New method of applying conformal group to quantum fields *

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Abstract: Most of previous work on applying the conformal group to quantum fields has emphasized its invariant aspects, whereas in this paper we find that the conformal group can give us running quantum fields, with some constants, vertex and Green functions running, compatible with the scaling properties of renormalization group method (RGM). We start with the renormalization group equation (RGE), in which the differential operator happens to be a generator of the conformal group, named dilatation operator. In addition we link the operator/spatial representation and unitary/spinor representation of the conformal group by inquiring a conformal-invariant interaction vertex mimicking the similar process of Lorentz transformation applied to Dirac equation. By this kind of application, we find out that quite a few interaction vertices are separately invariant under certain transformations (generators) of the conformal group. The significance of these transformations and vertices is explained. Using a particular generator of the conformal group, we suggest a new equation analogous to RGE which may lead a system to evolve from asymptotic regime to nonperturbative regime, in contrast to the effect of the conventional RGE from nonperturbative regime to asymptotic regime.

Key words: renormalization group equation, conformal group, unitary representation of conformal group, nonperturbation

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1 Introduction

The 4-dimensional conformal group [1, 2], with Poincare group as its subgroup, on the mathematical side has been investigated thoroughly from different aspects, and its applications to physics especially to quantum fields once were also widely considered. However the applications are not so satisfactory because hitherto no other perfect quantum system than photon field [3, 4] has been found so that the corresponding Lagrangian is conformal invariant, unless the masses of involved particles are zero [5–10]. Furthermore, one inference of the conformal invariance is that according to Noether's theorem, if a Lagrangian is invariant under scaling transformation, then the trace of the energy-momentum tensor should be null [6, 8]. These two factors become obstacles to apply the conformal group to most material fields. Other efforts were also experienced to search for an invariant fermion equation or scattering amplitude [7, 9, 11], and even to apply it to nonlocal action [12, 13]. None of the results is pertinent to known material fields. In this paper we investigate a tentative application by considering simultaneously the unitary representation and the coordinate representation of the conformal group, just mimicking the Lorentz group applied to Dirac equation. Then we find that the interaction vertices become running. It follows that the conformal group might make sense not for invariance, but for running.

In this paper we phenomenologically extract a scaling transformation from renormalization, under which physical quantities become running. The scaling transformation corresponds to one of the generators of conformal group. We then generalize such running effect to most of the other generators of conformal group. The renormalization group method (RGM) has its intrinsic relationship with scaling transformation if viewing the differentiating operator $\mu\dfrac{\mathrm{d}}{\mathrm{d}\mu}$ in the group equation as scaling operator. In RGM, for a function Λ that represents a vertex function, a wave function or a propagator, its renormalized form and unrenormalized form are linked as [14]

$$\Lambda = Z_{\mathrm{F}} \Lambda_{\mathrm{R}}.$$

Differentiating the above equation with respect to the renormalization parameter μ, since unrenormalized Λ is independent of μ, one immediately gets the renormalization group equation (RGE) [15],

$$\mu\frac{\mathrm{d}\Lambda_{\mathrm{R}}}{\mathrm{d}\mu} + \gamma_{\mathrm{F}} \Lambda_{\mathrm{R}} = 0, \qquad (1)$$

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where $\gamma_{\rm F}$ is the anomalous scaling dimension defined by

$$\gamma_{\rm F}=\mu\frac{\rm d}{{\rm d}\mu}\ln Z_{\rm F}.$$

In the Section 3 we will note that the operator

$$\mu\frac{\rm d}{{\rm d}\mu}$$

is just the scaling operator in its spatial representation, apart from a coefficient i. Eq. (1) is a special form of RGE, and a general one should be of the form [16, 17]

$$\left[M\frac{\partial}{\partial M}+\beta(\lambda)\frac{\partial}{\partial\lambda}+n\gamma(\lambda)\right]G^{(n)}(\{x_i\};M,\lambda)=0$$

which is for any Green's function of massless ϕ^4 theory. Supposing the function $\Lambda_{\rm R}$ has a dimension $\gamma_{\rm F}$ with respect to a scale parameter μ, then by such transformation $\mu\to\lambda\mu$, the $\Lambda_{\rm R}$ yields

$$\Lambda_{\rm R}(\mu,\text{other parameters})=\lambda^{\gamma_{\rm F}}\Lambda_{\rm R}\left(\frac{\mu}{\lambda},\text{other parameters}\right),$$

which is the essence of RGM. In this paper we shall use the scaling form of RGM, but we are free of the detailed calculation of renormalization.

The remaining parts of the paper are arranged as follows. In Section 2 we present the basic definition of conformal group. In Section 3 we focus on how to derive the spinor representation. In Section 4 we associate the two representations of conformal group by physical interaction vertices in conventional sense. Section 5 is dedicated to generalizing the above association to all kinds of vertices as well as explaining some of the physical meaning of the vertices and transformations. Then follows the conclusions and discussions.

