Minimum emittance in TBA and MBA lattices

XU Gang(徐刚) PENG Yue-Mei(彭月梅)
Key Laboratory of Particle Acceleration Physics and Technology, Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China

Abstract: For reaching a small emittance in a modern light source, triple bend achromats (TBA), theoretical minimum emittance (TME) and even multiple bend achromats (MBA) have been considered. This paper derived the necessary condition for achieving minimum emittance in TBA and MBA theoretically, where the bending angle of inner dipoles has a factor of $3^{1/3}$ bigger than that of the outer dipoles. Here, we also calculated the conditions attaining the minimum emittance of TBA related to phase advance in some special cases with a pure mathematics method. These results may give some directions on lattice design.

Key words: minimum emittance, dipole achromats, light source

PACS: 29.27.Bd, 41.75.-i, 29.20.dh **DOI:** 10.1088/1674-1137/39/3/037005

1 Introduction

In recent years, electron storage rings have been widely used as a synchrotron light source for research in atomic, molecular, condensed matter and solid state physics, chemistry, cell biology, etc. For many experiments, it is desirable to use a high brilliance light, which requires a small emittance of the beam.

Assuming negligible insertion-device contributions to the radiation integrals, the horizontal natural emittance of electron beams in a storage ring is given by [1]

$$\varepsilon_{x0}=C_{\mathrm{q}}\cdot\gamma^2\cdot\frac{\left\langle H/|\rho|^3\right\rangle_{\mathrm{dipole}}}{J_x\left\langle 1/\rho^2\right\rangle}=\frac{C_{\mathrm{q}}\cdot\gamma^2}{J_x}\cdot\frac{I_5}{I_2}, \tag{1}$$

here

$$C_{\mathrm{q}}=(55/32\sqrt{3})(\hbar/mc)=3.83\times10^{-13}\ \mathrm{m}$$

for electrons; γ is the Lorentz factor of beam energy; $J_x\approx1$ is the horizontal damping partition numbers, ρ is the dipole bending radius; the brackets $\langle\cdots\rangle$ indicate average over the bending magnets, and the dispersion action.

$$H=\gamma\eta^2+2\alpha\eta\eta'+\beta\eta'^2, \tag{2}$$

where β, α and γ are the Courant-Snyder parameters of horizontal betatron motion, η and η' are the dispersion and its derivative, respectively.

In the past, small emittance lattices are often double bend achromats (DBA), triple bend achromats (TBA), theoretical minimum emittance (TME), etc. Recently, a possibility of multiple bend achromats (MBA) has also been considered.

Well over a quarter of a century ago, it was realized that there is a theoretical minimum emittance achievable in electron storage rings [2]. Thorough analyses have been done which can serve as a guideline for realistic lattice design [3–5]. Since H is proportional to $L\theta^2$ and $L=\rho\theta$, (θ is the bending angle and L is the dipole length), the emittance obeys a scaling law

$$\varepsilon_{x0}=FC_{\mathrm{q}}\gamma^2\theta^3/J_x, \tag{3}$$

where the factor F depends on the lattice structure.

For a TME lattice using uniform field dipoles, there is no constraint on the optics parameters at both ends of the dipoles. The minimum emittance is

$$\varepsilon_{\mathrm{TME}}=C_{\mathrm{q}}\gamma^2\theta_{\mathrm{b}}^3/12\sqrt{15}J_x. \tag{4}$$

This is achieved when $\alpha_x=\eta_x'=0,\beta_x=L_{\mathrm{b}}/2\sqrt{15}$, and $\eta_x=L_{\mathrm{b}}\theta_{\mathrm{b}}/24$ at the center of the dipole.

For a DBA lattice using uniform field dipoles, η_x and η_x' is required zero at the entrance of the dipole. In this case, the minimum emittance is

$$\varepsilon_{\mathrm{DBA}}=C_{\mathrm{q}}\gamma^2\theta_{\mathrm{b}}^3/4\sqrt{15}J_x. \tag{5}$$

This is achieved when $\beta_x=6L_{\mathrm{b}}/\sqrt{15}$, and $\alpha_x=\sqrt{15}$ at the entrance of the dipole [6].

Some people generally believe that the theoretical minimum emittance in the TBA lattice is $\varepsilon_{\mathrm{TBA}}=(\varepsilon_{\mathrm{TME}}+2\varepsilon_{\mathrm{DBA}})/3$. In reality, the result is not like that.

This paper is organized as follows: In Section 2, we first list the basic formula which is used in the calculations. In Section 3, we calculate the minimum emittance in TBA and MBA. We will describe the methods used for this study, and then present our results in graphs in Section 4. In Section 5, we will give the results when the phase advance equals zero. We give a brief conclusion in Section 6.

Received May 2014

2 The basic formula

By minimizing I_5/I_2 in dipoles, a minimum emittance can be attained.

