

Neutrino mass ordering and μ - τ reflection symmetry breaking*Zhi-zhong Xing(邢志忠)^{1,2,3;1)} Jing-yu Zhu(朱景宇)^{1,2;2)}¹ Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China² School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China³ Center for High Energy Physics, Peking University, Beijing 100080, China

Abstract: If the neutrino mass spectrum turns out to be $m_3 < m_1 < m_2$, it may be relabeled as $m'_1 < m'_2 < m'_3$ such that all the masses of fundamental fermions with the same electrical charges are in order. In this case the columns of the 3×3 lepton flavor mixing matrix U should be reordered accordingly, and the resulting pattern U' may involve one or two large mixing angles in the standard parametrization or its variations. Since the Majorana neutrino mass matrix remains unchanged in such a mass relabeling, a possible μ - τ reflection symmetry is respected in this connection and its breaking effects are model-independently constrained at the 3σ level by using current experimental data.

Keywords: neutrinos, flavor symmetry, mass hierarchy

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1 Introduction

In a simple extension of the standard electroweak model which can generate finite but tiny masses for the three originally massless neutrinos, the phenomenon of lepton flavor mixing measures a nontrivial mismatch between the mass and flavor eigenstates of the charged leptons and neutrinos. Similar to the dynamics of quark flavor mixing, the conjecture that the lepton fields interact simultaneously with the scalar and gauge fields leads to both lepton flavor mixing and CP violation in the standard three-flavor scheme [1]. The 3×3 lepton and quark mixing matrices which manifest themselves in the weak charged-current interactions are referred to as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix U [2] and the Cabibbo-Kobayashi-Maskawa (CKM) matrix V [3], respectively:

$$-\mathcal{L}_{cc} = \frac{g}{\sqrt{2}} \left[\overline{(e \ \mu \ \tau)}_L \gamma^\mu U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L W_\mu^- + \overline{(u \ c \ t)}_L \gamma^\mu V \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L W_\mu^+ \right] + \text{h.c.} \quad (1)$$

with all the fermion fields being the mass eigenstates. By convention, U and V are defined to be associated with W^- and W^+ , respectively. In Eq. (1) the charged leptons and quarks with the same electric charges all have the “normal”³⁾ and strong mass hierarchies [4],

$$m_e \ll m_\mu \ll m_\tau, \quad m_u \ll m_c \ll m_t, \quad m_d \ll m_s \ll m_b. \quad (2)$$

For the time being, however, it remains unclear whether or not the three neutrinos also have a normal mass ordering $m_1 < m_2 < m_3$. Now that $m_1 < m_2$ has been fixed from the solar neutrino oscillation data by the $0 \leq \theta_{12} \leq \pi/4$ convention and with the help of matter effects [5], the only possible abnormal neutrino mass ordering is $m_3 < m_1 < m_2$, which has been referred to as the “inverted” mass ordering. Needless to say, the neutrino mass ordering is one of the central concerns in today’s neutrino physics. A combination of current atmospheric (Super-Kamiokande) and accelerator-based (T2K and NO ν A) neutrino oscillation data preliminarily favors the normal neutrino mass ordering up to the 2σ level [6, 7]. But one has to be very cautious and open-minded at this stage, because only the next-generation neutrino oscillation experiments can really pin down the true neutrino mass ordering [8].

If the masses of the three neutrinos end up in an

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1) E-mail: xingzz@ihep.ac.cn

2) E-mail: zhujingyu@ihep.ac.cn

3) Here “normal” means that the charged fermions in the first family are the lightest and those in the third family are the heaviest when their masses are renormalized to a common energy scale, as shown in Figure 1.



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inverted ordering, which is quite different to the mass spectra of their nine charged partners in the standard model (see Fig. 1 for illustration), one will have to explain or understand what is behind this “anomaly” from a theoretical point of view. From a purely phenomenological point of view, we may simply choose to relabel the three neutrino mass eigenstates in Eq. (1),

$$\begin{pmatrix} \nu'_1 \\ \nu'_2 \\ \nu'_3 \end{pmatrix} = S \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \quad \text{with} \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3)$$

and thus make the mass spectrum become “normal”: $m'_1 \equiv m_3 < m'_2 \equiv m_1 < m'_3 \equiv m_2$, as shown in Figure 2. The invariance of the weak charged-current interactions (i.e., \mathcal{L}_{cc}) under such a transformation requires that the PMNS matrix U transforms accordingly,

$$U' = \begin{pmatrix} U'_{e1} & U'_{e2} & U'_{e3} \\ U'_{\mu 1} & U'_{\mu 2} & U'_{\mu 3} \\ U'_{\tau 1} & U'_{\tau 2} & U'_{\tau 3} \end{pmatrix} = US^{-1} = \begin{pmatrix} U_{e3} & U_{e1} & U_{e2} \\ U_{\mu 3} & U_{\mu 1} & U_{\mu 2} \\ U_{\tau 3} & U_{\tau 1} & U_{\tau 2} \end{pmatrix}, \quad (4)$$

implying a striking change of the pattern of lepton flavor mixing. Provided U' is parametrized in the same way as U , the corresponding parameters must take different values. We shall show that U' may involve one or two large flavor mixing angles in the standard parametrization or its variations. We shall also illustrate that such a mass relabeling does not change the texture of the Majorana neutrino mass matrix M_ν by taking into account a possible μ - τ reflection symmetry, and model-independently constrain its breaking effects at the 3σ level with the help of current neutrino oscillation data.

Although relabeling $m_3 < m_1 < m_2$ as $m'_1 < m'_2 < m'_3$ does not add any new physical content, we stress that it represents a phenomenologically alternative and interesting way to describe the neutrino mass spectrum and the lepton flavor mixing pattern, especially when they are compared with (and even unified with) the “normal” quark mass spectra and the CKM quark flavor mixing matrix. That is why we intend to look into such an option in some detail, and hope that this kind of study may help make the underlying physics more transparent in some explicit model-building exercises towards deeper understanding of the origins of fermion masses and flavor mixing.

2 Possible patterns of U and U'

Let us focus on the intriguing case where the three massive neutrinos are of Majorana nature [10]. In this case the effective neutrino mass term reads

$$-\mathcal{L}_{\text{mass}} = \frac{1}{2} \overline{(\nu_e \ \nu_\mu \ \nu_\tau)_L} M_\nu \begin{pmatrix} \nu_e^c \\ \nu_\mu^c \\ \nu_\tau^c \end{pmatrix}_R + \text{h.c.}, \quad (5)$$

where the superscript “c” denotes the charge conjugation, and the 3×3 Majorana mass matrix M_ν is symmetric and can be expressed as

$$M_\nu = \begin{pmatrix} \langle m \rangle_{ee} & \langle m \rangle_{e\mu} & \langle m \rangle_{e\tau} \\ \langle m \rangle_{e\mu} & \langle m \rangle_{\mu\mu} & \langle m \rangle_{\mu\tau} \\ \langle m \rangle_{e\tau} & \langle m \rangle_{\mu\tau} & \langle m \rangle_{\tau\tau} \end{pmatrix}. \quad (6)$$

By diagonalizing M_ν one may transform the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$ into the neutrino mass eigenstates (ν_1, ν_2, ν_3) or (ν'_1, ν'_2, ν'_3) . In the basis where the flavor eigenstates of three charged leptons are identified with their mass eigenstates, the texture of M_ν can be reconstructed in terms of three neutrino masses and six flavor mixing parameters as follows:

$$M_\nu = U \widehat{M}_\nu U^T = U' S \widehat{M}'_\nu S^T U'^T = U' \widehat{M}'_\nu U'^T, \quad (7)$$

where “T” denotes the transpose, $\widehat{M}_\nu \equiv \text{Diag}\{m_1, m_2, m_3\}$ and $\widehat{M}'_\nu \equiv \text{Diag}\{m'_1, m'_2, m'_3\} = \text{Diag}\{m_3, m_1, m_2\}$. Equation (7) tells us that M_ν remains invariant under the neutrino mass relabeling. This observation is of course expectable because any physics must be unchanged by reordering the neutrino mass spectrum itself. It implies that a possible flavor symmetry of M_ν will give rise to the same constraints on the patterns of U and U' .

To see the above point of view in a more transparent way and illustrate the relevant physics, let us impose the flavor transformation

$$\nu_{eL} \longleftrightarrow \nu_{eR}^c, \quad \nu_{\mu L} \longleftrightarrow \nu_{\tau R}^c, \quad \nu_{\tau L} \longleftrightarrow \nu_{\mu R}^c \quad (8)$$

on the neutrino mass term $\mathcal{L}_{\text{mass}}$ in Eq. (5) and require the latter to be invariant. It is easy to see that the invariance of $\mathcal{L}_{\text{mass}}$ under such a μ - τ reflection transformation [11] cannot hold unless the elements of M_ν satisfy the four conditions [12]

$$\begin{aligned} \langle m \rangle_{ee} &= \langle m \rangle_{ee}^*, & \langle m \rangle_{e\mu} &= \langle m \rangle_{e\tau}^*, \\ \langle m \rangle_{\mu\mu} &= \langle m \rangle_{\tau\tau}^*, & \langle m \rangle_{\mu\tau} &= \langle m \rangle_{\mu\tau}^*. \end{aligned} \quad (9)$$

Such a special texture of M_ν can be diagonalized by the PMNS matrix¹⁾

1) In this parametrization $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$ (for $ij=12,13,23$) are defined, and all the three mixing angles are arranged to lie in the first quadrant. As for the CP -violating phases, δ varies from 0 to 2π , but ϕ and φ are only allowed to vary between 0 and π . Note that the sign convention of $U_{\mu i}$ (for $i=1,2,3$) in Eq. (10) is different from that advocated by the Particle Data Group [4], because the latter will lead to $\langle m \rangle_{e\mu} = -\langle m \rangle_{e\tau}^*$ which is in conflict with $\langle m \rangle_{e\mu} = \langle m \rangle_{e\tau}^*$ given in Eq. (9). In Eq. (12) the flavor mixing matrix U' is required to take the same new sign convention as U .