2 Basic definition of conformal group

In this section we introduce the basic definitions of the 4-dimensional conformal group [1, 6], including their generators and commutations in spatial representation [1, 18]. The conventional applications were mainly confined within this spatial representation, apart from a few exceptions, like Ref. [11]. We start with the null vector space (Euclidean space),

$$\eta_1^2+\eta_2^2+\eta_3^2+\eta_4^2+\eta_5^2+\eta_6^2=0\ . \tag{2}$$

reserving which gives the popular definition of conformal group [19]. A special expression of the differential forms in 4-dimensional spatial representation can be derived directly from the above equation. In derivation we need to apply the following variables [1, 2]

$$x_\mu = \frac{\eta_\mu}{K}\ , \text{ where } K=\eta_5+{\rm i}\eta_6,$$
$$\text{where } \mu = 1,2,3,4, \tag{3}$$

together with the differential form

$$\begin{aligned}\frac{\partial}{\partial\eta_a} = \frac{1}{K}\Big\{&[\delta_{a\mu}-(\delta_{a5}+{\rm i}\delta_{a6})x_\mu]\frac{\partial}{\partial x_\mu}\\ &+(\delta_{a5}+{\rm i}\delta_{a6})K\frac{\partial}{\partial K}\Big\}\ , \text{ where } a=1,2,\cdots,6,\end{aligned} \tag{4}$$

to the definition of 6-dimensional angular-momentum

$$M_{ab}={\rm i}\left(\eta_a\frac{\partial}{\partial\eta_b}-\eta_b\frac{\partial}{\partial\eta_a}\right), \text{ where } a,b=1,2,\cdots,6. \tag{5}$$

Then one gets the following generators for conformal group [1] [of which in Eq. (56)]

$$\begin{aligned}D &= {\rm i}M_{56}=-\left(\eta_5\frac{\partial}{\partial\eta_6}-\eta_6\frac{\partial}{\partial\eta_5}\right)\\ &= {\rm i}\left(x_\mu\frac{\partial}{\partial x_\mu}-K\frac{\partial}{\partial K}\right),\\ P_\mu &= M_{5\mu}+{\rm i}M_{6\mu}={\rm i}\frac{\partial}{\partial x_\mu},\ \ K_\mu=M_{5\mu}-{\rm i}M_{6\mu}\\ &= {\rm i}\left\{-x^2\frac{\partial}{\partial x_\mu}+2x_\mu x_\nu\frac{\partial}{\partial x_\nu}-2Kx_\mu\frac{\partial}{\partial K}\right\}.\end{aligned} \tag{6}$$

The projected form of Eq. (6) (making K as constant boundary of Minkowski space [2]) with Minkowski convention then is

$$\begin{aligned}D &= {\rm i}x_\mu\frac{\partial}{\partial x_\mu},\ M_{\mu\nu}={\rm i}\left(x_\mu\frac{\partial}{\partial x^\nu}-x_\nu\frac{\partial}{\partial x^\mu}\right),\\ P_\mu &= {\rm i}\frac{\partial}{\partial x^\mu},\ K_\mu=-{\rm i}\left(x^2\frac{\partial}{\partial x^\mu}-2x_\mu x^\nu\frac{\partial}{\partial x^\nu}\right),\end{aligned} \tag{7}$$

where $M_{\mu\nu}$ represents the components of conventional angular momentum in 4-dimensions. The corresponding commutation relation can be obtained by direct computation,

$$\begin{aligned}[][M_{\mu\nu},M_{\rho\sigma}] &= {\rm i}(g_{\nu\rho}M_{\mu\sigma}+g_{\mu\sigma}M_{\nu\rho}-g_{\mu\rho}M_{\nu\sigma}-g_{\nu\sigma}M_{\mu\rho}),\\ [M_{\mu\nu},P_\rho] &= {\rm i}(g_{\nu\rho}P_\mu-g_{\mu\rho}P_\nu),\\ [D,P_\mu] &= -{\rm i}P_\mu,\ [D,K_\mu]={\rm i}K_\mu,\\ [D,M_{\mu\nu}] &= 0,\\ [M_{\mu\nu},K_\rho] &= {\rm i}(g_{\nu\rho}K_\mu-g_{\mu\rho}K_\nu),\end{aligned} \tag{8}$$

$$[P_\mu,K_\rho] = -2{\rm i}(g_{\mu\rho}D+M_{\mu\rho}). \tag{9}$$

3 Spinor representations of the conformal group

In this section we derive the unitary/spinor representation of the 4-dimensional conformal group [1, 6] in the frame of group theory. The representation is derived by applying Cartan method [19] to $SO(6)$–$SU(4)$ transform. Before using the Cartan method to achieve its unitary representation, it is good to review first the steps of applying the Cartan method to $SO(3)$–$SU(2)$ [19][of which pp. 41-48]. To keep the invariance of $x_1^2+x_2^2+x_3^2=0$, one defines the matrix

$$X=\begin{pmatrix} x_3 & x_1-\mathrm{i}x_2 \\ x_1+\mathrm{i}x_2 & -x_3 \end{pmatrix}. \tag{10}$$

The trace $\mathrm{Tr}(X^\dagger X)$ is $x_1^2+x_2^2+x_3^2$. With U as an element of $SU(2)$ group, we define

$$X'=U^{-1}XU, \tag{11}$$

immediately we have

$$\mathrm{Tr}(X'^\dagger X')=\mathrm{Tr}(X^\dagger X), \tag{12}$$

thus $SU(2)$ group keeps the trace invariant, and by this way the group also keeps the metric $x_1^2+x_2^2+x_3^2$. With the knowledge that the $SO(3)$ group directly reserves the metric $x_1^2+x_2^2+x_3^2$, we conclude that Cartan matrix X acts as a mapping between $SO(3)$ and $SU(2)$. By the Cartan matrix X, one can define spinor $\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}$ by

$$X\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}=0, \tag{13}$$

with the solution $\xi_0=\pm\sqrt{\frac{x_1-\mathrm{i}x_2}{2}}$ and $\xi_1=\pm\sqrt{\frac{-x_1-\mathrm{i}x_2}{2}}$, and the reverse yields

$$\begin{aligned} x_1 &= \xi_0^2-\xi_1^2 \\ x_2 &= \mathrm{i}(\xi_0^2+\xi_1^2) \\ x_3 &= -2\xi_0\xi_1, \end{aligned} \tag{14}$$

which automatically satisfies $x_1^2+x_2^2+x_3^2=0$, from which we can define the spinor reversely.

