

Geometric sigma model of the Universe^{*}

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Abstract: The purpose of this work is to demonstrate how an arbitrarily chosen background of the Universe can be made a solution of a simple geometric sigma model. Geometric sigma models are purely geometric theories in which spacetime coordinates are seen as scalar fields coupled to gravity. Although they look like ordinary sigma models, they have the peculiarity that their complete matter content can be gauged away. The remaining geometric theory possesses a background solution that is predefined in the process of constructing the theory. The fact that background configuration is specified in advance is another peculiarity of geometric sigma models. In this paper, I construct geometric sigma models based on different background geometries of the Universe. Whatever background geometry is chosen, the dynamics of its small perturbations is shown to have a generic classical stability. This way, any freely chosen background metric is made a stable solution of a simple model. Three particular models of the Universe are considered as examples of how this is done in practice.

Keywords: cosmological models, sigma model, stability analysis

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1 Introduction

The latest astronomical observations have given a substantial boost to the development of modern cosmology [1–10]. In particular, the accelerating expansion of the Universe has drawn much attention. The early time acceleration is widely known as inflation, while the late time acceleration is usually referred to as the epoch of dark energy [11–14]. Presently, the Λ CDM model, in which the cosmological constant Λ plays the role of dark energy, is accepted as a standard cosmological model. There is an extensive body of literature on other forms of dark energy, too [15–25]. All in all, the number of dark energy models that can be found in the literature is enormous. The same holds for the inflationary models that have been constructed over the years.

In this paper, I shall describe a procedure which associates an action functional with an arbitrarily chosen background geometry of the Universe. Precisely, any desirable geometry of the Universe is made a solution of a particular geometric sigma model. Geometric sigma models are theories that possess two distinctive properties. First, their complete matter content can be gauged away. Second, any predefined geometry can be made a solution of a particular model. These models have first been proposed in Ref. [26] in the context of fermionic excitations of flat geometry. Here, I use them for model-

ing the dynamics of the Universe. To be more accurate, only geometry and dark energy are considered in this approach. The inclusion of ordinary matter is discussed separately.

The results obtained in this paper are summarized as follows. First, a class of purely geometric dark energy models has been constructed. Every particular model is defined as a geometric sigma model associated with a spatially flat, homogeneous and isotropic geometry. This way, an arbitrarily chosen geometry of this kind becomes a background solution of a particular geometric sigma model. Ultimately, one is provided with a class of dark energy models parametrized by their background geometries. The inflation and the late time acceleration have purely geometric origin. It is important to emphasize that, while the background metric can be chosen arbitrarily, the physics of its small perturbations cannot. In fact, the background metrics just parametrize geometric sigma models, very much the same as inflaton potentials parametrize the inflationary models.

The second result concerns the linear stability of the background solution in geometric sigma models. It has been proven true for almost all background geometries. Precisely, stability is guaranteed up to the existence of critical moments, where the perturbations may diverge. There, however, the linear analysis fails, and should be corrected by the inclusion of interaction terms.

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Finally, I have analyzed geometric sigma models coupled to ordinary matter. It has been shown that matter fields do not compromise the vacuum stability established earlier. In the case of minimal coupling to the metric, the linear stability of matter itself has been proven.

The layout of the paper is as follows. In Section 2, the construction of geometric sigma models, as defined in Ref. [26], is recapitulated and subsequently applied to spatially flat, homogeneous and isotropic geometries. As a result, a class of action functionals of the Universe is obtained. Each of these action functionals has a nontrivial background solution that describes the background geometry of a particular Universe. In Section 3, the dynamics of small perturbations of these nontrivial backgrounds is examined. In Section 4, the background solutions are proven stable for almost all spatially flat, homogeneous and isotropic geometries. In Section 5, geometric sigma modes are coupled to ordinary matter. It is shown that matter fields preserve the results obtained in the absence of matter. In Section 6, the examples of inflationary and bouncing Universes are used to demonstrate how geometric sigma models are constructed in practice. Section 7 is devoted to concluding remarks.

My conventions are as follows. The indexes μ, ν, \dots and i, j, \dots from the middle of the alphabet take values 0, 1, 2, 3. The indexes α, β, \dots and a, b, \dots from the beginning of the alphabet take values 1, 2, 3. The spacetime coordinates are denoted by x^μ , ordinary differentiation uses a comma ($X_{,\mu} \equiv \partial_\mu X$), and covariant differentiation uses a semicolon ($X_{;\mu} \equiv \nabla_\mu X$). The repeated indexes denote summation: $X_{\alpha\alpha} \equiv X_{11} + X_{22} + X_{33}$. The signature of the 4-metric $g_{\mu\nu}$ is $(-, +, +, +)$, and the curvature tensor is defined as $R^\mu{}_{\nu\lambda\rho} \equiv \partial_\lambda \Gamma^\mu{}_{\nu\rho} - \partial_\rho \Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\sigma\lambda} \Gamma^\sigma{}_{\nu\rho} - \Gamma^\mu{}_{\sigma\rho} \Gamma^\sigma{}_{\nu\lambda}$.

2 Geometric sigma models

The construction of a geometric sigma model begins with specifying a spacetime metric. I shall denote it by $g_{\mu\nu}^{(o)}(x)$. The metric $g_{\mu\nu}^{(o)}$ is freely chosen, and the coordinates x^μ are fully fixed. As a consequence, the functions $g_{\mu\nu}^{(o)}(x)$ are completely determined. In the next step, the corresponding Ricci tensor $R_{\mu\nu}^{(o)}(x)$ is calculated, and the following Einstein-like equation is postulated:

$$R_{\mu\nu} = R_{\mu\nu}^{(o)}(x). \tag{1}$$

Obviously, the metric $g_{\mu\nu}^{(o)}$ is a solution of Eq. (1). Its non-zero right hand side defines the matter content of the theory. Eq. (1) is an example of how an arbitrarily chosen metric can be made a solution of a simple model.

Equation (1) obviously lacks general covariance. To covariantize it, I introduce a new set of coordinates $\phi^i = \phi^i(x)$. In terms of these new coordinates, Eq. (1)

takes the form

$$R_{\mu\nu} = H_{ij}(\phi) \phi^i_{,\mu} \phi^j_{,\nu}, \tag{2}$$

where the functions $H_{ij}(\phi)$ are defined as

$$H_{ij}(\phi) \equiv R_{ij}^{(o)}(\phi). \tag{3}$$

In other words, the ten functions $H_{ij}(\phi)$ are obtained by replacing x with ϕ in ten components of the Ricci tensor $R_{\mu\nu}^{(o)}(x)$. Eq. (2) is generally covariant once the new coordinates ϕ^i are seen as scalar functions of the old coordinates x^μ . If the new coordinates are chosen to coincide with the old ones, $\phi^i(x) \equiv \delta^i_\mu x^\mu$, the covariant equation (2) is brought back to the non-covariant form (1). In what follows, the shorthand notation $\delta^i_\mu x^\mu \equiv x^i$ will be used.

Equation (2) has the form of Einstein's equation in which four scalar fields $\phi^i(x)$ of some nonlinear sigma model are coupled to gravity. The "matter field equations" are obtained by utilizing the Bianchi identities $\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right)_{;\nu} \equiv 0$. If the condition $\det \phi^i_{,\mu} \neq 0$ is fulfilled, one obtains

$$H_{ij} \nabla^2 \phi^j + \frac{1}{2} \left(\frac{\partial H_{ij}}{\partial \phi^k} - \frac{\partial H_{jk}}{\partial \phi^i} + \frac{\partial H_{ki}}{\partial \phi^j} \right) \phi^j_{,\mu} \phi^{k,\mu} = 0. \tag{4}$$

Eq. (4) is not an independent equation, as it follows from (2) and the Bianchi identities. It is straightforward to verify that Eqs. (2) and (4) can be derived from the action functional

$$I_g = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [R - H_{ij}(\phi) \phi^i_{,\mu} \phi^{j,\mu}]. \tag{5}$$

Here, the target metric $H_{ij}(\phi)$ is not an independent coefficient of the model. Instead, it is constructed out of the background metric $g_{\mu\nu}^{(o)}$, through its defining relation (3). This way, an action functional is associated with every freely chosen background metric. This action functional describes a nonlinear sigma model coupled to gravity, and possesses the nontrivial (let me call it vacuum) solution

$$\phi^i = x^i, \quad g_{\mu\nu} = g_{\mu\nu}^{(o)}. \tag{6}$$

Indeed, Eq. (2) with the target metric (3) is trivially satisfied if the scalars ϕ^i and the metric $g_{\mu\nu}$ are given by (6). Being a direct consequence of (2), the same is true for Eq. (4).