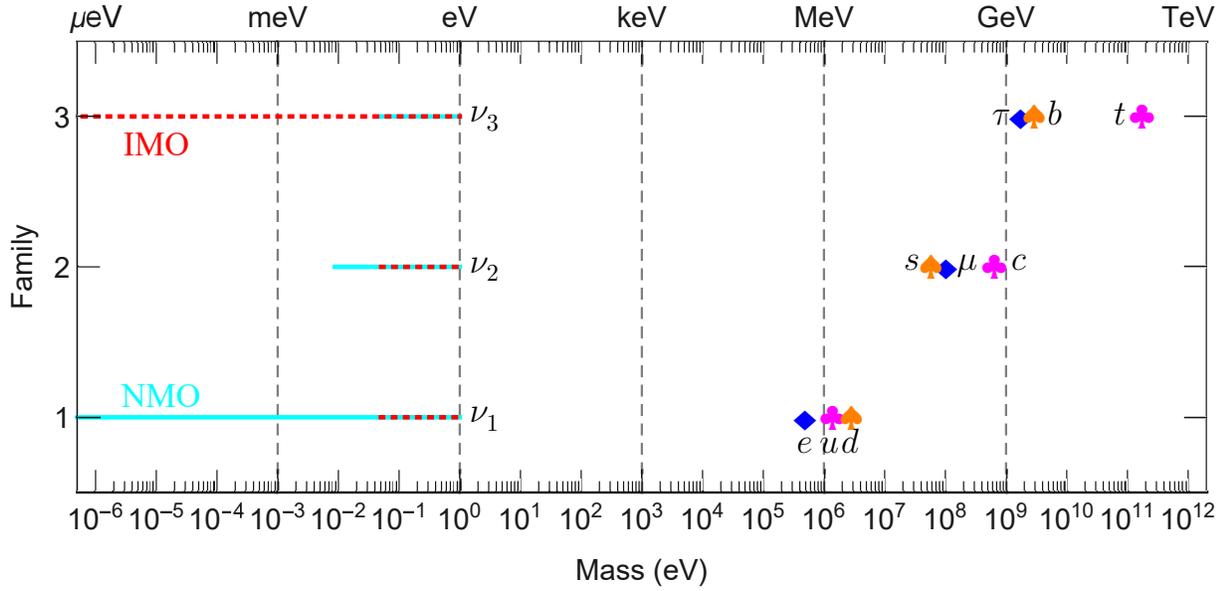


Fig. 1. (color online) A schematic illustration of the fermion mass spectra at the electroweak scale M_Z , where the allowed ranges of three neutrino masses are quoted from Ref. [6], and the typical values of three charged-lepton masses and six quark masses are quoted from Ref. [9]. For ν_1 , ν_2 and ν_3 , the horizontal cyan lines and red dashed lines stand for the normal mass ordering (NMO) and inverted mass ordering (IMO) cases, respectively. Note that the lower limit of m_1 in the NMO case or that of m_3 in the IMO case can be extended to zero.

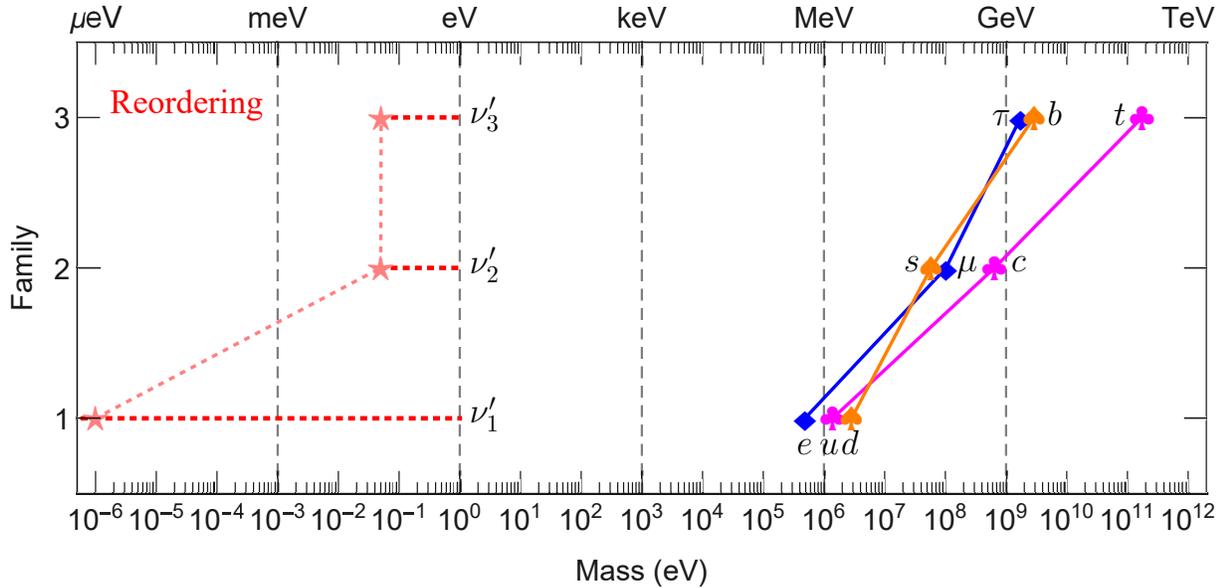


Fig. 2. (color online) A reordering of the IMO case in Fig. 1 is likely to make the neutrino mass spectrum look more or less similar to the mass spectra of the charged fermions. Here we have shown a simple example for illustration, by taking $m'_1 \sim 10^{-6}$ eV and $m'_2 \simeq m'_3 \sim 0.05$ eV.

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ s_{12}c_{23} + c_{12}s_{13}s_{23}e^{i\delta} & -c_{12}c_{23} + s_{12}s_{13}s_{23}e^{i\delta} & -c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23} \end{pmatrix} \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & e^{i\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (10)$$

via the transformation $U^\dagger M_\nu U^* = \widehat{M}_\nu$, where

$$\theta_{23} = \frac{\pi}{4}, \quad \delta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}, \quad \phi = 0 \text{ or } \frac{\pi}{2}, \quad \varphi = 0 \text{ or } \frac{\pi}{2}. \quad (11)$$

In view of Eq. (7), it is easy to show that the texture of M_ν constrained by Eq. (9) can also be diagonalized by

$$U' = \begin{pmatrix} c'_{12}c'_{13} & s'_{12}c'_{13} & s'_{13}e^{-i\delta'} \\ s'_{12}c'_{23} + c'_{12}s'_{13}s'_{23}e^{i\delta'} & -c'_{12}c'_{23} + s'_{12}s'_{13}s'_{23}e^{i\delta'} & -c'_{13}s'_{23} \\ s'_{12}s'_{23} - c'_{12}s'_{13}c'_{23}e^{i\delta'} & -c'_{12}s'_{23} - s'_{12}s'_{13}c'_{23}e^{i\delta'} & c'_{13}c'_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & e^{i\varphi} \end{pmatrix} \quad (12)$$

via the transformation $U'^{\dagger}M_{\nu}U'^{*} = \widehat{M}'_{\nu}$, where

$$\theta'_{23} = \frac{\pi}{4}, \quad \delta' = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}, \quad \phi = 0 \text{ or } \frac{\pi}{2}, \quad \varphi = 0 \text{ or } \frac{\pi}{2}. \quad (13)$$

Because of $U' = US^{-1}$, the locations of ϕ and φ in U' are different from those in U . We conclude that the μ - τ reflection symmetry of M_{ν} leads to both $|U_{\mu i}| = |U_{\tau i}|$ and $|U'_{\mu i}| = |U'_{\tau i}|$ (for $i=1,2,3$), although U and U' correspond to different neutrino mass orderings.

The generic relationships between the flavor mixing parameters $(\theta_{12}, \theta_{13}, \theta_{23}, \delta)$ in U and their counterparts in U' are given by

$$\begin{aligned} t'_{12} &= \left| \frac{U'_{e2}}{U'_{e1}} \right| = \left| \frac{U_{e1}}{U_{e3}} \right| = \frac{c_{12}c_{13}}{s_{13}}, \\ s'_{13} &= |U'_{e3}| = |U_{e2}| = s_{12}c_{13}, \\ t'_{23} &= \left| \frac{U'_{\mu 3}}{U'_{\tau 3}} \right| = \left| \frac{U_{\mu 2}}{U_{\tau 2}} \right| \\ &= \frac{\sqrt{c_{12}^2c_{23}^2 - 2c_{12}s_{12}s_{13}c_{23}s_{23}c_{\delta} + s_{12}^2s_{13}^2s_{23}^2}}{\sqrt{c_{12}^2s_{23}^2 + 2c_{12}s_{12}s_{13}c_{23}s_{23}c_{\delta} + s_{12}^2s_{13}^2c_{23}^2}}, \\ s'_{\delta} &= \frac{\text{Im}(U'_{e2}U'_{\mu 3}U'_{e3}U'^{*}_{\mu 2})}{c'_{12}s'_{12}c'^2_{13}s'_{13}c'_{23}s'_{23}} = \frac{\text{Im}(U_{e1}U_{\mu 2}U_{e2}U^*_{\mu 1})}{c'_{12}s'_{12}c'^2_{13}s'_{13}c'_{23}s'_{23}} \\ &= \frac{c_{12}s_{12}c_{13}s_{13}c_{23}s_{23}}{c'_{12}s'_{12}c'^2_{13}s'_{13}c'_{23}s'_{23}} s_{\delta}, \end{aligned} \quad (14)$$

where $t'_{ij} \equiv \tan\theta'_{ij}$, $c_{\delta} \equiv \cos\delta$, $s_{\delta} \equiv \sin\delta$ and $s'_{\delta} \equiv \sin\delta'$. It is easy to use Eq. (14) to check that $\theta'_{23} = \theta_{23} = \pi/4$ and $\delta' = \delta = \pi/2$ or $3\pi/2$ hold in the μ - τ reflection symmetry limit.