From the above Cartan matrix X we can extract the Pauli matrices σ_1, σ_2, σ_3 separately from the coefficients of x_1, x_2, x_3. Meanwhile Pauli matrices σ_1, σ_2, σ_3 act as the generators of $SU(2)$ group mentioned above. Furthermore it is easy to test that $SU(2)$ group reserves the metric

$$|\xi_0|^2+|\xi_1|^2=\Xi^\dagger\Xi. \tag{15}$$

Coincidentally the n-vectors form (defined in Eq. (22)) based on Pauli matrices do not generate new matrices, neither the multiplications nor the commutations among them, they themselves are closed. Now in what follows we would find the corresponding Cartan matrix from $SO(6)$ to $SU(4)/SU(2,2)$, namely the spinor representation for 4-dimensional conformal group.

To achieve its unitary/spinor representation in 4-dimensions, mimicking the relationship between the metric $x_1^2+x_2^2+x_3^2$ and that in Eq. (15), we shall associate the metric in Eq. (2) with the invariant quadratic form

$$|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2=Z^\dagger Z, \tag{16}$$

by the following matrix [20],

$$A=\begin{pmatrix} 0 & x_1+\mathrm{i}x_2 & x_3+\mathrm{i}x_4 & x_5+\mathrm{i}x_6 \\ -(x_1+\mathrm{i}x_2) & 0 & x_5-\mathrm{i}x_6 & -x_3+\mathrm{i}x_4 \\ -(x_3+\mathrm{i}x_4) & -x_5+\mathrm{i}x_6 & 0 & x_1-\mathrm{i}x_2 \\ -(x_5+\mathrm{i}x_6) & x_3-\mathrm{i}x_4 & -x_1+\mathrm{i}x_2 & 0 \end{pmatrix}. \tag{17}$$

Counting the degrees of freedom of the groups conserving separately Eq. (2) and Eq. (16), one finds they are both 15. Next we only need to extract the coefficients before x_i's to get the unitary matrices as generators of $SU(4)$, just like the method used in the three dimensional example Eq. (8)–(14). If we want to get the generators of $SU(2,2)$ we need only to change the signs before x_1 and x_2 and those ahead of corresponding matrices, which would change Eqs. (2) and (16) to

$$-x_1^2-x_2^2+x_3^2+x_4^2+x_5^2+x_6^2=0 \ . \tag{18}$$

and

$$-|z_1|^2-|z_2|^2+|z_3|^2+|z_4|^2=Z^\dagger Z. \tag{19}$$

The latter falls into a Dirac spinor like

$$\tilde{\psi}=(z_1,z_2,z_3,z_4).$$

It can be found that the matrix A in Eq. (17) meets the invariant expression

$$\mathrm{Tr}(A^\dagger A)=4(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2) \tag{20}$$

just like the above 3-dimensional example, while the $SU(4)$ group keeps the above trace $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2$=constant, and simultaneously reserves the metric Eq. (16). The above method of linking a real metric to a matrix is closely analogous to the Cartan method of constructing a spinor representation in any real space. Actually, the true spinor space for 4-d conformal group following Cartan method should be of 8-dimensions instead of 4-dimensions [19] [of which in pp. 88-89]. In what follows we would take over the process of deriving all of the n-vectors along the Cartan method [19], though we work in 4-dimensions rather than 8-dimensions. First we extract the matrices before the x_i's in Eq. (17), i.e.1-vectors,

$$\begin{aligned} B_1 &= \begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & \mathrm{i}\sigma_2 \end{pmatrix},\ B_2=\begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \\ B_3 &= \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},\ B_4=\begin{pmatrix} 0 & \mathrm{i}I \\ -\mathrm{i}I & 0 \end{pmatrix}, \\ B_5 &= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix},\ B_6=\begin{pmatrix} 0 & -\sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \end{aligned} \tag{21}$$

where the σ_i's are Pauli matrices. The definition of a k-vector is

$$B_{k\text{-vector}}=\sum_P(-1)^P B_{n_1}B_{n_2}\cdots B_{n_k}, \tag{22}$$

where P denotes different permutations. Applying the above formula to the 2-vector, and using the corresponding subscripts to denote the 1-vectors involved, then

$$B_{12} = B_1B_2-B_2B_1=\begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & \mathrm{i}\sigma_2 \end{pmatrix}\begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} - \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}\begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & \mathrm{i}\sigma_2 \end{pmatrix}=0.$$

Similarly, let us exhaust all possibilities, then obtain other nontrivial 2-vectors

$$\begin{aligned} B_{13} &= 2\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix},\ B_{15}=2\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \\ B_{35} &= 2\mathrm{i}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix},\ B_{36}=2\mathrm{i}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \\ B_{46} &= -2\mathrm{i}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix},\ B_{24}=2\mathrm{i}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \\ B_{23} &= -2\mathrm{i}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \end{aligned} \tag{23}$$

We note that the new ones which are independent of B_i's are just B_{23} and B_{36}. The same line can be followed to carry out the 3-vectors. Ignoring the repeating ones, we find the new 3-vectors independent of both 1-vectors and 2-vectors are

$$\begin{aligned} B_{123} &\sim \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix},\ B_{134}\sim\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \\ B_{145} &\sim \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix},\ B_{245}\sim\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \\ B_{345} &\sim \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix},\ B_{146}\sim\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \\ B_{124} &\sim \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \end{aligned} \tag{24}$$