The physics of small perturbations of the vacuum (6) does not violate the condition $\det \phi^i_{,\mu} \neq 0$, which enables one to interpret the scalars ϕ^i as spacetime coordinates. If this is the case, one is allowed to fix the gauge $\phi^i(x) = x^i$, which brings us back to the geometric equation (1). One should have in mind, however, that the action (5) has a non-geometric sector, too. It is

characterized by $\det \phi^i_{,\mu} = 0$, and includes trivial vacuum solutions such as $\phi^i = \text{const}$. In what follows, I shall restrict my considerations to purely geometric dynamics of small perturbations of the vacuum (6).

Before I continue, let me note that Eq. (1) is not the only geometric equation that allows the solution $g_{\mu\nu} = g_{\mu\nu}^{(o)}$. A simple generalization of this equation can be obtained by adding terms proportional to $g_{\mu\nu} - g_{\mu\nu}^{(o)}$. The simplest choice is the equation

$$R_{\mu\nu} = R_{\mu\nu}^{(o)}(x) + \frac{1}{2}V(x)(g_{\mu\nu} - g_{\mu\nu}^{(o)}). \quad (7)$$

It defines a class of geometric theories parametrized by metrics $g_{\mu\nu}^{(o)}$, and potentials V . The covariantization of the non-covariant Eq. (7) ultimately leads to the action functional

$$I_g = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [R - F_{ij}(\phi)\phi^i_{,\mu}\phi^{j,\mu} - V(\phi)], \quad (8)$$

where the target metric $F_{ij}(\phi)$ is defined by

$$F_{ij}(x) \equiv R_{ij}^{(o)}(x) - \frac{1}{2}V(x)g_{ij}^{(o)}(x). \quad (9)$$

The class of theories defined by Eq. (8) has the vacuum solution (6) for any choice of the potential $V(\phi)$. The physics of small perturbations of this vacuum allows the gauge condition $\phi^i = x^i$, which brings us back to the geometric Eq. (7).

In what follows, I shall associate a geometric sigma model with a given vacuum geometry of the Universe. Let me choose a spatially flat, homogeneous and isotropic metric $g_{\mu\nu}^{(o)}$, defined by

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \quad (10)$$

The corresponding Ricci tensor is straightforward to calculate. One obtains

$$R_{00}^{(o)} = -3\frac{\ddot{a}}{a}, \quad R_{0\alpha}^{(o)} = 0, \quad R_{\alpha\beta}^{(o)} = (a\ddot{a} + 2\dot{a}^2)\delta_{\alpha\beta},$$

where “dot” denotes the time derivative. Now, I am ready to construct the target metric $F_{ij}(\phi)$. In this paper, I choose the simplest model allowed by (9). It is obtained by noticing that the target metric F_{ij} can significantly be simplified by a proper choice of the potential V . Indeed, if the potential is chosen to have the form

$$V(t) = 2\left(2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right), \quad (11)$$

the component F_{00} remains the only nonzero component of the target metric. Precisely, one obtains

$$F(t) = 2\left(\frac{\dot{a}^2}{a^2} - \frac{\ddot{a}}{a}\right), \quad (12)$$

where the identification $F_{00} \equiv F$ is introduced for convenience. As a consequence, ϕ^0 is the only scalar field that enters the action functional (8). Let me simplify the notation by using the identification $\phi^0 \equiv \phi$. The action (8) then reduces to

$$I_g = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [R - F(\phi)\phi^\mu_{,\mu} - V(\phi)]. \quad (13)$$

It governs the dynamics of gravity coupled to a scalar field, and has the vacuum solution

$$\phi = t, \quad g_{\mu\nu} = g_{\mu\nu}^{(o)}. \quad (14)$$

The precise form of the target metric $F(\phi)$ and the potential $V(\phi)$ is determined once the function $a(t)$ is specified. Only the background geometry $g_{\mu\nu}^{(o)}$ is freely chosen. The dynamics of metric perturbations is governed by the corresponding action functional. The class of action functionals (13) represents a collection of dark energy models parametrized by the scale factors $a(t)$.

Similar attempts to derive a dark energy model out of the given scale factor already exist in literature. Take, for example, Refs. [27–36]. There, scalar field dark energy models have been constructed to mimic holographic dark energy. Every particular model is built with a separate effort to solve a particular problem. The procedure described in this paper, however, is the first systematic approach of the kind. It gives a precise prescription of how to construct the target metric and the potential of a stable dark energy model. As we shall see later, the generic classical stability is guaranteed for nearly any chosen background. Some specific models are considered in Section 6.

No ordinary matter has been considered so far. Luckily, the inclusion of matter fields does not compromise the basic predictions of dark energy models (13). This will be demonstrated later in Section 5. Besides, the prevailing form of matter in the Universe is believed to be dark energy. Thus, the class of dark energy models (13) can roughly be interpreted as a zero approximation of more realistic cosmologies.

The standard physical requirements that ensure the absence of ghosts and tachyons restrain the target metric $F(\phi)$ to be positive definite, and the potential $V(\phi)$ to be bounded from below. These restrictions, however, refer to trivial vacuums. Precisely, the positive definiteness of $F(\phi)$, and the fact that $\phi = \phi_0$ is a minimum of the potential $V(\phi)$, ensure stability of the vacuum $\phi = \phi_0$, $g_{\mu\nu} = \eta_{\mu\nu}$. In this paper, however, the vacuum of interest is the nontrivial vacuum (14). Its stability is not guaranteed by the above physical requirements, and I am led to check it by direct calculation.

3 Dynamics of small perturbations

In this section, I shall examine the dynamics of small perturbations of the vacuum (14), as governed by the action functional (13). The infinitesimal change of coordinates $x^\mu \rightarrow x^\mu + \xi^\mu(x)$ leaves this action invariant, and allows the gauge fixing $\phi = t$. In this gauge, the matter field equation is identically satisfied, and we are left with the gravitational field Eq. (7). The residual diffeomorphisms are defined by the constraint $\xi^0 = 0$.

The only variable in the gauge fixed theory is the metric perturbation $h_{\mu\nu}$, defined by

$$g_{\mu\nu} = g_{\mu\nu}^{(o)} + h_{\mu\nu}.$$

With respect to the residual diffeomorphisms, it transforms as

$$\begin{aligned} \delta_0 h_{00} &= 0, \\ \delta_0 h_{0\alpha} &= -a^2 \dot{\xi}_\alpha, \\ \delta_0 h_{\alpha\beta} &= -a^2 (\xi_{\alpha,\beta} + \xi_{\beta,\alpha}), \end{aligned}$$

where δ_0 is the form variation, and $\xi_\alpha \equiv \xi^\alpha$. It is seen that $h_{0\alpha}$ can also be gauged away. The gauge condition

$$h_{0\alpha} = 0$$

restrains the gauge parameters to be functions of spatial coordinates, only. Precisely, the residual gauge parameters are defined by

$$\xi^0 = 0, \quad \dot{\xi}^\alpha = 0.$$

In what follows, I shall demonstrate how the residual gauge symmetry can further be fixed.