In Table 1 we list the latest global-fit results of six neutrino oscillation parameters [6]. It is obvious that the best-fit value of δ is close to $\delta = 3\pi/2$, as indicated by the μ - τ reflection symmetry of M_{ν} . Hence we shall only take $\delta = 3\pi/2$ in the remaining part of this section when discussing the μ - τ reflection symmetry limit. With the help

of Table 1, we compute the 3σ ranges of the elements of U and U' in the inverted mass ordering case. The results are

$$|U| = \begin{pmatrix} 0.794 \rightarrow 0.858 & 0.494 \rightarrow 0.589 & 0.138 \rightarrow 0.156 \\ 0.194 \rightarrow 0.544 & 0.411 \rightarrow 0.728 & 0.612 \rightarrow 0.790 \\ 0.204 \rightarrow 0.550 & 0.425 \rightarrow 0.738 & 0.596 \rightarrow 0.777 \end{pmatrix}, \quad (15)$$

and

$$|U'| = \begin{pmatrix} 0.138 \rightarrow 0.156 & 0.794 \rightarrow 0.858 & 0.494 \rightarrow 0.589 \\ 0.612 \rightarrow 0.790 & 0.194 \rightarrow 0.544 & 0.411 \rightarrow 0.728 \\ 0.596 \rightarrow 0.777 & 0.204 \rightarrow 0.550 & 0.425 \rightarrow 0.738 \end{pmatrix}. \quad (16)$$

To be more specific, the four flavor mixing parameters in the standard parametrization of U or U' read as

$$\begin{aligned} \theta'_{12} &= 78.9^{\circ} \rightarrow 80.9^{\circ}, \quad \theta'_{13} = 29.6^{\circ} \rightarrow 36.1^{\circ}, \quad \theta'_{23} = 30.5^{\circ} \rightarrow 58.3^{\circ}; \\ \theta_{12} &= 30.0^{\circ} \rightarrow 36.5^{\circ}, \quad \theta_{13} = 7.9^{\circ} \rightarrow 8.9^{\circ}, \quad \theta_{23} = 38.3^{\circ} \rightarrow 52.9^{\circ}; \end{aligned} \quad (17)$$

together with

$$\delta \in [0^{\circ}, 27^{\circ}] \cup [124.2^{\circ}, 360^{\circ}], \quad \delta' \in [0^{\circ}, 59.2^{\circ}] \cup [150.5^{\circ}, 360^{\circ}]. \quad (18)$$

We see that θ'_{12} is more than two times larger than θ_{12} , and is even close to 90° . In addition, θ'_{13} is about four times larger than θ_{13} . But θ'_{23} and θ_{23} are roughly comparable in magnitude, and so are δ' and δ . Figure 3 gives a more intuitive comparison between the results of θ_{ij} and θ'_{ij} obtained in Eq. (17), where the fractional error bar of each flavor mixing angle is estimated from the difference between the upper and lower bounds of its 3σ range divided by its central value as listed in Table 1, and the corresponding errors translate from θ_{ij} to θ'_{ij} via Eq. (14).

Table 1. The best-fit values and 3σ ranges of six neutrino oscillation parameters obtained from a global fit of currently available experimental data [6].

	normal neutrino mass ordering		inverted neutrino mass ordering	
	best-fit	3σ range	best-fit	3σ range
θ_{12}	33.02°	$30^{\circ} - 36.51^{\circ}$	33.02°	$30^{\circ} - 36.51^{\circ}$
θ_{13}	8.43°	$7.92^{\circ} - 8.91^{\circ}$	8.45°	$7.92^{\circ} - 8.95^{\circ}$
θ_{23}	40.69°	$38.12^{\circ} - 51.65^{\circ}$	50.13°	$38.29^{\circ} - 52.89^{\circ}$
δ	248.4°	$0^{\circ} - 30.6^{\circ} \oplus 136.8^{\circ} - 360^{\circ}$	235.8°	$0^{\circ} - 27^{\circ} \oplus 124.2^{\circ} - 360^{\circ}$
$\frac{\Delta m_{21}^2}{10^{-5} \text{ eV}^2}$	7.37	6.93–7.96	7.37	6.93–7.96
$\frac{\Delta m_{31}^2}{10^{-3} \text{ eV}^2}$	2.56	2.45–2.69	-2.47	-2.59–-2.35

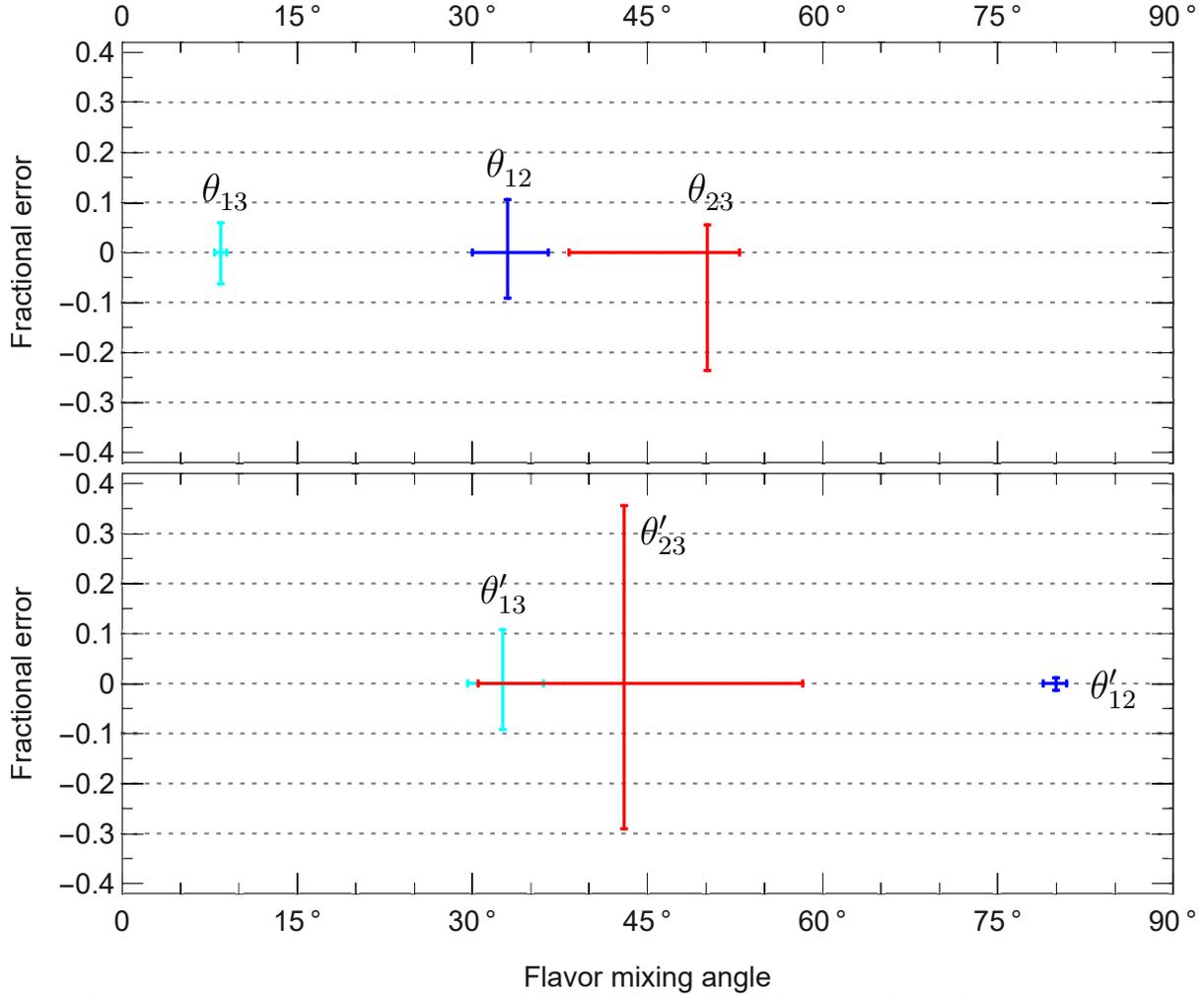


Fig. 3. (color online) An intuitive illustration of the numerical results of $(\theta_{12}, \theta_{13}, \theta_{23})$ and $(\theta'_{12}, \theta'_{13}, \theta'_{23})$ obtained in Eq. (17) for the IMO case.

What about the parameters of flavor mixing and CP violation in a different parametrization of U and U' ? To answer this question, we list all nine angle-phase parametrizations of U and U' [13] in Table 2 and calculate the best-fit values of the corresponding four flavor mixing parameters¹⁾. Some comments are in order.