Computing the 4-vectors and the higher ones will not give new independent matrices. Finally, we can rearrange all the above k-vector-produced matrices as follows [20],

$$\begin{aligned} U_i &= \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \\ V_\mu &= -\frac{1}{2}\begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix} \\ W_\mu &= \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix} \\ Y_\mu &= \frac{1}{2}\begin{pmatrix} 0 & \sigma_\mu \\ -\sigma_\mu & 0 \end{pmatrix}, \end{aligned} \tag{25}$$

where σ_i, $i=1,2,3$ are normal Pauli matrices and $\sigma_0=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The convention can be changed from Minkowski to Euclidean spaces while instead requiring $\sigma_\mu^2=-1$, i.e. making σ_0=i and replacing the definition of σ_i by those in [20]. The route of inquiring the concrete matrices following the Cartan method as above is a shortcut rarely mentioned in the literature. It can be checked that the commutations among U_i, V_μ, W_μ, Y_μ are just those for a conformal group [1, 18], accordingly the mapping from these matrices to differential forms turns out to be

$$\begin{aligned} U_i &\leftrightarrow \gamma_i\gamma_j\longrightarrow\mathrm{i}\left(x_j\frac{\partial}{\partial x^k}-x_k\frac{\partial}{\partial x^j}\right)\longrightarrow M_{jk} \\ W_i &\leftrightarrow \gamma_0\gamma_i\longrightarrow\mathrm{i}\left(x_i\frac{\partial}{\partial x^0}-x_0\frac{\partial}{\partial x^i}\right)\longrightarrow M_{0k} \\ W_0 &\leftrightarrow \gamma_5\longrightarrow\mathrm{i}x_\mu\frac{\partial}{\partial x_\mu}\longrightarrow D \\ V_\mu+Y_\mu &\leftrightarrow \gamma_\mu(1-\gamma_5)\longrightarrow\mathrm{i}\frac{\partial}{\partial x^\mu}\longrightarrow P_\mu \\ V_\mu-Y_\mu &\leftrightarrow \gamma_\mu(1+\gamma_5)\longrightarrow \\ &\qquad -\mathrm{i}\left(\frac{1}{2}x_\nu x^\nu\frac{\partial}{\partial x_\mu}-x_\mu x_\nu\frac{\partial}{\partial x_\nu}\right)\longrightarrow K_\mu. \end{aligned} \tag{26}$$

We use $\longrightarrow$ to represent the accurate mappings and $\leftrightarrow$ the equivalence, and the commutations have been examined by computer. Now we recognize that the role of operator $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}$ (or $x_\mu\frac{\partial}{\partial x_\mu}$) in the conformal group is equivalent to the scaling operator D, with its unitary form γ_5.

4 Physical relationship between the two representations of scaling transformation

In this section we use scaling transformation as a paradigm to investigate the effect of conformal group on the interaction vertex. Enlightened by utilizing Lorentz transformation to Dirac equation, we first try to link physically the spatial form of scaling transformation with its spinor/unitary form, the former representing the realistic expansions and contractions of space-time (dilatation and shrinkage means the same), the latter representing the intrinsic freedom very like spin angular momentum. Then in the next section we will generalize the method from scaling transformation to other generators of conformal group.

As for Lorentz transformation, the transformation matrix (Λ_μ^ν) for $j^\mu(y)=\bar{\psi}(y)\gamma^\mu\psi(y)$ corresponds to a complex transformation S for $\psi(y)$ so that the effect of the transformed result $\bar{\psi}(y)S^{-1}\gamma^\mu S\psi(y)$ is equivalent to

$\bar{\psi}(y)\Lambda^{\mu}_{\nu}\gamma^{\nu}\psi(y)$. Referencing the case of Lorentz transformation, our goal in this section is to search for the corresponding vertex-form Γ^{μ} so that it links with transformation S' by $S'^{-1}\Gamma^{\mu}S'=\Lambda'^{\mu}_{\nu}\Gamma^{\nu}$, where $S'=\mathrm{e}^{\frac{u}{2}\gamma_5}$, γ_5 is the spinor representation of the scaling operator D, and Λ'^{μ}_{ν} represent tensor's components of scaling transformation. A similar method was used in a previous paper [21], but for totally different motivation.

Usually we perform the spatial Lorentz transformation on the vectors A_{μ} and γ^{μ}. Obviously this combination brings about invariant formalism like $A^{\nu}(q^2)\bar{\psi}(p)\gamma_{\nu}\psi(p')$. We follow the convention that the same set of $\{\gamma^{\mu}\}$ is used in different coordinate systems, which naturally yields an equivalence transformation S satisfying [22, 23]

$$S^{-1}\gamma^{\mu}S=\Lambda^{\mu}_{\nu}\gamma^{\nu}=\gamma'^{\mu}, \tag{27}$$

where Λ^{μ}_{ν} stand for the tensors' components of the Lorentz transformation. Substituting Eq. (27) into $A_{\mu}(x)\bar{\psi}(x)\gamma^{\mu}\psi(x)$ yields

$$A'_{\mu}(y)\bar{\psi}'(y)S^{-1}\gamma^{\mu}S\psi'(y)=A'_{\mu}(y)\bar{\psi}(y)\gamma'^{\mu}\psi(y)\,. \tag{28}$$

While looking for Γ^{μ} we follow the same convention as that in the above paragraph, i.e., in different coordinate systems we use the same set of $\{\Gamma^{\mu}\}$. Then analogously, we use the form of the above formula Eq. (27) for scaling transformation as

$$S'^{-1}\Gamma^{\mu}S'=\Lambda'^{\mu}_{\nu}\Gamma^{\nu}\,, \tag{29}$$

where formally we have used Λ'^{μ}_{ν} to represent the scaling transformation to every coordinate component [7, 9, 11][of which Eq. (2)] instead of using the usual form $\mathrm{e}^{-\alpha}$ [18]. Different from the operator $\mu\dfrac{\mathrm{d}}{\mathrm{d}\mu}$ appearing in RGE, here the operator D has the usual form $D=\mathrm{i}x^{\nu}\partial_{\nu}$, being a hermitian. With the relation $\mathrm{e}^{-\mathrm{i}\alpha D}p_{\mu}\mathrm{e}^{\mathrm{i}\alpha D}=\mathrm{e}^{-\alpha}p_{\mu}$, i.e. $[D,p_{\mu}]=-\mathrm{i}p_{\mu}$ [18], we have

$$\begin{aligned}(\Gamma^{\mu}p_{\mu})'_{\text{scaling transform}} &= S'^{-1}\Gamma^{\mu}S'\Lambda'^{\nu}_{\mu}p_{\nu}\\ &= S'^{-1}\Gamma^{\mu}S'\mathrm{e}^{-\mathrm{i}\alpha D}p_{\mu}\mathrm{e}^{\mathrm{i}\alpha D}.\end{aligned} \tag{30}$$