Let me first linearize the field Eq. (7). After cumbersome, but straightforward, calculation one obtains

$$\partial_0 \left(\dot{h}_{\alpha\alpha} - 2 \frac{\dot{a}}{a} h_{\alpha\alpha} \right) + 3a\dot{a}\dot{h}_{00} + 2(a\ddot{a} + 2\dot{a}^2)h_{00} + h_{00,\alpha\alpha} = 0, \quad (15a)$$

$$\partial_0 \left[\frac{1}{a^2} (h_{\alpha\beta,\beta} - h_{\beta\beta,\alpha}) \right] - 2 \frac{\dot{a}}{a} h_{00,\alpha} = 0, \quad (15b)$$

$$\begin{aligned} \ddot{h}_{\alpha\beta} - \frac{\dot{a}}{a} \dot{h}_{\alpha\beta} - 2 \frac{\ddot{a}}{a} h_{\alpha\beta} + \frac{1}{a^2} (h_{\alpha\gamma,\gamma\beta} + h_{\beta\gamma,\gamma\alpha} - h_{\alpha\beta,\gamma\gamma} - h_{\gamma\gamma,\alpha\beta}) \\ + \left[\frac{\dot{a}}{a} \dot{h}_{\gamma\gamma} - 2 \frac{\dot{a}^2}{a^2} h_{\gamma\gamma} + a\dot{a}\dot{h}_{00} + 2(a\ddot{a} + 2\dot{a}^2)h_{00} \right] \delta_{\alpha\beta} \\ + h_{00,\alpha\beta} = 0. \end{aligned} \quad (15c)$$

It is immediately seen that Eq. (15b) implies

$$\partial_0 \left[\frac{1}{a^2} (h_{\alpha\gamma,\gamma\beta} - h_{\beta\gamma,\gamma\alpha}) \right] = 0,$$

which tells us that the expression in square brackets does not depend on time. As a consequence, this expression can be gauged away. Indeed, its transformation law reads

$$\delta_0 \left[\frac{1}{a^2} (h_{\alpha\gamma,\gamma\beta} - h_{\beta\gamma,\gamma\alpha}) \right] = \xi_{\alpha,\gamma\gamma\beta} - \xi_{\beta,\gamma\gamma\alpha}.$$

Both the expression in square brackets and the residual gauge parameters ξ_α are functions of spatial coordinates alone. This allows the gauge fixing

$$h_{\alpha\gamma,\gamma\beta} - h_{\beta\gamma,\gamma\alpha} = 0. \quad (16)$$

In what follows, I shall simplify the analysis by the assumption that metric perturbations are spatially localized. This means that the perturbation $h_{\mu\nu}$ is assumed to decrease sufficiently fast in spatial infinity. With this assumption, many expressions are simplified. For example, the equation $X_{,\alpha} = 0$ has the general solution $X = X(t)$, but the adopted boundary conditions imply $X = 0$. Similarly, the equation $X_{,\alpha\alpha} = 0$ has the unique solution $X = 0$. With this in mind, one easily determines the residual symmetry after the gauge condition (16) has been imposed. It is defined by

$$\xi_\alpha = \epsilon_{,\alpha},$$

where the new parameter ϵ is an arbitrary function of spatial coordinates.

Let me now extract the divergence-free parts of the variable $h_{\alpha\beta}$. To this end, I use the decomposition

$$h_{\alpha\beta} \equiv \tilde{h}_{\alpha\beta} + \tilde{h}_{\alpha,\beta} + \tilde{h}_{\beta,\alpha} + \tilde{h}_{,\alpha\beta},$$

where $\tilde{h}_{\alpha\beta}$ and \tilde{h}_α are, by definition, divergence-free ($\tilde{h}_{\alpha\beta,\beta} \equiv \tilde{h}_{\alpha,\alpha} \equiv 0$). In what follows, all the expressions will be rewritten in terms of the new variables $\tilde{h}_{\alpha\beta}$, \tilde{h}_α and \tilde{h} . Let me start with the gauge condition (16). With the help of the adopted boundary conditions, it is straightforward to verify that (16) becomes

$$\tilde{h}_\alpha = 0.$$

Now, I am ready to rewrite Eq. (15) in terms of the remaining variables h_{00} , $\tilde{h}_{\alpha\beta}$ and \tilde{h} . Let me first consider the scalar sector. As it turns out, there are only three independent scalar equations. The first two are the constraint equations

$$a\ddot{a} \left[\frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_{,0} + \frac{\dot{a}}{a} \left[\frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_{,\beta\beta} = 2\dot{a}^2 \left[\frac{1}{a^2} \tilde{h} \right]_{,0\beta\beta}, \quad (17a)$$

$$\left[\frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_{,0} + 2 \frac{\dot{a}}{a} h_{00} = 0. \quad (17b)$$

Eq. (17b) follows from (15b), and (17a) is a linear combination of the trace and divergence of (15c). It is seen that h_{00} is fully determined by other variables. Thus, it carries no degrees of freedom. On the other hand, the variable \tilde{h}/a^2 is determined up to a free function of the

spatial coordinates. This freedom, however, can readily be gauged away. Indeed, the variable \tilde{h}/a^2 transforms as

$$\delta_0 \left(\frac{1}{a^2} \tilde{h} \right) = -2\epsilon$$

with respect to the residual symmetry of the model. As $\tilde{h}_{\alpha\alpha}/a^2$ is gauge invariant, the residual parameter $\epsilon = \epsilon(\vec{x})$ is exactly what one needs to fix the free integration function of (17a). In summary, the constraint equations (17) tell us that neither h_{00} nor \tilde{h}/a^2 carry physical degrees of freedom.

The third scalar equation is Eq. (15a), or equivalently, the divergence of (15c). When its coefficients are expressed in terms of the Hubble parameter $H \equiv \dot{a}/a$, it takes the form

$$\begin{aligned} H\dot{H} \left[\frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_{,00} + \left(3\dot{H}H^2 - 2\dot{H}^2 + H\ddot{H} \right) \left[\frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_{,0} \\ - \frac{1}{a^2} H\dot{H} \left[\frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_{,\beta\beta} = 0. \end{aligned} \quad (18)$$

Equation (18) governs the dynamics of the unique scalar mode of the model.

Now, I am left with the traceless, divergence-free part of Eq. (15c), which governs the dynamics of the traceless, divergence-free part of $h_{\alpha\beta}$. The latter is defined by

$$\hat{h}_{\alpha\beta} \equiv \tilde{h}_{\alpha\beta} - \frac{1}{2} \tilde{h}_{\gamma\gamma} \delta_{\alpha\beta} + \frac{1}{2} \partial_\alpha \partial_\beta \left(\Delta^{-1} \tilde{h}_{\gamma\gamma} \right),$$

where Δ^{-1} stands for the inverse of the Laplacian $\Delta \equiv \delta^{\alpha\beta} \partial_\alpha \partial_\beta$. (The existence of Δ^{-1} is guaranteed by the adopted boundary conditions, which state that the perturbations $h_{\mu\nu}$ decrease sufficiently fast in spatial infinity.) Then, the tensor part of Eq. (15c) takes the simple form

$$\left[\frac{1}{a^2} \hat{h}_{\alpha\beta} \right]_{,00} + 3\frac{\dot{a}}{a} \left[\frac{1}{a^2} \hat{h}_{\alpha\beta} \right]_{,0} - \frac{1}{a^2} \left[\frac{1}{a^2} \hat{h}_{\alpha\beta} \right]_{,\gamma\gamma} = 0. \quad (19)$$

Being subject to the constraints $\hat{h}_{\alpha\alpha} = \hat{h}_{\alpha\beta,\beta} = 0$, the variable $\hat{h}_{\alpha\beta}$ carries two physical degrees of freedom.

The scalar Eq. (18) is identically satisfied if $H = \text{const}$. This corresponds to the choice $a \propto e^{Ht}$. If this is the case, the model (13) reduces to GR with the cosmological term—the model that carries only two physical degrees of freedom. In what follows, I shall restrict my considerations to the case $H \neq \text{const}$. Then, the scalar Eq. (18) is rewritten as

$$\left[\frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_{,00} + \left(3H - 2\frac{\dot{H}}{H} + \frac{\ddot{H}}{H} \right) \left[\frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_{,0} - \frac{1}{a^2} \left[\frac{1}{a^2} \tilde{h}_{\alpha\alpha} \right]_{,\beta\beta} = 0. \quad (20)$$

It is seen that the scalar mode of the geometric sigma model (13) is massless.