1) The benchmark parametrization of U or U' is P3, which is actually consistent with the standard parametrization given in Eq. (10) or Eq. (12) up to a rearrangement of the sign convention of $U_{\mu i}$ (for $i=1,2,3$), a rearrangement of the Dirac phase convention, and a neglect of the Majorana phase matrix. Starting from the best-fit values of $\theta_{12}, \theta_{13}, \theta_{23}$ and δ in Table 1, one may first determine the corresponding values of $\theta'_{12}, \theta'_{13}, \theta'_{23}$ and δ' of U' in the form of P3, and then translate the relevant results into the other eight parametrizations of U and U' as shown in Table 2.

2) Among the nine parametrizations of U and U' , only P1 [14] allows all three mixing angles of U' to be comparable in magnitude and smaller than 60° for the given inputs. In comparison, the outputs of four flavor mixing parameters of U' in P2 (the Kobayashi-Maskawa parametrization [3]) are quite similar to those in P3, with $\theta'_{12} \sim 80^\circ$. Another interesting parametrization of U' , P5 [15], contains $\theta'_{12} \gtrsim 70^\circ$. One can see that the parametrizations of U' under discussion may involve one or two large mixing angles in most cases, as a straightforward consequence of the reordering of the inverted neutrino mass hierarchy $(\nu_1, \nu_2, \nu_3) \rightarrow (\nu'_1, \nu'_2, \nu'_3) = (\nu_3, \nu_1, \nu_2)$.

When the μ - τ reflection symmetry is taken into account, as illustrated in Table 3, we find that only P2, P3 and P7 are suitable in this connection because they simply lead us to $\theta_{23} = \theta'_{23} = \pi/4$ and $\delta = \delta' = \pi/2$ or $3\pi/2$ as well as $\phi = 0$ or $\pi/2$ and $\varphi = 0$ or $\pi/2$. Al-

1) Because the Majorana phases ϕ and φ are completely insensitive to neutrino oscillations and completely unconstrained, here we only consider the Dirac CP -violating phase δ or δ' in our examples.

though $|U_{\mu i}| = |U_{\tau i}|$ and $|U'_{\mu i}| = |U'_{\tau i}|$ (for $i = 1, 2, 3$) also hold for the other six parametrizations of U and U' in the μ - τ reflection symmetry limit, the structures of these parametrizations make them less interesting in describing the phenomenology of neutrino oscillations.

At this point let us briefly comment on the so-called

quark-lepton complementarity relations $\theta_{12} + \theta_{12}^q = \pi/4$ [17] and $\theta_{23} \pm \theta_{23}^q = \pi/4$ [18] in the standard parametrization of the PMNS matrix U and the CKM matrix V . Given $\theta_{12}^q = 13.023^\circ \pm 0.038^\circ$, $\theta_{13}^q = 0.201^{+0.009^\circ}_{-0.008^\circ}$, $\theta_{23}^q = 2.361^{+0.063^\circ}_{-0.028^\circ}$ and $\delta_q = 69.21^{+2.55^\circ}_{-4.59^\circ}$ extracted from current experimental data on quark flavor mixing and CP violation [4], such

Table 2. The best-fit values of three flavor mixing angles and the Dirac CP -violating phase for the inverted neutrino mass ordering in each of the nine parametrizations of U and U' , where $s_{\theta_{ij}} \equiv \sin\theta_{ij}$, $s_{\vartheta_{ij}} \equiv \sin\vartheta_{ij}$, $c_{\theta_{ij}} \equiv \cos\theta_{ij}$ and $c_{\vartheta_{ij}} \equiv \cos\vartheta_{ij}$ (for $ij = 12, 13, 23$).