Now let us submit $S'=\mathrm{e}^{\frac{u}{2}\gamma_5}$ obtained from the last section (henceforth we use $\mathrm{e}^{\frac{u}{2}\gamma_5}$ instead of $\mathrm{e}^{\frac{u}{2}(1+\gamma_5)}$ as scaling transformation while no confusion occurs), where u is the infinitesimal parameter. Formally we get

$$\begin{aligned}S'^{-1}\Gamma^{\mu}S'\Lambda'^{\nu}_{\mu}p_{\nu} &= \mathrm{e}^{-\frac{u}{2}\gamma_5}\Gamma^{\mu}\mathrm{e}^{\frac{u}{2}\gamma_5}(p_{\mu})'_{\text{scaling transform}}\\ &= \mathrm{e}^{-\frac{u}{2}\gamma_5}\Gamma^{\mu}\mathrm{e}^{\frac{u}{2}\gamma_5}\mathrm{e}^{-\mathrm{i}\alpha D}p_{\mu}\mathrm{e}^{\mathrm{i}\alpha D}\\ &= \mathrm{e}^{-\frac{u}{2}\gamma_5}\Gamma^{\mu}\mathrm{e}^{\frac{u}{2}\gamma_5}\mathrm{e}^{-\alpha}p_{\mu}\\ &\doteq \mathrm{e}^{-\frac{u}{2}\gamma_5}\Gamma^{\mu}\mathrm{e}^{\frac{u}{2}\gamma_5}p_{\mu}(1-\alpha).\end{aligned} \tag{31}$$

From the experience of calculating γ-matrix and the following relations

$$\begin{aligned}\mathrm{e}^{-\frac{u}{2}\gamma_5}\gamma^{\mu}\mathrm{e}^{\frac{u}{2}\gamma_5} &\approx \left(1-\frac{u}{2}\gamma_5\right)\gamma^{\mu}\left(1+\frac{u}{2}\gamma_5\right)\\ &\approx \gamma^{\mu}+u\gamma^{\mu}\gamma_5,\end{aligned} \tag{32}$$

$$\begin{aligned}\mathrm{e}^{-\frac{u}{2}\gamma_5}\gamma^{\mu}\gamma_5\mathrm{e}^{\frac{u}{2}\gamma_5} &\approx \left(1-\frac{u}{2}\gamma_5\right)\gamma^{\mu}\gamma_5\left(1+\frac{u}{2}\gamma_5\right)\\ &\approx \gamma^{\mu}\gamma_5+u\gamma^{\mu},\end{aligned} \tag{33}$$

$$\begin{aligned}\mathrm{e}^{-\frac{u}{2}\gamma_5}\gamma^{\mu}(1\pm\gamma_5)\mathrm{e}^{\frac{u}{2}\gamma_5} &\approx \left(1-\frac{u}{2}\gamma_5\right)\gamma^{\mu}(1\pm\gamma_5)\left(1+\frac{u}{2}\gamma_5\right)\\ &\approx (1\pm u)\gamma^{\mu}(1\pm\gamma_5),\end{aligned} \tag{34}$$

we find out a possible form of Γ^{μ}

$$\Gamma^{\mu}=\gamma^{\mu}(1\pm\gamma_5)\ \text{while}\ \alpha\sim u\ . \tag{35}$$

The transformation in Eqs. (32, 33, 34) can be instead $\mathrm{e}^{-\frac{u}{2}(1+\gamma_5)}$ and $\mathrm{e}^{\frac{u}{2}(1+\gamma_5)}$, the results would obviously be the same. The coefficients $(1\pm u)$ of Eq. (34) can be contracted now to be 1 with coefficients $(1\mp u)$ that come from the transformation of p_{μ}. We note that the infinitesimal parameters u and α are not independent. In this way we set up the relationship between the operator D and $S'=\mathrm{e}^{\frac{u}{2}(1+\gamma_5)}$ directly.

We know $S'=\mathrm{e}^{\frac{u}{2}(1+\gamma_5)}$ is responsible for acting on the Dirac spinor as expected. The same transformation holds for vertex $\Gamma^{\mu}A_{\mu}$, as well as $\Gamma^{\mu}p_{\mu}$. The resultant vertex is different from that of Ref. [11] due to the choice of γ_5, since we have followed the convention of Quantum Field Theory. In fact we have extended the transformation, the interaction vertex and spinor space simultaneously, and these elements can be extended further while involving further the other generators of the conformal group.

