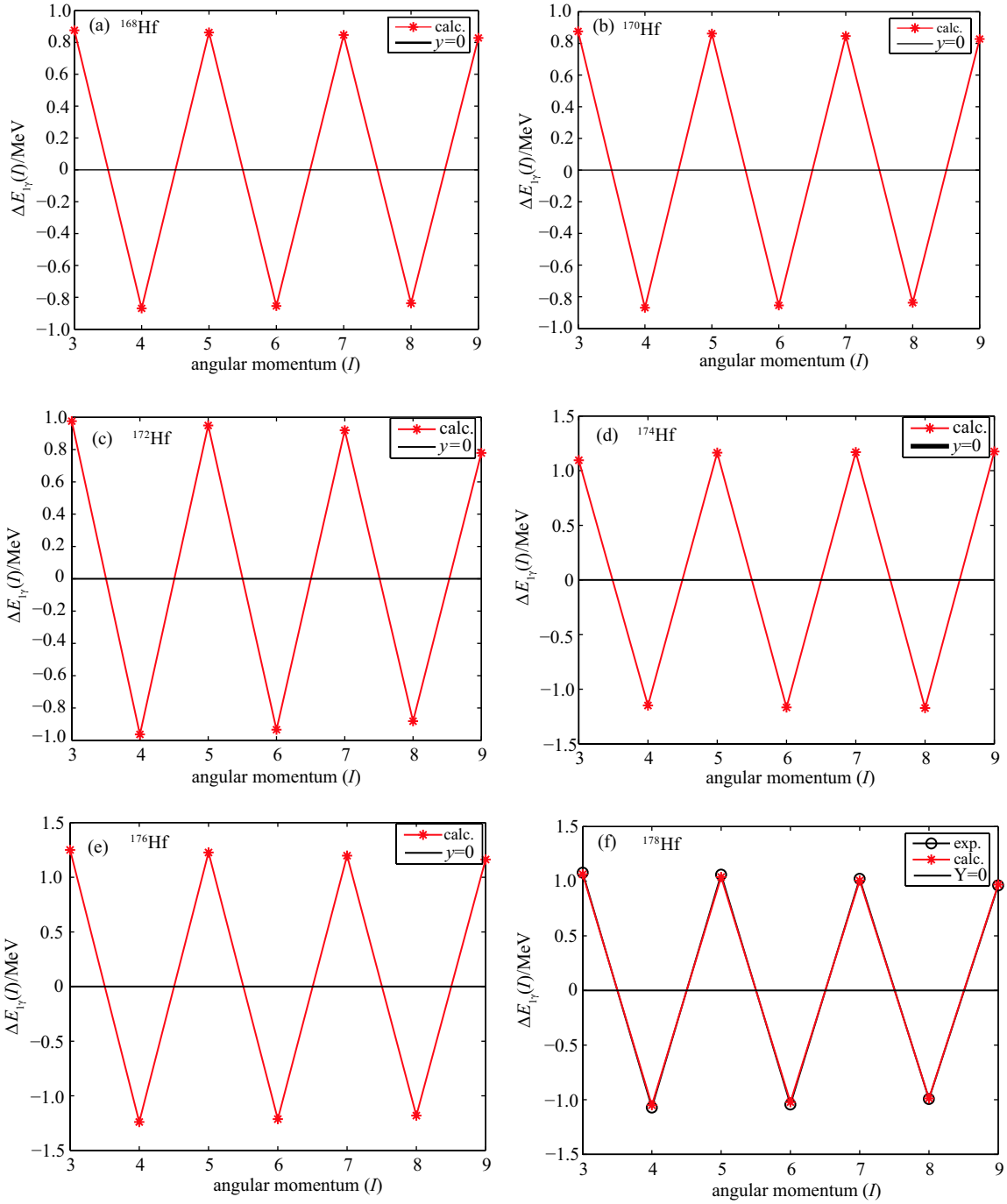

Fig. 4. (color online) Staggering calculated from Eq. (10) for Hf isotopes [20–24].