The transport matrix of an isomagnetic dipole with small angle approximation is

$$M_{\text{bend}}=\begin{pmatrix}1 & s & \frac{s^2}{2\rho}\\ 0 & 1 & \frac{s}{\rho}\\ 0 & 0 & 1\end{pmatrix}. \quad (6)$$

Giving the initial values β_0, α_0, γ_0, η_0 and η'_0 at the bending magnet entrance, the horizontal optics functions become formula (7)

$$\left.\begin{aligned}\beta(s,\alpha_0,\beta_0)&=\beta_0-2\alpha_0 s+\frac{1+\alpha_0^2}{\beta_0}s^2\\ \alpha(s,\alpha_0,\beta_0)&=\alpha_0-\frac{1+\alpha_0^2}{\beta_0}s\\ \gamma(s,\alpha_0,\beta_0)&=\gamma=\frac{1+\alpha_0^2}{\beta_0}\\ \eta(s,\eta_0,\eta'_0,\rho)&=\eta_0+\eta'_0 s+\frac{s^2}{2\rho}\\ \eta'(s,\eta_0,\eta'_0,\rho)&=\eta'_0+\frac{s}{\rho}\end{aligned}\right\}. \quad (7)$$

Therefore, the expressions of the radiation integrals I_5 in (1) is

$$\begin{aligned}I_5(\rho,L,\alpha_0,\beta_0,\eta_0,\eta'_0)\ =\ &\frac{L}{60\beta_0\rho^5}[(1+\alpha_0^2)(3L^4-20\eta_0L^2\rho+60\eta_0^2\rho^2)-5\alpha_0\beta_0(3L^3-12\eta_0L\rho+4\eta'_0L^2\rho\\ &-24\eta_0\eta'_0\rho^2)+20\beta_0^2(L^2+3\eta'_0L\rho+3\eta_0'^2\rho^2)]. \end{aligned} \quad (8)$$

The transfer matrix from s_0 to s_1 in beam line is

$$M(s_1/s_0)=\begin{pmatrix}\sqrt{\frac{\beta_1}{\beta_0}}(\cos\mu+\alpha_0\sin\mu) & \sqrt{\beta_1\beta_0}\sin\mu\\ -\frac{(-\alpha_0+\alpha_1)\cos\mu+(1+\alpha_0\alpha_1)\sin\mu}{\sqrt{\beta_1\beta_0}} & \sqrt{\frac{\beta_0}{\beta_1}}(\cos\mu+\alpha_1\sin\mu)\end{pmatrix}, \quad (9)$$

where β_1, α_1 and β_0, α_0 are values of betatron amplitude functions at s_1 and s_0, respectively, μ is the phase advance from s_0 to s_1

3 The theoretical minimum emittance in TBA and MBA

Assume that the TBA and MBA consists of two types of dipoles. One type is the first dipole and the end dipole, where the length is L_1, the bending angle is θ_1 and the radius is ρ_1. The other is the inner dipole (s), where the length is L_2, the bending angle is θ_2 and the radius is ρ_2. Fig. 1 shows the twiss functions in the entrance and end of dipoles in TBA.

To ensure every dipole can reach the achievable minimum emittance, the Twiss function at the first dipole entrance satisfied $\beta_1=6L_1/\sqrt{15}$, $\alpha_1=\sqrt{15}$ and the Twiss function at the middle of the inner dipole satisfied

$$\alpha_4=\eta'_4=0,\ \beta_4=L_2/2\sqrt{15},$$
$$\eta_4=L_2^2/(24\rho_2).$$

Here, we can get

$$\beta_2\ =\ \beta(L_1,\alpha_1,\beta_1)=\frac{16L_1}{\sqrt{15}}, \quad (10)$$

$$\alpha_2\ =\ \alpha(L_1,\alpha_1,\beta_1)=-5\sqrt{\frac{5}{3}}, \quad (11)$$

$$\alpha_3\ =\ -\alpha(L_2/2,\eta_4,0,\rho_2)=\sqrt{15}, \quad (12)$$

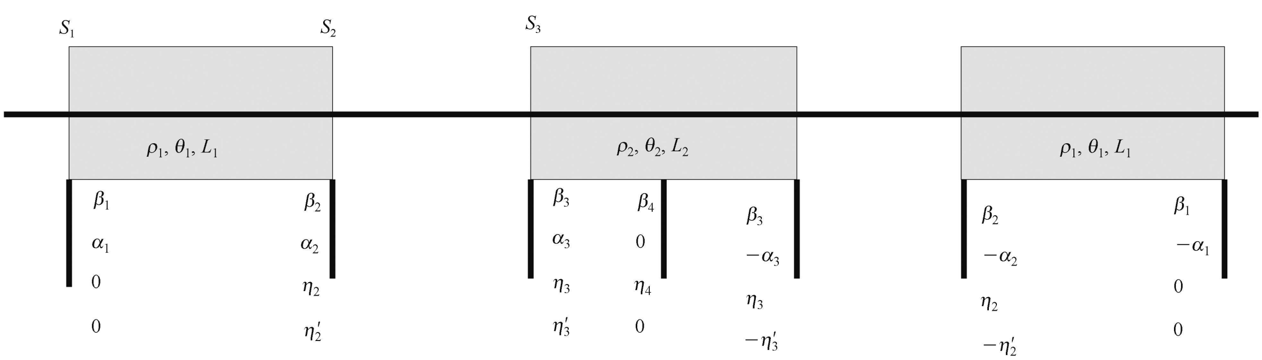


Fig. 1. (color online) Twiss functions in the entrance and end of dipoles in TBA.