4 Stability analysis

In this section, I shall examine the stability of the vacuum solution $h_{\mu\nu} = 0$. Let me start with the tensor Eq. (19). First, I introduce the collective variable

$$Q \equiv \frac{1}{a^2} \hat{h}_{\alpha\beta}.$$

In terms of Q , Eq. (19) takes the compact form

$$\ddot{Q} + 3\frac{\dot{a}}{a}\dot{Q} - \frac{1}{a^2}Q_{,\alpha\alpha} = 0. \quad (21)$$

The function $Q(x)$ is searched for in the form

$$Q = \text{Re} \int d^3k q(k, t) e^{i\vec{k}\cdot\vec{x}}, \quad (22)$$

whereupon Eq. (21) becomes

$$\ddot{q} + 3\frac{\dot{a}}{a}\dot{q} + \frac{k^2}{a^2}q = 0. \quad (23)$$

The stability of the vacuum solution $q = 0$ is examined by the canonical analysis of Eq. (23). In the first step, I notice that the Eq. (23) is obtained from the Lagrangian

$$\mathcal{L} = a(a^2\dot{q}^2 - k^2q^2).$$

Indeed, it is easily verified that its variation leads to the Eq. (23). The corresponding Hamiltonian is straightforwardly calculated to be

$$\mathcal{H} = a \left(\frac{1}{4a^4} p^2 + k^2 q^2 \right). \quad (24)$$

The Hamiltonian is positive for all the allowed values of the parameter $a(t)$, and all the values of the wave vector \vec{k} . Its minimum is located at $q = p = 0$. For Hamiltonians with no explicit time dependence, this would imply the stability of the vacuum $q = p = 0$. Indeed, owing to $\dot{\mathcal{H}} = \{\mathcal{H}, \mathcal{H}\} = 0$, the physical phase space trajectories coincide with the orbits $\mathcal{H} = \text{const}$. These orbits, on the other hand, are closed curves around the vacuum $q = p = 0$. As a consequence, a phase space trajectory which is initially close to the vacuum continues to be in the vicinity of the vacuum at all times.

Unfortunately, the Hamiltonian (24) depends on time through the free parameter $a(t)$. As a consequence,

$$\dot{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial t} + \{\mathcal{H}, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial t}, \quad (25)$$

and the Hamiltonian is not conserved. Still, the stability of the vacuum $q = p = 0$ is not compromised. To see this, note that the orbits $\mathcal{H} = \text{const}$ remain closed curves around $q = p = 0$, only this time they evolve in

time as shown in Fig. 1. It is seen from Eq. (25) that the change of the Hamiltonian along the phase space trajectory $(q(t), p(t))$ is the same as for the still point $(q, p) = \text{const}$. Thus, the general solution of Eq. (25) is the phase space trajectory which, at any time, touches the respective orbit $\mathcal{H} = \text{const}$. As these orbits are closed curves around $q = p = 0$, the phase space trajectories initially close to $q = p = 0$ remain close to $q = p = 0$ at all times. In other words, the vacuum solution $q = p = 0$ is stable against small perturbations. This holds true for any $a(t)$ and \vec{k} .

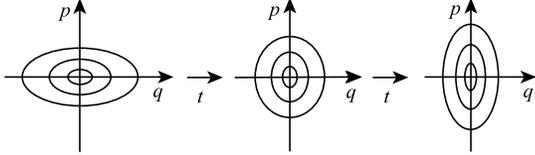


Fig. 1. Time evolution of orbits $H = \text{const}$.

The stability of the scalar Eq. (20) is examined analogously. Using the notation

$$Q \equiv \frac{1}{a^2} \tilde{h}_{\alpha\alpha},$$

Eq. (20) is rewritten as

$$\ddot{Q} + \left(3H - 2\frac{\dot{H}}{H} + \frac{\ddot{H}}{H} \right) \dot{Q} - \frac{1}{a^2} Q_{,\alpha\alpha} = 0, \quad (26)$$

and its Fourier transform, as defined by (22), becomes

$$\ddot{q} + \left(3H - 2\frac{\dot{H}}{H} + \frac{\ddot{H}}{H} \right) \dot{q} + \frac{k^2}{a^2} q = 0. \quad (27)$$

(This notation should not be confused with the same notation used in the analysis of tensor modes.) It is easily checked that Eq. (27) follows from the Lagrangian

$$\mathcal{L} \propto a \frac{\dot{H}}{H^2} (a^2 \dot{q}^2 - k^2 q^2),$$

or equivalently, from the Hamiltonian

$$\mathcal{H} \propto a \frac{\dot{H}}{H^2} \left(\frac{1}{4a^4} \frac{H^4}{\dot{H}^2} p^2 + k^2 q^2 \right). \quad (28)$$

The proportionality sign reminds us that \mathcal{L} and \mathcal{H} are determined only up to a multiplicative constant. If necessary, this freedom can be used to correct the overall sign of the Hamiltonian. As opposed to the Hamiltonian (24), the Hamiltonian (28) is not necessarily positive. However, it is only the overall sign of \mathcal{H} that can be negative. This implies that the vacuum $q = p = 0$ can only be a maximum or a minimum, and never a saddle point. As a consequence, the lines of constant \mathcal{H} are closed curves

around $q = p = 0$. From this point on, the stability analysis reduces to that of the tensor mode, leading to the conclusion that the vacuum $q = p = 0$ is stable against scalar perturbations, too. This holds true in a generic time interval, and for a generic choice of $a(t)$.

However, there can exist critical moments in the evolution of scalar perturbations, where the stability may be lost. As seen from Eq. (28), these are defined by $H = 0$ or $\dot{H} = 0$. In fact, the stability discussed in this section is proven only up to the presence of critical moments. Such critical moments can be found in many cosmological models, as will be demonstrated in Section 6. Some of them are benign, but some may cause divergent behavior. It is important to realize that the construction of geometric sigma models does not guarantee the absence of critical moments. As a consequence, not every choice of the scale factor $a(t)$ leads to an everywhere regular model. One should have in mind, however, that the linear analysis considered in this paper is inapplicable in the vicinity of critical moments. Indeed, the perturbations grow big there, and the interaction terms cannot be neglected.

Before I close this section, let me mention once more that matter fields have been excluded from the above analysis. As a consequence, the proven stability may be compromised. Indeed, the target metric $F(\phi)$ has been allowed to take negative values, and the potential $V(\phi)$ to be unbounded from below. Then, the conventional coupling to matter fields may lead to the appearance of ghosts and tachyons. In this paper, however, I am examining the linear stability alone. It will be demonstrated in the next section that matter fields cannot appear in the linearized Einstein's and scalar field equations. Thus, the established stability is not threatened. The stability of matter perturbations themselves, on the other hand, is studied separately.

5 Matter fields

In this section, I shall examine how the presence of matter fields influences the behavior of geometric sigma models. My starting point is the action

$$I = I_g + I_m, \quad (29)$$

where I_g is the geometric action (13), and I_m stands for the action of matter fields. Customarily, the matter Lagrangian is thought of as the Lagrangian of the standard model of elementary particles, minimally coupled to gravity, and possibly, to the inflaton field ϕ . In what follows, the matter fields will collectively be denoted by Ω .

Let me first consider the simplest case characterized by the absence of direct matter–inflaton couplings. Instead, the matter fields are minimally coupled to the metric alone. This choice is justified by the fact that

minimal coupling to the metric already contains a simple coupling to the inflaton itself. Indeed, it will shortly be shown that matter fields do not compromise the gauge fixing procedure of the preceding sections. This procedure leaves us with three dynamical metric components, one of which is reminiscent of the original inflaton field. Of course, one can always change this simple scenario by employing direct couplings to the inflaton field. It will be shown later that most of the direct inflaton couplings preserve the results of the preceding sections.

5.1 Vacuum solution

In this subsection, I shall examine how the sigma model vacuum (14) is influenced by the presence of matter fields. Let me start with the analysis of the matter field equations

$$\frac{\delta I_m}{\delta \Omega} = 0. \quad (30)$$

Owing to the minimal coupling to the metric, these equations have a vacuum solution that coincides with the well known vacuum of the standard model of elementary particles. Indeed, the standard model equations are trivially satisfied when matter fields take proper constant values

$$\Omega = \Omega_0,$$

and so are Eq. (30). This holds true for any value of the metric that appears in the field equations. In particular, the vacuum value of the stress-energy tensor

$$T_m^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta I_m}{\delta g_{\mu\nu}} \quad (31)$$

is zero, irrespectively of the presence of $g_{\mu\nu}$. Formally,

$$\Omega = \Omega_0 \quad \Rightarrow \quad T_m^{\mu\nu} = 0$$

for any $g_{\mu\nu}$. With this, the inflaton and Einstein's equations take the form of the sigma model equations considered in the preceding sections. Indeed, owing to the absence of the matter-inflaton coupling, the inflaton equation is the same as that of the sigma model,

$$\frac{\delta I}{\delta \phi} = \frac{\delta I_g}{\delta \phi} = 0.$$

Einstein's equations, on the other hand, reduce to the sigma model equations once the matter fields take their vacuum values,

$$\Omega = \Omega_0 \quad \Rightarrow \quad R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = T_\phi^{\mu\nu}.$$

It has already been shown in Section 2 that geometric sigma model have a vacuum solution (14). As a consequence, the field equations that follow from the action (29) are satisfied by

$$\Omega = \Omega_0, \quad \phi = t, \quad g_{\mu\nu} = g_{\mu\nu}^{(o)}. \quad (32)$$

This is the vacuum, or shall we say, the background solution of the model (29). The presence of matter fields does not compromise the sigma model vacuum of the preceding sections. This result is summarized in the sentence

- matter fields do not violate the sigma model vacuum.