Now we are interested in what happens if we perform the scaling transformation S' successively N times upon the vector vertex-form γ^{μ}. Different from Eqs. (32, 33, 34), we now employ the following formulism without approximation

$$\begin{aligned}&(\mathrm{e}^{-\frac{u}{2}(1+\gamma_5)})^{N}\gamma^{\mu}(\mathrm{e}^{\frac{u}{2}(1+\gamma_5)})^{N}\\ &=\gamma^{\mu}\cosh Nu+\gamma^{\mu}\gamma_5\sinh Nu,\end{aligned} \tag{36}$$

from which one notes that the vector vertex arrives at its limits $\gamma^{\mu}(1\pm\gamma_5)$ only if $\dfrac{\cosh Nu}{\sinh Nu}\rightarrow\pm1$, i.e. $Nu\rightarrow\pm\infty$. $Nu\rightarrow\pm\infty$ means one carrying out enough steps of inflating or shrinking transformation. We call such states involving interaction vertices $\gamma^{\mu}(1\pm\gamma_5)$ "extreme states", which evolve from the interaction vertex γ^{μ} with the scale constantly changing. The variation of coupling constant is assumed to be absorbed into the coupling constant. It turns out that such scaling transformation does not conserve the vector-dominant interaction, or alternatively, the transformation tends to make the coupling constant as well as interaction vertex running.

5 Generalization to other generators of conformal group, and running effect

In this section we will apply the method of the last section to other generators of the conformal group. By this way we analyze the structure of the conformal group and its relationship with the vertices of conventional quantum field theory. Finally we present a table form for generators (transformations) and vertices, by which we can judge the concrete physical significance of every block of the conformal group.

In the last section we have derived the limits $\gamma^\mu(1\pm\gamma_5)$ with the relation $e^{-i\alpha D}p_\mu e^{i\alpha D}=e^{-\alpha}p_\mu$, i.e. $[D,p_\mu]=-ip_\mu$[18]. The P_μ plays the role of the production operator or annihilation operator of D from the point of view of quantum mechanics. Analogously, we may notice another similar commutation among conformal group, $[M_{\mu\nu},K_\rho]=i(g_{\nu\rho}K_\mu-g_{\mu\rho}K_\nu)$. However we fail to get the variation of vertex $\gamma^\rho(1\pm\gamma_5)$ with the transformation $e^{\frac{u}{2}\gamma_\mu\gamma_\nu}$ (operation of $M_{\mu\nu}$ in spinor space). So the analog cannot be followed by any other commutations of conformal group. The details are as follows in Eq. (38). While the angular momentum transformation is written to be $S=e^{\frac{u}{2}\gamma_\mu\gamma_\nu}$, its effect on vector vertex γ^μ [we label it by Γ^μ] is

$$\begin{aligned}\Gamma^{\mu\prime} &= e^{-\frac{u}{2}\gamma_\rho\gamma_\sigma}\Gamma^\mu e^{\frac{u}{2}\gamma_\rho\gamma_\sigma}\approx\left(1-\frac{u}{2}\gamma_\rho\gamma_\sigma\right)\Gamma^\mu\left(1+\frac{u}{2}\gamma_\rho\gamma_\sigma\right)\\ &\approx \gamma^\mu+u\left(g^\mu_\rho\gamma_\sigma-g^\mu_\sigma\gamma_\rho\right),\\ P'_\mu &= e^{-iaM_{\rho\sigma}}P_\mu e^{iaM_{\rho\sigma}}=P_\mu+ia[P_\mu,M_{\rho\sigma}]\\ &= P_\mu-a(g_{\mu\rho}P_\sigma-g_{\mu\sigma}P_\rho).\end{aligned}\tag{37}$$

Combining the above transformations yields

$$\begin{aligned}(\Gamma^\mu P_\mu)' &= \left[\gamma^\mu+u\left(g^\mu_\rho\gamma_\sigma-g^\mu_\sigma\gamma_\rho\right)\right][P_\mu-a(g_{\mu\rho}P_\sigma-g_{\mu\sigma}P_\rho)]\\ &= \Gamma^\mu P_\mu+u\left(g^\mu_\rho\gamma_\sigma-g^\mu_\sigma\gamma_\rho\right)P_\mu-a\gamma^\mu(g_{\mu\rho}P_\sigma-g_{\mu\sigma}P_\rho)\\ &= \Gamma^\mu P_\mu+(u+a)(\gamma_\sigma P_\rho-\gamma_\rho P_\sigma),\end{aligned}\tag{38}$$

when $u\sim -a$, one obtains $(\Gamma^\mu P_\mu)'=\Gamma^\mu P_\mu$. If replacing γ^μ with axil-vector vertex $\gamma^\mu(1\pm\gamma_5)$ [we use the same label Γ^μ for this vertex], then we get the same form

$$\begin{aligned}\Gamma^{\mu\prime} &= e^{-\frac{u}{2}\gamma_\rho\gamma_\sigma}\Gamma^\mu e^{\frac{u}{2}\gamma_\rho\gamma_\sigma}\approx\left(1-\frac{u}{2}\gamma_\rho\gamma_\sigma\right)\Gamma^\mu\left(1+\frac{u}{2}\gamma_\rho\gamma_\sigma\right)\\ &\approx \Gamma^\mu+u\left(g^\mu_\rho\Gamma_\sigma-g^\mu_\sigma\Gamma_\rho\right),\\ P'_\mu &= e^{-iaM_{\rho\sigma}}P_\mu e^{iaM_{\rho\sigma}}=P_\mu+ia[P_\mu,M_{\rho\sigma}]\\ &= P_\mu-a(g_{\mu\rho}P_\sigma-g_{\mu\sigma}P_\rho).\end{aligned}\tag{39}$$

Combining the above transformations yields

$$(\Gamma^\mu P_\mu)'=\Gamma^\mu P_\mu+(u+a)(\Gamma_\sigma P_\rho-\Gamma_\rho P_\sigma).\tag{40}$$

Just for this result we conclude that the commutation $[D,p_\mu]=-ip_\mu$ is unique in the conformal group.