The odd–even staggering or $\Delta I=1$ staggering can be calculated by the quantity [29]:

$$\Delta E_{1,\gamma}(I) = 1/16[6E_{1,\gamma}(I)-4E_{1,\gamma}(I-1)-4E_{1,\gamma}(I+1) +E_{1,\gamma}(I-2)+E_{1,\gamma}(I+2)], \quad (10)$$

where $E_{1,\gamma}(I)=E_{1,\gamma}(I+1)-E(I)$.

The quantity $\Delta E_{1,\gamma}(I)$ exhibits values of alternating sign over an extended region of the angular momentum. Odd–even staggering starts from relatively high values and then decreases with increasing angular momentum. In the vanishing value ($\Delta E_{1,\gamma}(I)=0$), the staggering starts rising and then drops again. A phase change occurs [9, 19] when the staggering reaches a vanishing value. The staggering results are shown in Fig. 4 for $^{168-178}$Hf nuclei. From this figure, the calculated results

are roughly constant with increasing I. The staggering curves do not approach zero and it confirms the stability properties $SU(3)$ of these isotopes.

3.2 B(E2) values

Under the framework of IBM, the electrical transition can be also analyzed and the most general E2 transition operator can be written as [1, 8, 18]:

$$T^{\rm E2}=\alpha_2[d^\dagger s+s^\dagger d]^{(2)}+\beta_2[d^\dagger d]^{(2)}=e_B\hat{Q}, \qquad (11)$$

where ($s^\dagger$, $d^\dagger$) and (s, d) are creation and annihilation operators for s and d bosons, respectively. α_2 and β_2 are two parameters, with $\beta_2=\chi\alpha_2$, $\alpha_2=e_B$ (effective charge of boson), and the quadrupole operator Q is shown in Eq. (3). The matrix elements of the $T^{\rm E2}$ operator can give reduced transition rates as [8, 30, 31]:

$$B((\mathrm{E2})L_{\rm i}\to L_{\rm f})=\frac{1}{2L_{\rm i}+1}\left|\langle L_{\rm f}\|T^{(\mathrm{E2})}\|L_{\rm i}\rangle\right|^2. \qquad (12)$$

The Hf isotopes are an excellent example to study the behavior of the total low-lying E2 strengths. The experimental $B(\mathrm{E2});2_1^+\to 0_1^+$ values are used to determined effective charge, $\alpha_2=e_B$. The values of the α_2 and β_2 parameters are presented in Table 4. The absolute B(E2) transition rates for experimental values (where available) and IBM results are listed in Table 5 for the Hf isotopes.

Table 4. Parameters (in eb) used to reproduce B(E2) values for the Hf isotopes.

isotope	N	α_2	β_2
^{168}Hf	12	0.1144	−0.3383
^{170}Hf	13	0.1156	−0.3419
^{172}Hf	14	0.1004	−0.2969
^{174}Hf	15	0.0941	−0.2784
^{176}Hf	16	0.0982	−0.2905
^{178}Hf	15	0.0985	−0.2914

Table 5. IBM-1 and experimental values [20–24] of B(E2) (in e^2 b^2) for Hf isotopes.