parametrization	U	U'
P1: $U = R_{12}(\theta_{12})R_{23}(\theta_{23}, \delta)R_{12}^{-1}(\vartheta_{12}) =$ $\begin{pmatrix} s_{\theta_{12}} s_{\vartheta_{12}} c_{\theta_{23}} + c_{\theta_{12}} c_{\vartheta_{12}} e^{-i\delta} & s_{\theta_{12}} c_{\vartheta_{12}} c_{\theta_{23}} - c_{\theta_{12}} s_{\vartheta_{12}} e^{-i\delta} & s_{\theta_{12}} s_{\theta_{23}} \\ c_{\theta_{12}} s_{\vartheta_{12}} c_{\theta_{23}} - s_{\theta_{12}} c_{\vartheta_{12}} e^{-i\delta} & c_{\theta_{12}} c_{\vartheta_{12}} c_{\theta_{23}} + s_{\theta_{12}} s_{\vartheta_{12}} e^{-i\delta} & c_{\theta_{12}} s_{\theta_{23}} \\ -s_{\vartheta_{12}} s_{\theta_{23}} & -c_{\vartheta_{12}} s_{\theta_{23}} & c_{\theta_{23}} \end{pmatrix}$	$\theta_{12} = 11.0^\circ$ $\vartheta_{12} = 37.2^\circ$ $\theta_{23} = 50.6^\circ$ $\delta = 308.3^\circ$	$\theta'_{12} = 43.2^\circ$ $\vartheta'_{12} = 53.6^\circ$ $\theta'_{23} = 52.0^\circ$ $\delta' = 197.1^\circ$
P2: $U = R_{23}(\theta_{23})R_{12}(\theta_{12}, \delta)R_{23}^{-1}(\vartheta_{23}) =$ $\begin{pmatrix} c_{\theta_{12}} & s_{\theta_{12}} c_{\vartheta_{23}} & -s_{\theta_{12}} s_{\vartheta_{23}} \\ -s_{\theta_{12}} c_{\vartheta_{23}} & c_{\theta_{12}} c_{\vartheta_{23}} + s_{\theta_{23}} s_{\vartheta_{23}} e^{-i\delta} & -c_{\theta_{12}} c_{\vartheta_{23}} + s_{\theta_{23}} c_{\vartheta_{23}} e^{-i\delta} \\ s_{\theta_{12}} s_{\vartheta_{23}} & -c_{\theta_{12}} s_{\vartheta_{23}} + c_{\theta_{23}} s_{\vartheta_{23}} e^{-i\delta} & c_{\theta_{12}} s_{\vartheta_{23}} + c_{\theta_{23}} c_{\vartheta_{23}} e^{-i\delta} \end{pmatrix}$	$\theta_{12} = 34.0^\circ$ $\theta_{23} = 56.7^\circ$ $\vartheta_{23} = 15.3^\circ$ $\delta = 297.4^\circ$	$\theta'_{12} = 81.5^\circ$ $\theta'_{23} = 39.9^\circ$ $\vartheta'_{23} = 33.0^\circ$ $\delta' = 235.8^\circ$
P3: $U = R_{23}(\theta_{23})R_{31}(\theta_{13}, \delta)R_{12}(\theta_{12}) =$ $\begin{pmatrix} c_{\theta_{12}} c_{\theta_{13}} & s_{\theta_{12}} c_{\theta_{13}} & s_{\theta_{13}} \\ -c_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} - s_{\theta_{12}} c_{\theta_{23}} e^{-i\delta} & -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{12}} c_{\theta_{23}} e^{-i\delta} & c_{\theta_{13}} s_{\theta_{23}} \\ -c_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} + s_{\theta_{12}} s_{\theta_{23}} e^{-i\delta} & -s_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} - c_{\theta_{12}} s_{\theta_{23}} e^{-i\delta} & c_{\theta_{13}} c_{\theta_{23}} \end{pmatrix}$	$\theta_{12} = 33.0^\circ$ $\theta_{13} = 8.5^\circ$ $\theta_{23} = 50.1^\circ$ $\delta = 235.8^\circ$	$\theta'_{12} = 80.0^\circ$ $\theta'_{13} = 32.6^\circ$ $\theta'_{23} = 43.0^\circ$ $\delta' = 305.3^\circ$
P4: $U = R_{12}(\theta_{12})R_{31}(\theta_{13}, \delta)R_{23}^{-1}(\vartheta_{23}) =$ $\begin{pmatrix} c_{\theta_{12}} c_{\theta_{13}} & c_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + s_{\theta_{12}} c_{\theta_{23}} e^{-i\delta} & c_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} - s_{\theta_{12}} s_{\theta_{23}} e^{-i\delta} \\ -s_{\theta_{12}} c_{\theta_{13}} & -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{12}} c_{\theta_{23}} e^{-i\delta} & -s_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} - c_{\theta_{12}} s_{\theta_{23}} e^{-i\delta} \\ -s_{\theta_{13}} & c_{\theta_{13}} s_{\theta_{23}} & c_{\theta_{13}} c_{\theta_{23}} \end{pmatrix}$	$\theta_{12} = 20.3^\circ$ $\theta_{13} = 27.9^\circ$ $\theta_{23} = 44.2^\circ$ $\delta = 333.2^\circ$	$\theta'_{12} = 79.0^\circ$ $\theta'_{13} = 39.4^\circ$ $\theta'_{23} = 37.2^\circ$ $\delta' = 308.3^\circ$
P5: $U = R_{31}(\theta_{13})R_{12}(\theta_{12}, \delta)R_{31}^{-1}(\vartheta_{13}) =$ $\begin{pmatrix} c_{\theta_{12}} c_{\theta_{13}} c_{\vartheta_{13}} + s_{\theta_{13}} s_{\vartheta_{13}} e^{-i\delta} & s_{\theta_{12}} c_{\theta_{13}} & -c_{\theta_{12}} c_{\theta_{13}} s_{\vartheta_{13}} + s_{\theta_{13}} c_{\vartheta_{13}} e^{-i\delta} \\ -s_{\theta_{12}} c_{\vartheta_{13}} & c_{\theta_{12}} & s_{\theta_{12}} s_{\vartheta_{13}} \\ -c_{\theta_{12}} s_{\theta_{13}} c_{\vartheta_{13}} + c_{\theta_{13}} s_{\vartheta_{13}} e^{-i\delta} & -s_{\theta_{12}} s_{\theta_{13}} & c_{\theta_{12}} s_{\theta_{13}} s_{\vartheta_{13}} + c_{\theta_{13}} c_{\vartheta_{13}} e^{-i\delta} \end{pmatrix}$	$\theta_{12} = 54.9^\circ$ $\theta_{13} = 48.8^\circ$ $\vartheta_{13} = 68.0^\circ$ $\delta = 336.2^\circ$	$\theta'_{12} = 72.2^\circ$ $\theta'_{13} = 29.4^\circ$ $\vartheta'_{13} = 37.1^\circ$ $\delta' = 207.9^\circ$
P6: $U = R_{12}(\theta_{12})R_{23}(\theta_{23}, \delta)R_{31}(\theta_{13}) =$ $\begin{pmatrix} -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{12}} c_{\theta_{13}} e^{-i\delta} & s_{\theta_{12}} c_{\theta_{23}} & s_{\theta_{12}} c_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{12}} s_{\theta_{13}} e^{-i\delta} \\ -c_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} - s_{\theta_{12}} c_{\theta_{13}} e^{-i\delta} & c_{\theta_{12}} c_{\theta_{23}} & c_{\theta_{12}} c_{\theta_{13}} s_{\theta_{23}} - s_{\theta_{12}} s_{\theta_{13}} e^{-i\delta} \\ -s_{\theta_{13}} c_{\theta_{23}} & -s_{\theta_{23}} & c_{\theta_{13}} c_{\theta_{23}} \end{pmatrix}$	$\theta_{12} = 43.2^\circ$ $\theta_{13} = 36.4^\circ$ $\theta_{23} = 38.0^\circ$ $\delta = 197.1^\circ$	$\theta'_{12} = 69.7^\circ$ $\theta'_{13} = 45.8^\circ$ $\theta'_{23} = 27.9^\circ$ $\delta' = 333.2^\circ$
P7: $U = R_{23}(\theta_{23})R_{12}(\theta_{12}, \delta)R_{31}^{-1}(\theta_{13}) =$ $\begin{pmatrix} c_{\theta_{12}} c_{\theta_{13}} & s_{\theta_{12}} & -c_{\theta_{12}} s_{\theta_{13}} \\ -s_{\theta_{12}} c_{\theta_{13}} c_{\theta_{23}} + s_{\theta_{13}} s_{\theta_{23}} e^{-i\delta} & c_{\theta_{12}} c_{\theta_{23}} & s_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} + c_{\theta_{13}} s_{\theta_{23}} e^{-i\delta} \\ s_{\theta_{12}} c_{\theta_{13}} s_{\theta_{23}} + s_{\theta_{13}} c_{\theta_{23}} e^{-i\delta} & -c_{\theta_{12}} s_{\theta_{23}} & -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{13}} c_{\theta_{23}} e^{-i\delta} \end{pmatrix}$	$\theta_{12} = 32.6^\circ$ $\theta_{13} = 10.0^\circ$ $\theta_{23} = 47.0^\circ$ $\delta = 305.3^\circ$	$\theta'_{12} = 56.0^\circ$ $\theta'_{13} = 74.7^\circ$ $\theta'_{23} = 56.7^\circ$ $\delta' = 297.4^\circ$
P8: $U = R_{31}(\theta_{13})R_{12}(\theta_{12}, \delta)R_{23}(\theta_{23}) =$ $\begin{pmatrix} c_{\theta_{12}} c_{\theta_{13}} & s_{\theta_{12}} c_{\theta_{13}} c_{\theta_{23}} - s_{\theta_{13}} s_{\theta_{23}} e^{-i\delta} & s_{\theta_{12}} c_{\theta_{13}} s_{\theta_{23}} + s_{\theta_{13}} c_{\theta_{23}} e^{-i\delta} \\ -s_{\theta_{12}} & c_{\theta_{12}} c_{\theta_{23}} & c_{\theta_{12}} s_{\theta_{23}} \\ -c_{\theta_{12}} s_{\theta_{13}} & -s_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} - c_{\theta_{13}} s_{\theta_{23}} e^{-i\delta} & -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{13}} c_{\theta_{23}} e^{-i\delta} \end{pmatrix}$	$\theta_{12} = 17.8^\circ$ $\theta_{13} = 29.4^\circ$ $\theta_{23} = 52.9^\circ$ $\delta = 207.9^\circ$	$\theta'_{12} = 49.4^\circ$ $\theta'_{13} = 77.0^\circ$ $\theta'_{23} = 61.9^\circ$ $\delta' = 294.5^\circ$
P9: $U = R_{31}(\theta_{13})R_{23}(\theta_{23}, \delta)R_{12}^{-1}(\theta_{12}) =$ $\begin{pmatrix} -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{12}} c_{\theta_{13}} e^{-i\delta} & -c_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} - s_{\theta_{12}} c_{\theta_{13}} e^{-i\delta} & s_{\theta_{13}} c_{\theta_{23}} \\ s_{\theta_{12}} c_{\theta_{13}} & c_{\theta_{12}} c_{\theta_{23}} & s_{\theta_{23}} \\ -s_{\theta_{12}} c_{\theta_{13}} s_{\theta_{23}} - c_{\theta_{12}} s_{\theta_{13}} e^{-i\delta} & -c_{\theta_{12}} c_{\theta_{13}} s_{\theta_{23}} + s_{\theta_{12}} s_{\theta_{13}} e^{-i\delta} & c_{\theta_{13}} c_{\theta_{23}} \end{pmatrix}$	$\theta_{12} = 28.1^\circ$ $\theta_{13} = 13.0^\circ$ $\theta_{23} = 49.4^\circ$ $\delta = 294.5^\circ$	$\theta'_{12} = 68.0^\circ$ $\theta'_{13} = 41.2^\circ$ $\theta'_{23} = 35.1^\circ$ $\delta' = 336.2^\circ$

Table 3. The best-fit values of θ_{12} and θ_{13} together with $\theta_{23}=\pi/4$ and $\delta=3\pi/2$ for P3 in the μ - τ reflection symmetry limit [16] and in the inverted neutrino mass ordering, and their variations for the other eight parametrizations in the same case.