In what follows, I shall examine the stability of the vacuum (32) against its small perturbations. Owing to their smallness, the dynamics of vacuum perturbations is governed by the linearized field equations. Therefore, the linear stability of the model (29) is examined in the next subsection.

5.2 Stability analysis

The linear stability of the vacuum (32) is examined by inspecting the linearized field equations of the action (29). It is immediately seen that, after linearization, the inflaton and Einstein's equations reduce to those of the geometric sigma model of the preceding sections. Indeed, the stress-energy tensor $T_m^{\mu\nu}$, being at least quadratic in perturbations of matter fields, does not appear on the r.h.s. of the linearized Einstein's equations. At the same time, the inflaton does not couple to matter fields, at all. Hence, the linearized inflaton and metric equations of motion remain unchanged by the inclusion of matter. They are diffeomorphism invariant, so that the complete gauge fixing procedure of Section 3 is still valid. In this gauge, the tensor and scalar equations (19) and (20) are exactly what one obtains from the linearized equations $\delta I/\delta \phi = 0$ and $\delta I/\delta g_{\mu\nu} = 0$.

The generic linear stability of the sigma model vacuum has already been established in the preceding section. What remains to be shown is the linear stability of the matter field equations. To this end, notice that the minimal coupling to matter fields implies the validity of the principle of equivalence. Indeed, the matter fields of the field equations $\delta I/\delta \Omega = 0$ are coupled to at most first derivatives of the metric $g_{\mu\nu}$. In a local inertial frame, the metric derivatives vanish, and the metric itself becomes the Minkowski metric $\eta_{\mu\nu}$. In this way, the matter field equations take their special relativistic form—the standard model of elementary particles in flat spacetime. The latter is known to be stable against perturbations of its trivial vacuum. As a consequence, the linearized matter field equations have the stable vacuum $\Omega = \Omega_0$. To summarize, the vacuum (32) has a generic linear stability against perturbations governed by the action (29). The dynamics of its geometric part remains the same as found in Section 3. Therefore,

- the presence of matter fields does not violate the established linear dynamics of geometric sigma models.

What remains to be found is the dynamics of the linearized matter field equations. There are three types of matter fields that appear in the action: fermion fields, gauge fields and the Higgs. In the linearized theory, fermion fields are governed by the Dirac equation, gauge fields obey Maxwell equations, and the Higgs is subject to the Klein-Gordon equation. All of these are minimally coupled to the external curved background.

5.2.1 Scalar fields

The scalar φ , minimally coupled to the external metric $g_{\mu\nu}^{(o)}$, obeys the Klein-Gordon equation $(\square - m^2)\varphi = 0$. The scalar field mass is denoted by m , and \square stands for the covariant d'Alembertian of the vacuum metric $g_{\mu\nu}^{(o)}$. With $g_{\mu\nu}^{(o)}$ of the form (10), the Klein-Gordon equation becomes

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + \left(m^2 - \frac{1}{a^2}\Delta\right)\varphi = 0,$$

where $\Delta \equiv \delta^{\alpha\beta}\partial_\alpha\partial_\beta$. Now, one can use the Fourier decomposition (22) of the preceding section to rewrite the above equation in the form

$$\ddot{q} + 3\frac{\dot{a}}{a}\dot{q} + \left(m^2 + \frac{k^2}{a^2}\right)q = 0. \quad (33)$$

This equation differs from the scalar perturbation Eq. (27) not only by the presence of mass, but also by the different friction coefficient. During the inflationary phase, when the Hubble parameter is approximately constant, the friction coefficient of Eq. (27) is, in most cases, close to that of Eq. (33). In some cases, however, the scalar modes of the geometric sigma model may have significantly higher friction, as will be demonstrated in the next section. In such situations, the rapid expansion of the Universe makes the inflaton decay much faster than scalars of the matter Lagrangian.

5.2.2 Gauge fields

In the linear approximation, gauge fields obey the equation $\nabla_\mu F^{\mu\nu} = 0$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the linearized gauge field strength, and ∇_μ stands for the covariant derivative. To simplify calculations, I shall work in the Coulomb gauge. Then, the time component A_0 is constrained to be zero, while the spatial components A_α satisfy

$$\ddot{A}_\alpha + \frac{\dot{a}}{a}\dot{A}_\alpha - \frac{1}{a^2}\Delta A_\alpha = 0, \quad \partial_\alpha A_\alpha = 0.$$

In terms of their Fourier components q_α , the field equations take the form

$$\ddot{q}_\alpha + \frac{\dot{a}}{a}\dot{q}_\alpha + \frac{k^2}{a^2}q_\alpha = 0, \quad k_\alpha q_\alpha = 0. \quad (34)$$

It is seen that the friction coefficient of Eq. (34) is three times smaller than that of Eq. (23). Thus, in the expanding Universe, the gauge fields outlast the excitations of the gravitational field.

5.2.3 Dirac field

The evolution of the Dirac field minimally coupled to gravity is given by the equation

$$(i\gamma^k\nabla_k - m)\psi = 0, \quad (35)$$

where γ^k are Dirac gamma matrices ($\{\gamma^i, \gamma^j\} = -2\eta^{ij}$), and ∇_k stands for the covariant derivative

$$\nabla_k\psi = e_k{}^\mu(\partial_\mu + \omega_\mu)\psi.$$

The gravitational variables that enter this equation are the tetrad $e^k{}_\mu$, and the spin connection $\omega^{ij}{}_\mu$. The tetrad $e^k{}_\mu$ is the inverse of $e_k{}^\mu$, while $\omega^{ij}{}_\mu$ are the components of ω_μ in the basis of Lorentz generators $\sigma_{ij} \equiv \frac{1}{4}[\gamma_i, \gamma_j]$,

$$\omega_\mu = \frac{1}{2}\omega^{ij}{}_\mu\sigma_{ij}.$$

The metric $g_{\mu\nu}$ and the connection $\Gamma^\lambda{}_{\mu\nu}$, which are used in this paper, are related to $e^k{}_\mu$ and $\omega^{ij}{}_\mu$ through the equations [37]

$$g_{\mu\nu} = \eta_{ij}e^i{}_\mu e^j{}_\nu, \quad e^i{}_{\mu,\nu} + \omega^i{}_{j\nu}e^j{}_\mu - \Gamma^\lambda{}_{\mu\nu}e^i{}_\lambda = 0. \quad (36)$$

The first equation is the very definition of the orthonormal tetrad, while the second represents the metricity condition. Given the metric $g_{\mu\nu}$, the tetrad is determined only up to the local Lorentz rotations. These, however, are a symmetry of the Dirac Eq. (35), and can be gauge fixed. Thus, starting with the background metric $g_{\mu\nu}^{(o)}$, the simplest solution for the background tetrad is found to have the diagonal form

$$e^0{}_0 = 1, \quad e^1{}_1 = e^2{}_2 = e^3{}_3 = a.$$

Then, the second Eq. (36) yields the background value of the spin connection. The only non-zero components turn out to be

$$\omega^{b0}{}_\alpha = -\omega^{0b}{}_\alpha = \dot{a}\delta_\alpha^b.$$

With the known background values of the tetrad and spin connection, the Dirac Eq. (35) is rewritten as

$$\dot{\psi} + \frac{1}{a}\gamma^0\gamma^\alpha\psi_{,\alpha} + \frac{1}{2}\left(3\frac{\dot{a}}{a} + 2im\gamma^0\right)\psi = 0.$$