isotopes	^{168}Hf		^{170}Hf		^{172}Hf	
$J_{\rm i}\to J_{\rm f}$	EXP.	IBM-1	EXP.	IBM-1	EXP.	IBM-1
$2_1^+\to 0_1^+$	0.8484	0.8478	1.0074	1.0058	0.8754	0.8748
$3_1^+\to 2_2^+$		1.1749		1.4228		1.2582
$4_1^+\to 2_1^+$	1.3442	1.1919	1.4551	1.4178		1.2353
$4_2^+\to 2_2^+$		0.3859		1.0636		0.4149
$4_2^+\to 3_1^+$		0.8789				0.9404
$5_1^+\to 4_2^+$		0.6174		0.7502		0.6654
$6_1^+\to 4_1^+$		1.5701	1.2757	1.5238		1.332
$6_2^+\to 4_2^+$				0.8998		0.8001
$6_2^+\to 5_1^+$		0.4626		0.5598		0.4950
$6_3^+\to 4_3^+$				1.1837		1.0571
$8_1^+\to 6_1^+$	1.9282	1.2794	1.679	1.538		1.3518
$8_2^+\to 6_2^+$				1.0438		0.9333
$8_2^+\to 7_1^+$						0.2953
$10_1^+\to 8_1^+$	2.0384	1.2389	1.735	1.503		1.3299
$12_1^+\to 10_1^+$	1.7629	1.1663	1.5671	1.4332		1.2806
$14_1^+\to 12_1^+$	1.3222	1.0669	1.679	1.3348		1.2088
isotopes	^{174}Hf		^{176}Hf		^{178}Hf	
$J_{\rm i}\to J_{\rm f}$	EXP.	IBM-1	EXP.	IBM-1	EXP.	IBM-1
$2_1^+\to 0_1^+$	0.877	0.8754	1.0797	1.0733	0.960	0.9591
$2_3^+\to 0_1^+$	0.0278	0	0.023	0	0.0043	0
$2_4^+\to 0_1^+$					0.0023	0
$4_1^+\to 2_1^+$		1.2378				1.3562
$4_3^+\to 2_1^+$					0.00066	0
$4_4^+\to 2_1^+$					$0.36{*}10^{-4}$	0
$4_4^+\to 2_2^+$					0.0426	0
$6_1^+\to 4_1^+$		1.338		1.6451	1.314	1.4661
$6_2^+\to 4_1^+$			$0.0186{*}10^{-7}$	0	$0.75{*}10^{-6}$	0
$6_2^+\to 4_2^+$		0.8113			0.000225	0
$8_1^+\to 6_1^+$		1.3625		1.6796	1.422	1.4928
$10_1^+\to 8_1^+$		1.3481		1.6679	1.542	1.477
$12_1^+\to 10_1^+$		1.3073		1.6252	0.72	1.4324
$14_1^+\to 12_1^+$		1.2455		1.5578	1.74	1.3647

3.3 Potential energy surface (PES)

The potential energy surface considered is important to give the final shape of nuclei. In the IBM, the Hamiltonian (Eq. (1)) can be derived using the intrinsic condensate state as [8, 32]:

$$|N,\beta,\gamma=1/\sqrt{N!}(b_{\rm c}^{\dagger})^{N}|0\rangle, \tag{13}$$

where $|0\rangle$ denotes the boson vacuum, and

$$b_{\rm c}^{\dagger}=(1+\beta^2)^{-1/2}\left\{s^{\dagger}+\beta\left[\cos\gamma(d_0^{\dagger})+\sqrt{1/2}\sin\gamma(d_2^{\dagger}+d_{-2}^{\dagger})\right]\right\}, \tag{14}$$

N is the boson number, β is a measure of the total deformation of nucleus, and γ measures the deviation from axial symmetry, which determines the geometrical shape of the nucleus.