parametrization	U	U'
P1: $U=R_{12}(\theta_{12})R_{23}(\theta_{23},\delta)R_{12}^{-1}(\vartheta_{12})=$ $\begin{pmatrix} s_{\theta_{12}} s_{\vartheta_{12}} c_{\theta_{23}} + c_{\theta_{12}} c_{\vartheta_{12}} e^{-i\delta} & s_{\theta_{12}} c_{\vartheta_{12}} c_{\theta_{23}} & -c_{\theta_{12}} s_{\vartheta_{12}} e^{-i\delta} & s_{\theta_{12}} s_{\theta_{23}} \\ c_{\theta_{12}} s_{\vartheta_{12}} c_{\theta_{23}} - s_{\theta_{12}} c_{\vartheta_{12}} e^{-i\delta} & c_{\theta_{12}} c_{\vartheta_{12}} c_{\theta_{23}} + s_{\theta_{12}} s_{\vartheta_{12}} e^{-i\delta} & c_{\theta_{12}} s_{\vartheta_{12}} c_{\theta_{23}} & c_{\theta_{12}} s_{\theta_{23}} \\ -s_{\vartheta_{12}} s_{\theta_{23}} & -c_{\vartheta_{12}} s_{\theta_{23}} & c_{\vartheta_{12}} \end{pmatrix}$	$\theta_{12}=11.9^\circ$ $\vartheta_{12}=33.6^\circ$ $\theta_{23}=45.6^\circ$ $\delta=277.3^\circ$	$\theta'_{12}=42.1^\circ$ $\vartheta'_{12}=60.5^\circ$ $\theta'_{23}=53.4^\circ$ $\delta'=203.7^\circ$
P2: $U=R_{23}(\theta_{23})R_{12}(\theta_{12},\delta)R_{23}^{-1}(\vartheta_{23})=$ $\begin{pmatrix} c_{\theta_{12}} & s_{\theta_{12}} c_{\vartheta_{23}} & -s_{\theta_{12}} s_{\vartheta_{23}} \\ -s_{\theta_{12}} c_{\theta_{23}} & c_{\theta_{12}} c_{\vartheta_{23}} + s_{\theta_{12}} s_{\vartheta_{23}} e^{-i\delta} & -c_{\theta_{12}} c_{\theta_{23}} s_{\vartheta_{23}} + s_{\theta_{12}} c_{\vartheta_{23}} e^{-i\delta} \\ s_{\theta_{12}} s_{\theta_{23}} & -c_{\theta_{12}} s_{\theta_{23}} c_{\vartheta_{23}} + c_{\theta_{12}} s_{\vartheta_{23}} e^{-i\delta} & c_{\theta_{12}} s_{\theta_{23}} s_{\vartheta_{23}} + c_{\theta_{12}} c_{\vartheta_{23}} e^{-i\delta} \end{pmatrix}$	$\theta_{12}=34.0^\circ$ $\theta_{23}=45.0^\circ$ $\vartheta_{23}=15.3^\circ$ $\delta=270.0^\circ$	$\theta'_{12}=81.5^\circ$ $\theta'_{23}=45.0^\circ$ $\vartheta'_{23}=33.0^\circ$ $\delta'=270.0^\circ$
P3: $U=R_{23}(\theta_{23})R_{31}(\theta_{13},\delta)R_{12}(\theta_{12})=$ $\begin{pmatrix} c_{\theta_{12}} c_{\theta_{13}} & s_{\theta_{12}} c_{\theta_{13}} & s_{\theta_{13}} \\ -c_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} - s_{\theta_{12}} c_{\theta_{23}} e^{-i\delta} & -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{12}} c_{\theta_{23}} e^{-i\delta} & c_{\theta_{13}} s_{\theta_{23}} \\ -c_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} + s_{\theta_{12}} s_{\theta_{23}} e^{-i\delta} & -s_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} - c_{\theta_{12}} s_{\theta_{23}} e^{-i\delta} & c_{\theta_{13}} c_{\theta_{23}} \end{pmatrix}$	$\theta_{12}=33.0^\circ$ $\theta_{13}=8.5^\circ$ $\theta_{23}=45.0^\circ$ $\delta=270.0^\circ$	$\theta'_{12}=80.0^\circ$ $\theta'_{13}=32.6^\circ$ $\theta'_{23}=45.0^\circ$ $\delta'=270.0^\circ$
P4: $U=R_{12}(\theta_{12})R_{31}(\theta_{13},\delta)R_{23}^{-1}(\theta_{23})=$ $\begin{pmatrix} c_{\theta_{12}} c_{\theta_{13}} & c_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + s_{\theta_{12}} c_{\theta_{23}} e^{-i\delta} & c_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} - s_{\theta_{12}} s_{\theta_{23}} e^{-i\delta} \\ -s_{\theta_{12}} c_{\theta_{13}} & -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{12}} c_{\theta_{23}} e^{-i\delta} & -s_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} - c_{\theta_{12}} s_{\theta_{23}} e^{-i\delta} \\ -s_{\theta_{13}} & c_{\theta_{13}} s_{\theta_{23}} & c_{\theta_{13}} c_{\theta_{23}} \end{pmatrix}$	$\theta_{12}=25.5^\circ$ $\theta_{13}=23.3^\circ$ $\theta_{23}=40.4^\circ$ $\delta=329.1^\circ$	$\theta'_{12}=78.1^\circ$ $\theta'_{13}=44.4^\circ$ $\theta'_{23}=33.6^\circ$ $\delta'=277.3^\circ$
P5: $U=R_{31}(\theta_{13})R_{12}(\theta_{12},\delta)R_{31}^{-1}(\vartheta_{13})=$ $\begin{pmatrix} c_{\theta_{12}} c_{\theta_{13}} c_{\vartheta_{13}} + s_{\theta_{13}} s_{\vartheta_{13}} e^{-i\delta} & s_{\theta_{12}} c_{\theta_{13}} & -c_{\theta_{12}} c_{\theta_{13}} s_{\vartheta_{13}} + s_{\theta_{13}} c_{\vartheta_{13}} e^{-i\delta} \\ -s_{\theta_{12}} c_{\vartheta_{13}} & c_{\theta_{12}} & s_{\theta_{12}} s_{\vartheta_{13}} \\ -c_{\theta_{12}} s_{\theta_{13}} c_{\vartheta_{13}} + c_{\theta_{13}} s_{\vartheta_{13}} e^{-i\delta} & -s_{\theta_{12}} s_{\theta_{13}} & c_{\theta_{12}} s_{\theta_{13}} s_{\vartheta_{13}} + c_{\theta_{13}} c_{\vartheta_{13}} e^{-i\delta} \end{pmatrix}$	$\theta_{12}=53.4^\circ$ $\theta_{13}=47.9^\circ$ $\vartheta_{13}=60.5^\circ$ $\delta=336.3^\circ$	$\theta'_{12}=66.7^\circ$ $\theta'_{13}=25.5^\circ$ $\vartheta'_{13}=40.4^\circ$ $\delta'=210.9^\circ$
P6: $U=R_{12}(\theta_{12})R_{23}(\theta_{23},\delta)R_{31}(\theta_{13})=$ $\begin{pmatrix} -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{12}} c_{\theta_{13}} e^{-i\delta} & s_{\theta_{12}} c_{\theta_{23}} s_{\theta_{13}} + c_{\theta_{12}} s_{\theta_{13}} e^{-i\delta} \\ -c_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} - s_{\theta_{12}} c_{\theta_{13}} e^{-i\delta} & c_{\theta_{12}} c_{\theta_{23}} c_{\theta_{13}} - s_{\theta_{12}} s_{\theta_{13}} e^{-i\delta} \\ -s_{\theta_{13}} c_{\theta_{23}} & -s_{\theta_{23}} & c_{\theta_{13}} c_{\theta_{23}} \end{pmatrix}$	$\theta_{12}=42.1^\circ$ $\theta_{13}=29.5^\circ$ $\theta_{23}=36.6^\circ$ $\delta=203.7^\circ$	$\theta'_{12}=64.5^\circ$ $\theta'_{13}=49.6^\circ$ $\theta'_{23}=23.3^\circ$ $\delta'=329.1^\circ$
P7: $U=R_{23}(\theta_{23})R_{12}(\theta_{12},\delta)R_{31}^{-1}(\theta_{13})=$ $\begin{pmatrix} c_{\theta_{12}} c_{\theta_{13}} & s_{\theta_{12}} & -c_{\theta_{12}} s_{\theta_{13}} \\ -s_{\theta_{12}} c_{\theta_{13}} c_{\theta_{23}} + s_{\theta_{13}} s_{\theta_{23}} e^{-i\delta} & c_{\theta_{12}} c_{\theta_{23}} & s_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} + c_{\theta_{13}} s_{\theta_{23}} e^{-i\delta} \\ s_{\theta_{12}} c_{\theta_{13}} s_{\theta_{23}} + s_{\theta_{13}} c_{\theta_{23}} e^{-i\delta} & -c_{\theta_{12}} s_{\theta_{23}} & -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{13}} c_{\theta_{23}} e^{-i\delta} \end{pmatrix}$	$\theta_{12}=32.6^\circ$ $\theta_{13}=10.0^\circ$ $\theta_{23}=45.0^\circ$ $\delta=270.0^\circ$	$\theta'_{12}=56.0^\circ$ $\theta'_{13}=74.7^\circ$ $\theta'_{23}=45.0^\circ$ $\delta'=270.0^\circ$
P8: $U=R_{31}(\theta_{13})R_{12}(\theta_{12},\delta)R_{23}(\theta_{23})=$ $\begin{pmatrix} c_{\theta_{12}} c_{\theta_{13}} & s_{\theta_{12}} c_{\theta_{13}} c_{\theta_{23}} - s_{\theta_{13}} s_{\theta_{23}} e^{-i\delta} & s_{\theta_{12}} c_{\theta_{13}} s_{\theta_{23}} + s_{\theta_{13}} c_{\theta_{23}} e^{-i\delta} \\ -s_{\theta_{12}} & c_{\theta_{12}} c_{\theta_{23}} & c_{\theta_{12}} s_{\theta_{23}} \\ -c_{\theta_{12}} s_{\theta_{13}} - s_{\theta_{12}} s_{\theta_{13}} c_{\theta_{23}} - c_{\theta_{13}} s_{\theta_{23}} e^{-i\delta} & -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{13}} c_{\theta_{23}} e^{-i\delta} \end{pmatrix}$	$\theta_{12}=23.3^\circ$ $\theta_{13}=25.5^\circ$ $\theta_{23}=49.6^\circ$ $\delta=210.9^\circ$	$\theta'_{12}=44.4^\circ$ $\theta'_{13}=78.1^\circ$ $\theta'_{23}=56.4^\circ$ $\delta'=262.7^\circ$
P9: $U=R_{31}(\theta_{13})R_{23}(\theta_{23},\delta)R_{12}^{-1}(\theta_{12})=$ $\begin{pmatrix} -s_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} + c_{\theta_{12}} c_{\theta_{13}} e^{-i\delta} & -c_{\theta_{12}} s_{\theta_{13}} s_{\theta_{23}} - s_{\theta_{12}} c_{\theta_{13}} e^{-i\delta} & s_{\theta_{13}} c_{\theta_{23}} \\ s_{\theta_{12}} c_{\theta_{23}} & c_{\theta_{12}} c_{\theta_{23}} & s_{\theta_{23}} \\ -s_{\theta_{12}} c_{\theta_{13}} s_{\theta_{23}} - c_{\theta_{12}} c_{\theta_{13}} e^{-i\delta} & -c_{\theta_{12}} c_{\theta_{13}} s_{\theta_{23}} + s_{\theta_{12}} s_{\theta_{13}} e^{-i\delta} & c_{\theta_{13}} c_{\theta_{23}} \end{pmatrix}$	$\theta_{12}=33.6^\circ$ $\theta_{13}=11.9^\circ$ $\theta_{23}=44.4^\circ$ $\delta=262.7^\circ$	$\theta'_{12}=60.5^\circ$ $\theta'_{13}=42.1^\circ$ $\theta'_{23}=36.6^\circ$ $\delta'=336.3^\circ$

relations seem acceptable as a phenomenological conjecture¹⁾. In a similar way, one may conjecture relations like $\theta'_{12} + \theta_{12} = \pi/2$ and $\theta'_{23} \pm \theta_{23} = \pi/4$ [19] when

reordering the inverted neutrino mass hierarchy and taking account of the standard parametrization of U' , but they seem unlikely to shed light on the underlying dy-

1) Note that U and V are associated respectively with W^- and W^+ , and hence it is more appropriate to compare U and V^\dagger in some sense [19]. But V itself is approximately symmetric, so the three mixing angles of V^\dagger are equal to those of V to a good degree of accuracy.

namics of flavor mixing.