Its Fourier expansion then yields

$$\dot{q} + \frac{1}{2}\left[3\frac{\dot{a}}{a} + 2i\gamma^0\left(m + \frac{1}{a}\vec{\gamma}\cdot\vec{k}\right)\right]q = 0, \quad (37)$$

where $q(\vec{k}, t)$ are defined by

$$\psi(x) = \int d^3k q(\vec{k}, t) e^{i\vec{k}\cdot\vec{x}}.$$

Using the notation

$$q = \begin{pmatrix} u \\ v \end{pmatrix}, \quad u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}, \quad v = \begin{pmatrix} v_+ \\ v_- \end{pmatrix},$$

the 4-component Eq. (37) is rewritten as a pair of 2-component equations. The first is the constraint equation

$$v = \frac{i}{m} \left[\dot{u} + \frac{1}{2a} (3\dot{a} - 2i\vec{\sigma} \cdot \vec{k}) u \right],$$

which tells us that v carries no degrees of freedom. The second is the dynamical equation

$$\ddot{u} + 3H\dot{u} + \left[m^2 + \frac{\vec{k}}{a} \cdot \left(\frac{\vec{k}}{a} + iH\vec{\sigma} \right) + \frac{3}{4} (2\dot{H} + 3H^2) \right] u = 0.$$

If the coordinates are rotated so that \vec{k} is directed along the z -axis, the above equation turns into a system of two one-component equations

$$\ddot{u}_\pm + 3H\dot{u}_\pm + \left[m^2 + \frac{k}{a} \left(\frac{k}{a} \pm iH \right) + \frac{3}{4} (2\dot{H} + 3H^2) \right] u_\pm = 0. \tag{38}$$

The friction coefficient coincides with that of the scalar field, whereas the mass term is different. In the early stage of inflation, when a is still very small, the inequality $k \gg aH$ holds for a wide range of k . In this regime, the mass term is close to that of the scalar field. At the end of inflation, however, the scale factor is large, so that most k satisfy $k \ll aH$. Then, the mass term takes the form $m^2 + (3H/2)^2$. As the inflationary value of H is typically much larger than masses of elementary fermions, Eq. (38) is practically independent of m . This means that the production and propagation of elementary fermions are the same for all the fermion species.

5.3 Matter-inflaton coupling

So far, the considered matter fields have been assumed to couple to the metric alone. It has been shown that the results of the previous sections are not compromised by the inclusion of such matter. In this subsection, I shall discuss direct matter-inflaton couplings.

Let me start with the verification of the vacuum solution (32). It is immediately seen that interaction terms which are at least quadratic in perturbations of matter fields do not compromise the vacuum (32). Indeed, their contribution to the field equations disappears when $\psi = \psi_0$, thereby making (32) a valid vacuum solution. The sigma model vacuum (14), on the other hand, satisfies the field equations even when the couplings are linear in perturbations of matter fields. Thus,

- typical matter-inflaton couplings preserve the sigma model vacuum.

The examples of possible matter-inflaton couplings are

$$f(\phi)g(\chi), \quad f(\phi)\bar{\psi}\psi, \quad f(\phi)F_{\mu\nu}F^{\mu\nu}$$

and so on. It is seen that symmetries of the standard model Lagrangian restrict most of these interaction terms to be at least quadratic in matter fields. This is fortunate because such matter-inflaton couplings preserve the results of the preceding sections. Indeed, the linearized Einstein's and inflaton equations remain the same as those obtained in the absence of matter fields. Thus,

- the analysis of the preceding sections is not compromised by the presence of interaction terms which are at least quadratic in matter fields.

What does change, however, is the form of the linearized matter field equations. As an example, let me consider Dirac field with the inflaton coupling of the form $f(\phi)\bar{\psi}\psi$. Then, the linearized equations (38) take the form

$$\ddot{u}_\pm + A\dot{u}_\pm + \left[M^2 + \frac{k}{a} \left(\frac{k}{a} \pm iB \right) + \frac{3}{4} H (2C + 3H) \right] u_\pm = 0, \tag{39}$$

where $M(t) \equiv m - f(t)$, and

$$A \equiv 3H - \frac{\dot{M}}{M}, \quad B \equiv H + \frac{\dot{M}}{M}, \quad C \equiv \frac{\dot{H}}{H} - \frac{\dot{M}}{M}.$$

It is seen that Eq. (39) has singular points defined by $M(t) = 0$. The easiest way to get rid of these is to restrain the couplings to satisfy $f(\phi) < m$, such as, for example, the coupling $f = \xi\phi^2$ when $\xi < 0$. The simplest choice $f = \lambda\phi$ makes the perturbations u_\pm diverge at $t = m/\lambda$. However, this coupling is perfectly acceptable in cosmological models whose initial singularity is located at $t \geq m/\lambda$. For example, if the Universe is born at $t = 0$, the coupling $f = \lambda\phi$ with $\lambda < 0$ yields an everywhere regular dynamics.

Before I close this section, let me note that matter-inflaton coupling can significantly modify the dynamics of matter fields. In particular, the particle production rate can be increased. For example, the coupling $f = \lambda\phi$ with $\lambda < 0$ diminishes the value of the friction coefficient to

$$A = 3H + \frac{\lambda}{m - \lambda t}.$$

If $|\lambda|$ is large enough, there is a time interval in which the friction becomes negative. During that time, the initial matter perturbations keep growing, despite the rapid expansion of the Universe. Obviously, this makes the production of particles more effective. Unfortunately, the negative friction in this example does not last long. It is more than one hundred times shorter than the typical inflationary period. Still, it is always possible to adjust the matter-inflaton coupling to obtain the desired particle production rate.

6 Examples

In this section, I shall analyze three specific choices of the Universe dynamics. In the first, a toy model is used to demonstrate how the described procedure works in practice. In the second, I propose an inflationary model close to the standard model of the Universe. The third is a model of a bouncing Universe. Neither of these models considers matter fields. They are solely used for the demonstration of how a dark energy action is associated with a chosen background geometry of the Universe. Hopefully, the right choice of the scale factor $a(t)$ will ultimately lead to a realistic cosmological model.

6.1 Toy model

Let me consider a homogeneous, isotropic and spatially flat geometry (10) with the scale factor of the simple form

$$a(t) = \frac{1}{\cosh \omega t}. \quad (40)$$

Its graph is displayed in Fig. 2. It describes an ever-existing Universe, whose inflationary epoch begins at infinite past and lasts infinitely many e -folds. The exit from inflation happens at $t \approx -1/\omega$. The constant ω is a free parameter of the model.

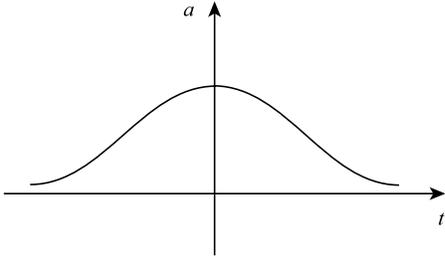


Fig. 2. Toy model.