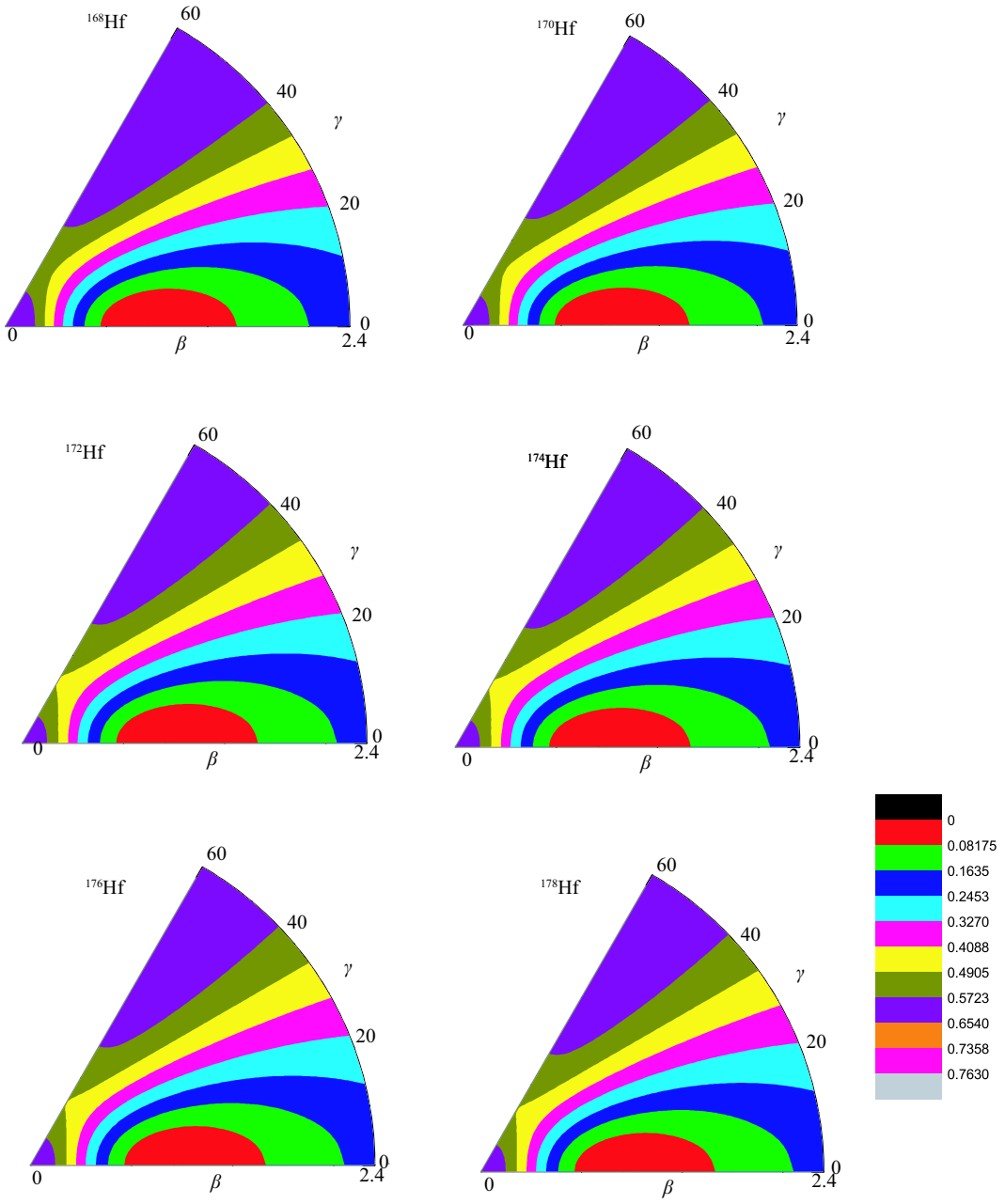

Fig. 5. (color online) Potential energy surfaces for Hf isotopes.

Here, $\beta \geqslant 0$ and $0<\gamma<\pi/3$. β and γ are given by [8]:

$$E(N,\,\beta,\,\gamma)=a_2N(N-1)\frac{1+\frac{3}{4}\beta^4-\sqrt{2}\beta^3\cos3\gamma}{2(1+\beta^2)^2}. \quad (15)$$

The calculated potential energy surfaces are shown in Fig. 5 for the Hf isotopes. This figure shows that all the isotopes under study have a rotational $SU(3)$ shape.

4 Conclusions

In conclusion, the energy levels of positive parity states and electric transition probabilities B(E2) have been calculated using IBM-1 and SEF. The ratio $r((I+2)/I)$ and the E-GOS curves of the GSB have been calculated for Hf isotopes. These curves indicated that these isotopes have a rotational property $SU(3)$. We have demonstrated that the ground and γ bands exhibit $\Delta I = 1$ staggering, and the vanishing value of the staggering $\Delta E1,\ \gamma(I) = 0$ has not been reached. The potential energy surfaces for Hf isotopes have been calculated using the simplified form of Interacting Boson Model (IBM) with an intrinsic coherent state. All isotopes studied have dynamical symmetry $SU(3)$ characteristics.