3 μ - τ symmetry breaking

At the level of the PMNS matrix U or U' , a deviation of θ_{23} or θ'_{23} from $\pi/4$ is the so-called *octant* problem, which is an important concern in current neutrino oscillation experiments. On the other hand, a departure of δ or δ' from $3\pi/2$ can be referred to as the *quadrant* problem, provided the present best-fit value of δ is essentially true. Both issues actually measure the effects of μ - τ reflection symmetry breaking, and hence it makes sense to examine such effects with the help of Table 1.

At the level of the Majorana neutrino mass matrix M_ν , the μ - τ reflection symmetry is equivalent to the four relations given in Eq. (9). That is why some authors [20] have used the imaginary parts of $\langle m \rangle_{ee}$ and $\langle m \rangle_{\mu\tau}$ and the differences $\langle m \rangle_{e\mu} - \langle m \rangle_{e\tau}^*$ and $\langle m \rangle_{\mu\mu} - \langle m \rangle_{\tau\tau}^*$ to measure the symmetry breaking effects. However, these four quantities depend on the phase convention of the charged-lepton fields, and hence they are not fully physical.

Here we illustrate the effects of μ - τ reflection symmetry breaking in a somewhat different way. We first calculate the profiles of $|\langle m \rangle_{\alpha\beta}|$ (for $\alpha, \beta = e, \mu, \tau$) versus the lightest neutrino mass (either m_1 in the normal ordering or m_3 in the inverted ordering) by inputting the 3σ ranges of the relevant neutrino oscillation parameters listed in Table 1. For each of the six profiles, we find the much narrower areas fixed by the μ - τ reflection symmetry conditions $\theta_{23} = \pi/4$, $\delta = \pi/2$ or $3\pi/2$, $\phi = 0$ or $\pi/2$ and $\varphi = 0$ or $\pi/2$ in the standard parametrization of U . Note that the case of $\delta = \pi/2$ is equivalent to that of $\delta = 3\pi/2$, because $|\langle m \rangle_{\alpha\beta}|$ only contains $\cos\delta$. Therefore, we are left with four different situations with $(\phi, \varphi) = (0, 0)$, $(\pi/2, 0)$, $(0, \pi/2)$ and $(\pi/2, \pi/2)$, as shown in Fig. 4 for the normal neutrino mass ordering or in Fig. 5 for the inverted mass ordering. Some discussion is in order.

1) In either Fig. 4 or Fig. 5, the μ - τ reflection symmetry reflected by the relationships $|\langle m \rangle_{e\mu}| = |\langle m \rangle_{e\tau}|$ and $|\langle m \rangle_{\mu\mu}| = |\langle m \rangle_{\tau\tau}|$ can clearly be seen. Given $m_1 \lesssim 0.1$ eV for the normal mass ordering or $m_3 \lesssim 0.1$ eV for the inverted mass ordering, a reasonable upper bound extracted from current cosmological data [21], we find that either $|\langle m \rangle_{\mu\tau}|$ in Fig. 4 or $|\langle m \rangle_{ee}|$ in Fig. 5 can be most stringently constrained. Hence the effect of μ - τ reflection symmetry breaking for either of them is relatively modest even at the 3σ level. In comparison, the symmetry breaking effects are completely unconstrained at the 3σ level for $|\langle m \rangle_{e\mu}|$, $|\langle m \rangle_{\mu\mu}|$, $|\langle m \rangle_{e\tau}|$ and $|\langle m \rangle_{\tau\tau}|$ in

the inverted mass ordering case, as shown in Fig. 5. For $m_1 \lesssim 2 \times 10^{-2}$ eV in the normal mass ordering case, Fig. 4 shows that $|\langle m \rangle_{\mu\mu}|$ and $|\langle m \rangle_{\tau\tau}|$ are well constrained (around a few $\times 10^{-2}$ eV), and the corresponding μ - τ reflection symmetry breaking effects are quite modest.

2) The size of $|\langle m \rangle_{ee}|$ determines the rate of neutrinoless double-beta ($0\nu 2\beta$) decay for some even-even nuclei, which is the only feasible way at present to probe the Majorana nature of massive neutrinos [22]. The inverted mass ordering is more favorable for this purpose, because even the lower limit of $|\langle m \rangle_{ee}|$ is above 0.01 eV, which is experimentally accessible in the foreseeable future [23]. As for the normal neutrino mass ordering, what interests us most is the possibility of significant cancellation among the three components of $\langle m \rangle_{ee}$ even in the μ - τ reflection symmetry case (the green and red regions in the profile of $|\langle m \rangle_{ee}|$, as shown in Fig. 4). To understand this observation, we follow Ref. [24] to express $|\langle m \rangle_{ee}|$ in the following way in terms of two Majorana phases ρ and σ :

$$|\langle m \rangle_{ee}| = |m_1 |U_{e1}|^2 e^{i\rho} + m_2 |U_{e2}|^2 + m_3 |U_{e3}|^2 e^{i\sigma}|, \quad (19)$$

where $\rho = 2(\phi - \varphi)$ and $\sigma = -2(\delta + \varphi)$ in the phase notations defined in Eq. (10). As pointed out in Ref. [24], a necessary condition for $|\langle m \rangle_{ee}| \rightarrow 0$ is $\rho = \pm\pi$, which is equivalent to $(\phi, \varphi) = (\pi/2, 0)$ (the green region) and $(\phi, \varphi) = (0, \pi/2)$ (the red region) in Fig. 4, respectively. That is to say, among the four possible combinations of the two Majorana phases ϕ and φ in the μ - τ reflection symmetry limit, two of them are likely to make $|\langle m \rangle_{ee}|$ fall into a well (i.e., make the rate of a $0\nu 2\beta$ decay too small to be observable) if the value of m_1 happens to lie in a “wrong” region¹.

Of course, it is very difficult to pin down the effects of μ - τ reflection symmetry breaking at the neutrino mass matrix level unless the octant issue of θ_{23} is experimentally resolved, the absolute neutrino mass scale is fixed or at least further constrained, and the three CP -violating phases are determined to an acceptable degree of accuracy.

Even the preliminary results obtained in Figs. 4 and 5, however, can tell us something useful. For example, one may draw the conclusion that the well-known Fritzsch texture [25],

$$M_\nu^{(F)} = \begin{pmatrix} \mathbf{0} & \langle m \rangle_{e\mu} & \mathbf{0} \\ \langle m \rangle_{e\mu} & \mathbf{0} & \langle m \rangle_{\mu\tau} \\ \mathbf{0} & \langle m \rangle_{\mu\tau} & \langle m \rangle_{\tau\tau} \end{pmatrix}, \quad (20)$$

must not be applicable to the inverted neutrino mass ordering even if the latter is relabeled to be normal. The reason is simply that $\langle m \rangle_{ee}$ is not only nonzero but also

¹ As for the inverted neutrino mass ordering, $|\langle m \rangle_{\mu\tau}| \rightarrow 0$ is also possible in the μ - τ reflection symmetry limit with $\phi = \pi/2$ and $\varphi = 0$ (the green region in Fig. 5).