We have to use P_μ to make the vertex $\gamma^\mu(1+\gamma_5)$ running and K_μ to make the vertex $\gamma^\mu(1-\gamma_5)$ running. For example, for $\Gamma^\mu=\gamma^\mu(1-\gamma_5)$ we use $K_\mu=\gamma_\mu(1+\gamma_5)$ to transform it,

$$\begin{aligned}\Gamma^{\mu\prime} &= e^{\frac{u}{2}\gamma_\rho(1+\gamma_5)}\gamma^\mu(1-\gamma_5)e^{-\frac{u}{2}\gamma_\rho(1+\gamma_5)}\\ &= [1+\frac{u}{2}\gamma_\rho(1+\gamma_5)]\gamma^\mu(1-\gamma_5)[1-\frac{u}{2}\gamma_\rho(1+\gamma_5)]\\ &= \gamma^\mu(1-\gamma_5)+\frac{u}{2}[\gamma_\rho(1+\gamma_5),\gamma^\mu(1-\gamma_5)]\\ &\quad -\frac{u^2}{4}\gamma_\rho(1+\gamma_5)\gamma^\mu(1-\gamma_5)\gamma_\rho(1+\gamma_5)\\ &= \gamma^\mu(1-\gamma_5)+2u\left[g^\mu_\rho(1-\gamma_5)-\gamma^\mu\gamma_\rho\right]\\ &\quad -u^2\gamma_\rho\gamma^\mu\gamma_\rho(1+\gamma_5),\end{aligned}\tag{41}$$

when $u\longrightarrow 0$, $\Gamma^{\mu\prime}\longrightarrow\left(\gamma^\mu+2ug^\mu_\rho\right)(1-\gamma_5)-2u\gamma^\mu\gamma_\rho$ and when $u\longrightarrow\infty$, $\Gamma^{\mu\prime}\longrightarrow -u^2\gamma_\rho\gamma^\mu\gamma_\rho(1+\gamma_5)$. Then combine the above equation with the spatial transformation $P'_\mu=e^{-iaK_\rho}P_\mu e^{iaK_\rho}\approx[1-iaK_\rho]P_\mu[1+iaK_\rho]=P_\mu+ia[P_\mu,K_\rho]=P_\mu+2a(g_{\mu\rho}D-M_{\mu\rho})$, we have

$$\begin{aligned}(\Gamma^\mu P_\mu)' &\approx [(\gamma^\mu+2ug^\mu_\rho)(1-\gamma_5)\\ &\qquad -2u\gamma^\mu\gamma_\rho][P_\mu+2a(g_{\mu\rho}D-M_{\mu\rho})]\\ &\approx \Gamma^\mu P_\mu+2u[g^\mu_\rho(1-\gamma_5)-\gamma^\mu\gamma_\rho]P_\mu\\ &\qquad +2a\gamma^\mu(1-\gamma_5)(g_{\mu\rho}D-M_{\mu\rho}).\end{aligned}\tag{42}$$

The generator P_μ would have similar effect on the vertex $\gamma^\mu(1+\gamma_5)$. In this way we make the vertex $\gamma^\mu(1+\gamma_5)$ running. So far we are also curious about all of the vertices γ_5, γ^μ, $\gamma^\mu(1\pm\gamma_5)$, $\gamma^\mu\gamma_5$, $\gamma^\mu\gamma^\nu$ as well as the ones combined with A_μ like $\gamma^\mu A_\mu$. When do these vertices remain conserved and when do they become running (we assume that the field A_μ transforming as P_μ and use φ to mark a scalar field)? In what follows we list the table form to show the relationship of generators (transformations) and vertices, by which we can recognize whether they are running or conserved.

The above "conserved" means the results as in Eq. (38), and "running" means the result like in Eq. (42), in which there is no way to arrange the value of a and u so that the redundant terms are cancelled.

By observing the table, we note γ_5 is pseudo-scalar and relates to chirality, so it is conserved under D and M transformation, but runs under P_μ and K_μ. P_μ and K_μ plays the role like acceleration additional to Lorentz group, so it is reasonable to affect the chirality (more impressive in the above extreme case). γ^μ and $\gamma^\mu\gamma_5$ runs as expected since it corresponds to (pseudo-)vector vertex, the same with $\gamma^\mu A_\mu$ and $\gamma^\mu\gamma_5 A_\mu$. $\gamma^\mu(1+\gamma_5)$ (or $\gamma^\mu(1-\gamma_5)$) conserves under K_μ(P_μ) transformation for its commutation relationship. But only when A_μ performs as K_μ(P_μ) in all of the commutators, can $\gamma^\mu(1+\gamma_5)A_\mu$ (or $\gamma^\mu(1-\gamma_5)A_\mu$) be conserved. Different from former, $\gamma^\mu(1+\gamma_5)A_\mu$ (or$\gamma^\mu(1-\gamma_5)A_\mu$) only runs under P_μ(K_μ) transformation. $\gamma^\mu\gamma^\nu A_\mu A_\nu$ is a scalar, having something to do with rotation, so it is similar to γ_5

in Table 1.

Table 1. The relationship between generators (transformations) and vertices.

	D	P_μ	K_μ	$M_{\mu\nu}$
$\gamma_5(\varphi)$	conserved	running	running	conserved
γ^μ	running	running	running	running
$\gamma^\mu A_\mu$	running	running	running	conserved
$\gamma^\mu\gamma_5$	running	running	running	running
$\gamma^\mu\gamma_5 A_\mu$	running	running	running	conserved
$\gamma^\mu(1+\gamma_5)$	running	running	conserved	running
$\gamma^\mu(1+\gamma_5)A_\mu$	conserved	running	conserved	conserved
$\gamma^\mu(1-\gamma_5)$	running	conserved	running	running
$\gamma^\mu(1-\gamma_5)A_\mu$	conserved	conserved	running	conserved
$\gamma^\mu\gamma^\nu$	conserved	running	running	running
$\gamma^\mu\gamma^\nu A_\mu A_\nu$	conserved	running	running	conserved