We thank the University of Mosul, College of Education for Pure Science, Department of Physics and University of Kerbala, College of Science, Department of Physics for supporting this work.

References

1 F. Iachello and A. Arima, *The Interacting Boson Model*, (Cambridge University Press, Cambridge) (1987)
2 F. Iachello, Phys. Rev. Lett., **87**: 052502 (2001)
3 F. Iachello, Phys. Rev. Lett., **85**: 3580 (2000)
4 P. Cejnar, J. Jolie and R. F. Casten, Rev. Mod. Phys., **82**: 2155 (2010)
5 R. F. Casten and E. A. McCutchan, Jou. Phys. G, **34**: R285 (2007)
6 A. Bohr and B. R. Mottelson, *Nuclear Structure*, (Singapore: World Scientific) Vol. II, p748 (1998)
7 K. S. Krane, *Introductory Nuclear Physics* (New York: John Wiley and Sons) (1987)
8 F. Iachello, Phys. Rev. Lett., **44**: 772 (1980)
9 I. Mamdouh and M. Al-Jubbori, Indian J. Phys., **89**: 1085 (2015)
10 S. N. Abood and M. A. Al-Jubbori, Commun. Theor. Phys., **60**: 335(2013)
11 E. H. Hagemann, Nucl. Phys. A, **161**: 449 (1971)
12 T. Morikawa et al, Phys. Lett. B, **350**: 169 (1995)
13 K. Nomura et al, Phys .Rev. C, **83**: 041302 (2011)
14 H. H. Kassim, A. A. Mohammed, F. I. Sharrad, I. Hossain, and K. S. Jassim, Iran. J. Sci. Tech. Tran. Sci., DOI 10.1007/s40995-016-0104-x
15 A. Arima and F. Iachello, Ann. Phys. N.Y., **111**: 201 (1978)
16 K. Abrahams, K. Allaart, and A. E. L. Dieperink, *Nuclear Structure*, Plenum press, New York and London (1981)
17 F. Iachello, Phys. Rev. Lett., **44**: 772 (1980)
18 R. F. Casten and D. D. Warner, Rev. Mod. Phys., **60**: 389 (1988)
19 M. A. Al-Jubbori, H. H. Kassim, F. I. Sharrad, and I. Hossain, Nucl. Phys. A, **955**: 101 (2016)
20 M. Baglin, Nuclear Data Sheets, **111**: 1807 (2002, 2010)
21 B. Singh, Nuclear Data Sheets, **75**: 199 (1995)
22 E. Browne and H. Junde, Nuclear Data Sheets, **87**: 15 (1999)
23 M. S. Basunia, Nuclear Data Sheets, **107**: 791 (2006)
24 E. Achterberg, O. A. Capurro, and G. V. Marti, Nuclear Data Sheets, **110**: 1473 (2009)
25 P. H. Regan et al, Phys. Rev. Lett., **90**: 152502 (2003)
26 D. Bonatsos and L. D. Skouras, Phys. Rev. C, **43**: 952R (1991)
27 A. M. Khalaf and A. M. Ismail, Prog. Phys., **2**: 98 (2013)
28 O. Scholten, Computer code PHINT, KVI; Groningen, Holland, (1980)
29 E. A. McCutchan and N. V. Zamfi, Phys. Rev. C, **71**: 054306 (2005)
30 H. H. Khudher, A. K. Hasan, and F. I. Sharrad, Mode. Appl. Scie., **10**: 181 (2016)
31 H. H. Kassim and F. I. Sharrad, Inter. J. of Mod. Phys. E, **23**: 1450070 (2014)
32 A. E. L. Dieperink, O. Scholten, and F. Iachello, Phys. Rev. Lett., **44**: 1747 (1980)