The scale factor (40) is a solution of the sigma model (13) in which the potential $V(\phi)$ and the target metric $F(\phi)$ are calculated from (11) and (12). This procedure straightforwardly leads to

$$F(\phi) = \frac{2\omega^2}{\cosh^2 \omega \phi}, \quad V(\phi) = \frac{6 \sinh^2 \omega \phi - 2}{\cosh^2 \omega \phi} \omega^2. \quad (41)$$

With $F(\phi)$ and $V(\phi)$ defined by Eq. (41), the action functional (13) has the solution (14), in which $g_{\mu\nu}^{(o)}$ is defined by Eq. (10) and (40). The model can further be simplified by a suitable redefinition of the scalar field. Specifically, the redefinition $\phi \rightarrow \chi(\phi)$ of the form

$$\chi \equiv 2 \arctan(\sinh \omega \phi) \quad (42)$$

brings the action (13) to the form

$$I_g = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} \chi'^{\mu} \chi_{,\mu} - U(\chi) \right].$$

The new potential $U(\chi) = V(\phi(\chi))$ reads

$$U(\chi) = 2\omega^2 \left(4 \sin^2 \frac{\chi}{2} - 1 \right).$$

In terms of χ , the nontrivial vacuum solution $\phi = t$ becomes

$$\chi(t) = 2 \arctan(\sinh \omega t), \quad (43)$$

while the metric $g_{\mu\nu} = g_{\mu\nu}^{(o)}$ remains the same. The graphs of the potential $U(\chi)$, and the solution (43) are displayed in Figs. 3 and 4. It is seen that the potential $U(\chi)$ is a periodic function of χ , with the period 2π . Thus, the scalar field χ lives on a circle. The soliton solution (43) is one-to-one mapping $R^1 \rightarrow S^1$. As for the trivial solutions, the theory accommodates two of them. The first is given by $\chi = 0$, and has an anti-de Sitter metric. The second has $\chi = \pm\pi$, and the metric is de Sitter. Only the first solution is stable, because $\chi = 0$ is the minimum of the potential $U(\chi)$, and the anti-de Sitter metric is a stable solution of the corresponding geometric equation [38]. The unstable solution $\chi = \pm\pi$ is the limiting case of the stable soliton solution when $t \rightarrow \pm\infty$. It seems as if a stable soliton is asymptotically unstable. However, there is no contradiction in this unusual situation. This is because the stability analysis of Section 4 deals with infinitesimal perturbations, which preserve the monotonic character of the soliton solution. As a consequence, the scalar χ can be gauged away. This is not the case with the trivial vacuum $\chi = \pm\pi$. No matter how small the perturbations of this vacuum are, they can not be gauged away. This is why the stability of $\chi = \pm\pi$ cannot be a substitute for the asymptotic stability of the soliton vacuum.

Regarding perturbations of the background solution (40), it is immediately seen that $t = 0$ is the only critical point of this model. In this point, the coefficients of the tensor equation (23) are regular, but the friction coefficient of the scalar equation (27) diverges as $-2/t$. Luckily, this singularity turns out not to be harmful. Indeed, a careful analysis shows that scalar perturbations formed in the past regularly pass the critical point $t = 0$.

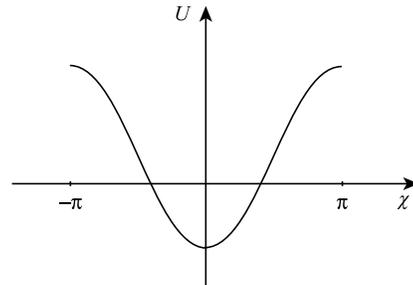


Fig. 3. Potential function $U(\chi)$.

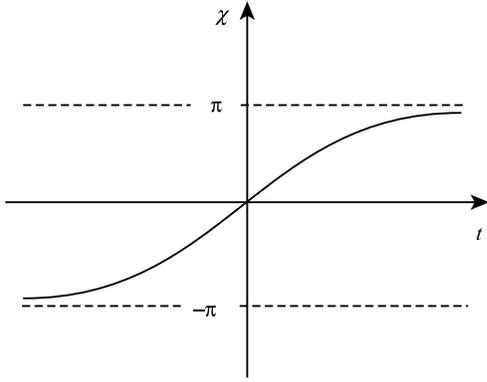


Fig. 4. Soliton solution.

Finally, let me calculate the friction coefficient of the scalar mode of this model. A simple calculation shows that it approaches the value 5ω when $t \rightarrow -\infty$. In the same epoch, the friction coefficient of matter scalars (which obey Eq. (33)) has the value 3ω . Thus, during rapid expansion of the Universe, the inflaton decays faster than matter scalars. Moreover, if the model is defined by

$$a(t) = (\cosh \omega t)^{-\frac{1}{n}},$$

the ratio of the two friction coefficients becomes $1+2n/3$. When $n \gg 1$, the inflaton decay rate becomes much larger than that of matter scalars.

6.2 Inflationary Universe

In the second example, I shall examine a scale factor of the form

$$a(t) = \left[1 + \tanh(8\omega t) \right] \ln \left(1 + \exp \frac{\omega t - 4}{4} \right). \quad (44)$$

As seen from its graph in Fig. 5, it mimics the standard model of the Universe. Indeed, all the expected phases of the cosmological evolution are there. The inflationary epoch begins at infinite past, and lasts infinitely many e -folds. The exit from inflation happens at $\omega t \approx 0$, when the early acceleration stops. The Universe continues to expand slowly, until it reaches the moment when the late time acceleration begins. The present epoch is located at $\omega t \approx 7.7$. I shall demonstrate later how the parameter

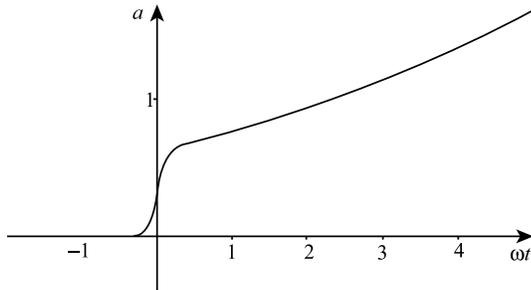
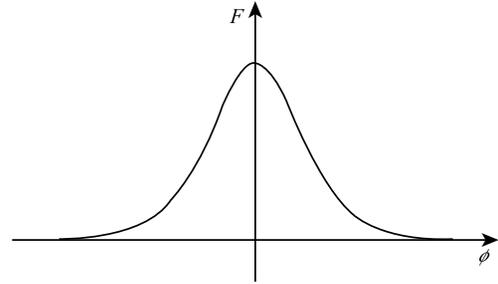
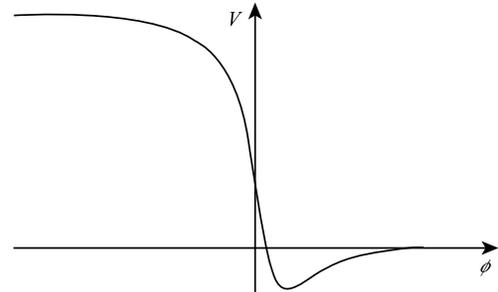


Fig. 5. Inflationary Universe.

ω and the present time t_{now} are calculated from the known values of the Hubble and deceleration parameters.

The action functional whose vacuum solution is defined by Eq. (44) has the form (13), with $F(\phi)$ and $V(\phi)$ calculated from Eqs. (11) and (12). A straightforward procedure leads to cumbersome expressions which I choose not to display here. Instead, their graphs are presented. The action (13), with $F(\phi)$ and $V(\phi)$ depicted in Figs. 6 and 7, has the soliton solution $\phi = t$.


 Fig. 6. Target metric $F(\phi)$.

 Fig. 7. Potential $V(\phi)$.

The parameter ω can be determined from the known values of the Hubble and deceleration parameters. These are defined as

$$H \equiv \frac{\dot{a}}{a}, \quad q \equiv -\frac{\ddot{a}}{aH^2}.$$

With the scale factor given by Eq. (44), the parameters H/ω and q depend on ω and t only through the combination ωt . The direct calculation yields the functions whose graphs are displayed in Fig. 8. Using these, one easily finds that the astronomically observed values

$$H = 0.075 \text{ Gyr}^{-1}, \quad q = -0.5$$

imply $\omega t = 7.68$ and $H/\omega = 0.14$. Thus,

$$\omega = 0.53 \text{ Gyr}^{-1}, \quad t_{\text{now}} = 14.55 \text{ Gyr}.$$

The obtained value of t_{now} is the time coordinate measured from the arbitrarily chosen origin $t = 0$. This is not what one would like to have. Instead, the present epoch should be measured relative to a physically significant moment in the history of the Universe. This cannot be the initial singularity, as our model does not have one. Instead, my choice is the end of inflation. The end of inflation t_{inf} is naturally defined as the moment when the early acceleration stops. It is seen from the graph in Fig. 5 that $\omega t_{\text{inf}} = 0.003$. This leads to

$$t_{\text{now}} - t_{\text{inf}} = 14.54 \text{ Gyr.} \quad (45)$$

The time interval (45) is a substitute for what is commonly called the age of the Universe.