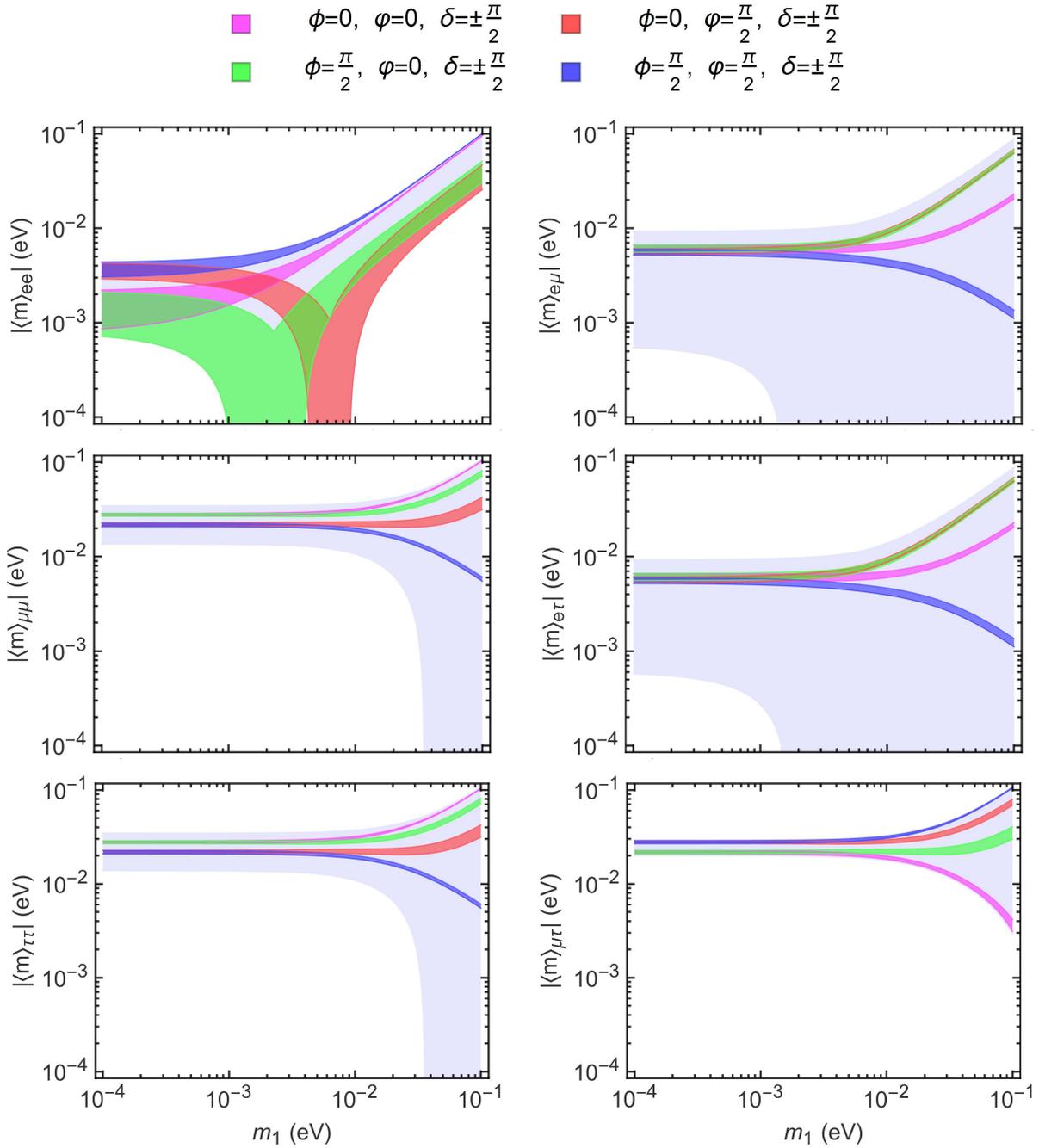


Fig. 4. (color online) The profiles of $|\langle m \rangle_{\alpha\beta}|$ versus the lightest neutrino mass m_1 for the **normal** neutrino mass ordering, where the 3σ ranges of six neutrino oscillation parameters [6] have been input. The pink, red, green and blue regions are fixed by the μ - τ reflection symmetry with $\theta_{23}=\pi/4$ and the special values of δ , ϕ and φ listed on top of the figure.

sizeable in this case, as shown in Fig. 5. Another interesting observation is that there is strong tension between $M_\nu^{(F)}$ and current experimental data at the 3σ level even in the normal mass ordering case, as Fig. 4 shows that the matrix elements $\langle m \rangle_{ee}$ and $\langle m \rangle_{\mu\mu}$ cannot simultane-

ously vanish¹⁾.

Another instructive example is the following two-zero texture of M_ν , which is usually referred to as “Pattern C” and favored with the inverted neutrino mass ordering [27, 28]:

1) This result is in conflict with previous studies based on very preliminary neutrino oscillation data [26].

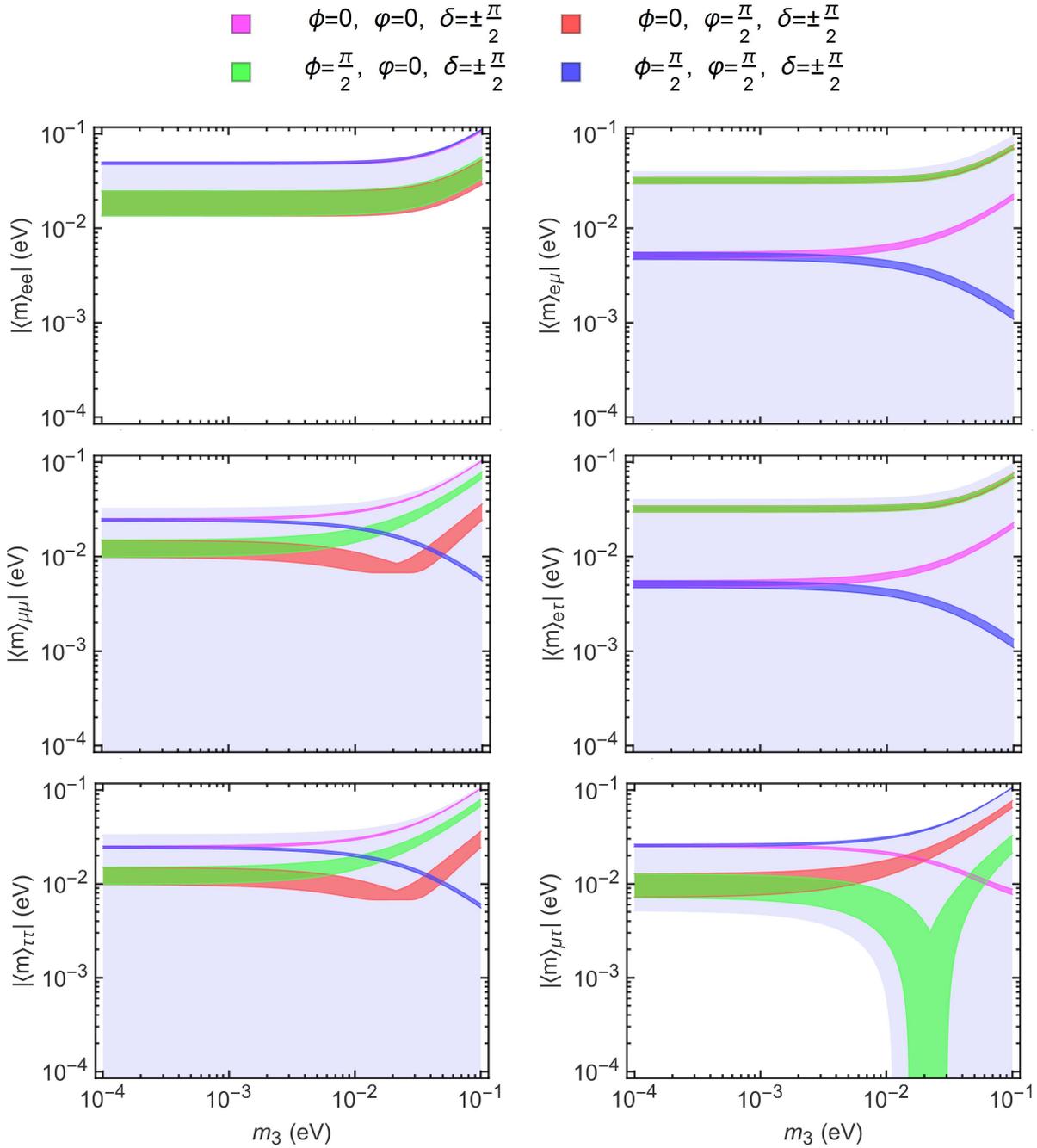


Fig. 5. (color online) The profiles of $|\langle m \rangle_{\alpha\beta}|$ versus the lightest neutrino mass m_3 for the **inverted** neutrino mass ordering, where the 3σ ranges of six neutrino oscillation parameters [6] have been input. The pink, red, green and blue regions are fixed by the μ - τ reflection symmetry with $\theta_{23}=\pi/4$ and the special values of δ , ϕ and φ listed on top of the figure.

$$M_\nu = \begin{pmatrix} \langle m \rangle_{ee} & \langle m \rangle_{e\mu} & \langle m \rangle_{e\tau} \\ \langle m \rangle_{e\mu} & \mathbf{0} & \langle m \rangle_{\mu\tau} \\ \langle m \rangle_{e\tau} & \langle m \rangle_{\mu\tau} & \mathbf{0} \end{pmatrix}. \quad (21)$$

A comparison of Eq. (21) with Fig. 5 tells us that such a special texture of M_ν remains compatible with current neutrino oscillation data at the 3σ level, but it does not

respect the μ - τ reflection symmetry, which definitely forbids vanishing or very small (generously say, $\lesssim 10^{-3}$ eV) values of $\langle m \rangle_{\mu\mu}$ and $\langle m \rangle_{\tau\tau}$.

Therefore, we expect that our model-independent results shown in Figs. 4 and 5 can help with model-building exercises in connection to the neutrino mass spectrum, flavor mixing and CP violation, whether one

follows the flavor symmetry approach or the texture-zero approach, or a mixture of the two.

4 Summary

We have asked ourselves what to do if the neutrino mass spectrum is finally found to be inverted. A straightforward and phenomenologically meaningful choice is certainly to relabel $m_3 < m_1 < m_2$ as $m'_1 < m'_2 < m'_3$, such that the latter is as normal as the mass spectra of those charged leptons and quarks of the same electrical charges. But in this case the columns of the PMNS matrix U must be reordered accordingly, and the resulting pattern U' turns out to involve an especially large mixing angle in the standard parametrization. We have examined the other angle-phase parametrizations of U and U' , and reached a similar observation. Given the fact that the Majorana neutrino mass matrix M_ν stays unchanged in such a mass relabeling exercise, we have considered the

intriguing μ - τ reflection symmetry to illustrate the texture of M_ν and the effects of its symmetry breaking at the 3σ level by using current neutrino oscillation data. We expect our model-independent results to be helpful for building specific neutrino mass models.

What is really behind the normal or inverted neutrino mass ordering remains an open question, and whether there is a definite correlation between the fermion mass spectra and flavor mixing patterns is also a big puzzle. The underlying flavor theory, which might be based on a certain flavor symmetry and its spontaneous or explicit breaking mechanism, may finally answer the above questions. To approach such a theory, we are now following a phenomenological path from the bottom up. Much more effort in this connection is needed.

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