6 Conclusions and discussions

By investigating Cartan's matrices and the generators of the conformal group, we recognize that there must be some relationship between conformal group and Dirac algebra. We find the correspondence by comparing their commutators, and γ matrices are isomorphic to the algebra of conformal group. Using this representation from Cartan's matrices, we associate transformations (generators) with physical interaction vertex in quantum fields, just as linking Lorentz transformation with vector vertex rooted in Dirac equation. Then we find some of the generators of conformal group can lead to running of some kinds of vertex, just as shown in the above table. The running effect is very similar to the effect in renormalization, and in this paper we have amplified the running effect in renormalization of perturbative dynamics. We hope by extending such running effect we can step into the nonperturbative regime with new physics. As stated above, the tool we rely on is conformal group. In the historic research path, there were successive failures to find a conformal invariance dynamics. However this disadvantage of conformal group becomes an advantage for our physical objective. The conformal group may just exist for running (evolution with scales), but not for invariance. With the differential operator formed by these conformal group generators, we may construct equations like RGE to push some of our results from the perturbative regime to the nonperturbative regime, where the strong interaction or intermediate strong interaction works.

We know that from conformal field theory, the operator K_μ comes from inversion-translation-inversion transform [2, 24], as

$$x'_\mu=\frac{x_\mu}{x^2},\ x''_\mu=x'_\mu+c_\mu,\ \bar{x}_\mu=\frac{x''_\mu}{(x''_\mu)^2}, \tag{43}$$

it follows that

$$\bar{x}_\mu=\frac{x_\mu+c_\mu x^2}{1+2c\cdot x+c^2x^2}, \tag{44}$$

which leads directly to

$$\bar{x}_\mu\bar{x}^\mu=x^2/\lambda(x), \tag{45}$$

here $\lambda(x) = 1+2c\cdot x+c^2x^2$. It is clear that the infinitesimal transformation from x_μ to $\bar{x}_\mu$ is $K_\mu = -\mathrm{i}\left(x^2\frac{\partial}{\partial x^\mu}-2x_\mu x^\nu\frac{\partial}{\partial_{x^\nu}}\right)$ [24]. We do not need to get involved in the concrete transformation form, just by comparing Eqs. (43, 44, 45), especially Eq. (45), we can note that K_μ includes the part of dilatation transform as the effect of generator D. So by comparing with the renormalization group equation Eq. (1), i.e. $\mu\frac{\mathrm{d}\Lambda_\mathrm{R}}{\mathrm{d}\mu}+\gamma_\mathrm{F}\Lambda_\mathrm{R}=0$ (γ_F is the anomalous dimension), we may surmise a new evolving equation which may lead us to a nonperturbative region,

$$\begin{aligned}&(\not{K}-m\tilde{\gamma}_\mathrm{F})\tilde{\Lambda}_\mathrm{R}\\ &\quad=0,\ (\tilde{\gamma}_\mathrm{F}\text{ is a kind of anomalous dimension}),\end{aligned} \tag{46}$$

where $\not{K}=\gamma^\mu K_\mu$ (γ_μ is the Dirac matrices of common sense), the differential operators in K_μ become momentum dependent rather than coordinate dependent to be compatible with operator $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}$ in RGM equation, like $K_\mu\sim\frac{1}{2}p_\nu p^\nu\frac{\partial}{\partial p^\mu}-p_\mu p_\nu\frac{\partial}{\partial p_\nu}$. The m just represents a mass dimension, temporally we may regard it as a fermion mass, and we should compute the $\tilde{\gamma}_\mathrm{F}$ in a conventional manner, but with a different differential manner as

$$\tilde{\gamma}_\mathrm{F}\sim\left(\frac{1}{2}p_\nu p^\nu\frac{\partial}{\partial p^\mu}-p_\mu p_\nu\frac{\partial}{\partial p_\nu}\right)\ln Z_\mathrm{F},$$

not like

$$\gamma_\mathrm{F}\sim\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln Z_\mathrm{F}\ ,$$

where Z_F is the renormalization constant relating to quantity $\tilde{\Lambda}_\mathrm{R}$. This generalization differs from the conventional ones [25–28] in that the conventional ones emphasize too much the curvature of deformed space-time metric. This bold assumption may be reasonable since the non-Abelian strong interaction has been verified to become asymptotically free while energy increases. At the high energy limit it happens that the above equation can reduce back to a normal RGE since

$$\begin{aligned}\gamma^\mu K_\mu \quad &\sim\quad \gamma^\mu\left(\frac{1}{2}p_\nu p^\nu\frac{\partial}{\partial p^\mu}-p_\mu p_\nu\frac{\partial}{\partial p_\nu}\right)\\ &\overset{p^2\to\infty}{\sim}\ \left(\frac{1}{2}p^2\gamma^\mu\frac{\partial}{\partial p^\mu}-\not{p}p_\nu\frac{\partial}{\partial p_\nu}\right)\\ &\to\quad \frac{1}{2}m^2\,\not{\partial}-mp_\nu\frac{\partial}{\partial p_\nu}\to-m\mu\frac{\partial}{\partial\mu},\end{aligned} \tag{47}$$

where $\partial\!\!\!/=\gamma^{\mu}\dfrac{\partial}{\partial p^{\mu}}$ and the on-shell boundary condition $p\!\!\!/=p=m$ is applied, which may be proportional to renormalization point [14]. At ultra-high energy limit the mass constant m is viewed as a one order infinitesimal constant. Thus Eq. (46) can reduce to the familiar one $\mu\dfrac{\mathrm{d}\Lambda_{\mathrm{R}}}{\mathrm{d}\mu}+\gamma_{\mathrm{F}}\Lambda_{\mathrm{R}}=0$. We attempt to use the Eq. (46) to describe the opposite direction from asymptotic regime to nonperturbative regime. The next step on numerical details is in progress.

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