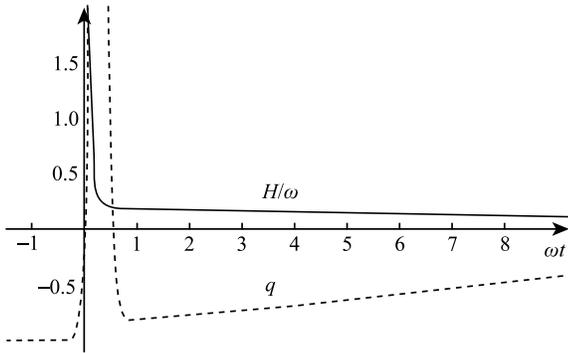


Fig. 8. Hubble and deceleration parameters.

So far, I have discussed everywhere regular cosmological models. However, these can easily be modified to become models with an initial singularity. For example, the model under consideration can be redefined by replacing its scale factor $a(t)$ with

$$\tilde{a}(t) = a(t) - a(t_0).$$

The new scale factor describes a Universe which is born at $t = t_0$, and lives regularly ever after. As opposed to the inflationary epoch of the everywhere regular model (44), the inflationary epoch of the new model lasts a finite number of e -folds. This number can be made arbitrarily large by letting $t_0 \rightarrow -\infty$. It can also be shown that the new model has no singularities other than $t = t_0$.

Finally, let me briefly discuss the propagation of small perturbations. It is seen from Figs. 5 and 6 that neither the Hubble parameter H , nor the target metric F have zeros. As a consequence, the model under consideration has no critical moments. This means that both tensor and scalar perturbations have everywhere regular dynamics. In particular, their friction coefficients are everywhere finite. While tensor fluctuations have always positive friction, the friction of the scalar fluctuations can become negative. Indeed, a straightforward analysis shows that, after the inflation, the scalar friction

abruptly drops and becomes negative. The period of negative friction does not last long. When the late time acceleration begins, it already has a small positive value which gradually approaches zero as $t \rightarrow \infty$. In the next subsection, I shall present the example of a model whose scalar fluctuations have critical moments.

6.3 Bouncing Universe

In this example, I shall consider a scale factor of the form

$$a(t) = (1 + \omega^2 t^2)^{\frac{1}{4}}. \quad (46)$$

Its graph is displayed in Fig. 9. It defines a bouncing Universe which begins in the infinite past, slowly shrinks to its minimal size, and then bounces to an expanding phase. The action functional whose vacuum solution is defined by Eq. (46) has the form (13), with $F(\phi)$ and $V(\phi)$ calculated from Eqs. (11) and (12). A straightforward procedure leads to

$$F(\phi) = \omega^2 \frac{\omega^2 \phi^2 - 1}{(\omega^2 \phi^2 + 1)^2}, \quad V(\phi) = \omega^2 \frac{\omega^2 \phi^2 + 2}{2(\omega^2 \phi^2 + 1)^2}. \quad (47)$$

The action (13), with $F(\phi)$ and $V(\phi)$ defined by Eq. (47), has the spatially homogeneous and isotropic solution

$$\phi(t) = t, \quad ds^2 = -dt^2 + \sqrt{1 + \omega^2 t^2} (dx^2 + dy^2 + dz^2).$$

It is seen that the background spacetime is flat in the infinite past and future. Indeed, all the curvature invariants are shown to fall off as $1/t^2$ or faster as $|t| \rightarrow \infty$. Thus, the Universe in this example evolves out of the flat spacetime.

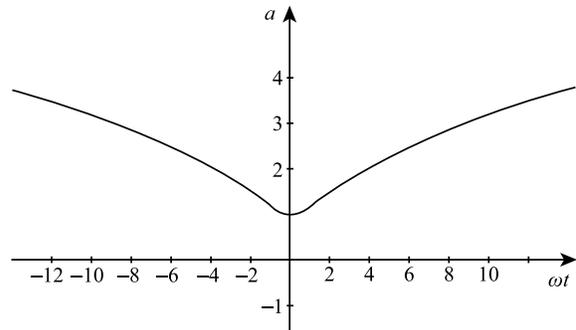


Fig. 9. Bouncing Universe.

How do metric perturbations propagate in this background? A simple analysis shows that tensor perturbations are everywhere regular, whereas scalar ones have critical points. It is straightforward to verify that there are three critical points: the two zeroes of \dot{H} , located in $t = \pm 1/\omega$, and the zero of H , located in $t = 0$. These are the singularities of the friction coefficient in Eq. (27). Its behavior in the vicinity of the critical points is given by

$$3H-2\frac{\dot{H}}{H}+\frac{\ddot{H}}{\dot{H}} \sim \begin{cases} -\frac{2}{t} & \text{in the vicinity of } t=0, \\ \frac{1}{t \pm 1/\omega} & \text{in the vicinity of } t = \mp 1/\omega. \end{cases}$$

As has already been mentioned, the singularity of the form $-2/t$ does not violate the regular propagation of scalar perturbations. However, this is not the case with the singularities $t = \pm 1/\omega$. These are shown to act as barriers to the scalar perturbations coming from the past. Indeed, the scalar perturbations go to infinity as they approach $t = \pm 1/\omega$. This is an undesirable property, which calls for the rejection of the model. Still, one should be aware of the fact that our linear analysis makes no sense if the fields are too strong. In such situations, the interaction terms should be taken into account. Hopefully, this could cure the singularity problem.

7 Recapitulation

The purpose of this work has been to demonstrate how an arbitrarily chosen background of the Universe can be made a solution of a simple model. To this end, I made use of the concept of geometric sigma models. These models possess two distinctive features. The first is that any metric can be made a solution of a particular geometric sigma model. The second ensures that the complete matter content can be gauged away. In this paper, a geometric sigma model is associated with an arbitrary homogeneous, isotropic and spatially flat geometry. In its simplest form, the model describes one scalar field in interaction with Einstein's gravity. It possesses the vacuum solution $\phi = t$, $g_{\mu\nu} = g_{\mu\nu}^{(o)}$. It is important to emphasize that, while the background metric $g_{\mu\nu}^{(o)}$ can be chosen arbitrarily, the physics of its small perturbations can not. In fact, the role of the background metrics $g_{\mu\nu}^{(o)}$ is to parametrize the class of models presented in this paper. This way, the search for a viable cosmological model reduces to the proper choice of the background metric.

The present work began with a recapitulation of the concept of geometric sigma models. The construction of the generic model was presented, and subsequently applied to the homogeneous, isotropic and spatially flat geometry of the Universe. Then, the dynamics of small, localized perturbations of the vacuum was examined. It was demonstrated that all but three degrees of freedom can be gauged away. The classical stability of the gauge fixed linearized theory was proven in Sec. 4. This was

done by direct calculation, as stability of the vacuum solution is not guaranteed by the very construction of geometric sigma models. The vacuum stability against matter fluctuations was considered in Section 5. It was shown that the inclusion of matter fields does not compromise the results of the preceding sections. The analysis of the stability against matter perturbations concluded the general considerations of the paper.

The rest of the paper was devoted to examples. The first was a toy model used to demonstrate how the method works in practice. The second was the example of an inflationary Universe, and the third was a bouncing model. The corresponding action functionals were constructed along the lines described in Section 2. The obtained target metric $F(\phi)$, and the potential $V(\phi)$ were presented through their graphs. The graphical method was also used for numerical calculations. In particular, it was demonstrated how the parameters of the model are calculated from the known values of the Hubble and deceleration parameters.

I have also given a brief insight into the propagation of small perturbations. It is argued that the generic linear stability proven in Section 4 fails in the vicinity of critical moments $\dot{H} = 0$. There, the scalar perturbations diverge, so that a consistent stability analysis must go beyond the linear approximation. A more elaborate analysis of this topic is left for future investigation. In particular, nonperturbative analysis near the critical moments, and the study of more complex geometric sigma models, is planned.

At the end, let me point out that the procedure described in this paper misses an important ingredient. It concerns the back reaction of quantum vacuum fluctuations on the background geometry. Indeed, the only matter that the geometric sigma models of Section 2 deal with is a specific dark energy of purely geometric origin. Although it successfully generates any desirable geometry of the Universe, the influence of quantum fluctuations of ordinary matter has not been taken into account. Instead, it has been demonstrated in Section 5 that matter fields preserve the classical linear stability established in Section 6. The completion of the incomplete cosmology presented in this paper requires the inclusion of quantum fluctuations of both geometry and matter. Until then, it is comforting to know that dark energy is commonly believed to dominate all other forms of matter in the Universe. Owing to this, the predictions of this work may not be far from realistic, after all.

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