$$\beta_3 = \frac{L_2(1+\alpha_3^2)}{2\alpha_3}. \tag{13}$$

$$\begin{pmatrix}\eta_2\\ \eta_2'\end{pmatrix} = \begin{pmatrix}\eta(L_1,0,0,\rho_1)\\ \eta'(L_1,0,0,\rho_1)\end{pmatrix}, \tag{14}$$

$$\begin{pmatrix}\eta_3\\ \eta_3'\end{pmatrix} = \begin{pmatrix}\eta(L_2/2,\eta_4,0,\rho_2)\\ -\eta'(L_2/2,\eta_4,0,\rho_2)\end{pmatrix}. \tag{15}$$

The phase advance from s_2 to s_3 is μ

$$\begin{pmatrix}\eta_3\\ \eta_3'\end{pmatrix} = M(s_3/s_2)\begin{pmatrix}\eta_2\\ \eta_2'\end{pmatrix}, \tag{16}$$

where $M(s_3/s_2)$ is the transfer matrix (9).

Solving formula (16), we obtain

$$\mu=\arccos\left(-\sqrt{\frac{3}{8}}\right), \tag{17}$$

$$\rho_1=\frac{\sqrt{3}L_1^{3/2}}{L_2^{3/2}}\rho_2. \tag{18}$$

Let $\theta_2=r\theta_1$, from (18), we get

$$\rho_2=\frac{3}{r^3}\rho_1. \tag{19}$$

If the ring has n TBA cells, we know

$$2\theta_1+\theta_2=\frac{2\pi}{n}. \tag{20}$$

The radiation integral of a TBA is

$$\begin{aligned} I_5 &= 2I_5(\rho_1,L_1,\alpha_1,\beta_1,0,0)\\ &\quad +2I_5(\rho_2,L_2/2,0,\beta_4,\eta_4,0), \end{aligned} \tag{21}$$

$$I_2 = 2L_1/\rho_1^2+L_2/\rho_2^2, \tag{22}$$

we obtain

$$\frac{I_5}{I_2}=\frac{2\pi^3(18+r^7)}{3\sqrt{15}n^3(2+r)^3(6+r^4)}. \tag{23}$$

The minimum emittance is then obtained by imposing the condition

$$\partial_r(I_5/I_2)=\frac{4\pi^3(-3+r^3)(18+14r^3+7r^4+r^7)}{15n^3(2+r)^4(6+r^4)^2}=0. \tag{24}$$

Solving (24), we get

$$r=3^{1/3}. \tag{25}$$

So

$$\left(\frac{I_5}{I_2}\right)_{\mathrm{TBA\,min}}=\frac{1}{4\sqrt{15}(2+3^{1/3})^3}\left(\frac{2\pi}{n}\right)^3. \tag{26}$$

For an MBA ring, the number of MBA cells in the storage ring is n

$$2\theta_1+(M-2)\theta_2=\frac{2\pi}{n}, \tag{27}$$

$$\begin{aligned} I_5 &= 2\times I_5(\rho_1,L_1,\alpha_1,\beta_1,0,0)+(M-2)\\ &\quad \times 2\times I_5(\rho_2,L_2/2,0,\beta_4,\eta_4,0), \end{aligned} \tag{28}$$

$$I_2 = 2L_1/\rho_1^2+(M-2)L_2/\rho_2^2. \tag{29}$$

Using the same process of that in TBA, we get the condition that the minimum emittance of MBA which can be achieved is also $r=3^{1/3}$.

So

$$\left(\frac{I_5}{I_2}\right)_{\mathrm{MBA\,min}}=\frac{1}{4\sqrt{15}(2-2\times3^{1/3}+M3^{1/3})^3}\left(\frac{2\pi}{n}\right)^3. \tag{30}$$

From these calculations, we find that the necessary condition for reaching the minimum emittance of TBA and MBA is the bending angle of inner dipoles which has a factor of $3^{1/3}$ bigger than that of the outer dipoles.

4 The attained minimum emittance in TBA with various conditions

Let the total bending angle of TBA be θ, $\beta_2=\beta_\mathrm{r}L_1$, $\rho_2=\rho_\mathrm{r}\rho_1$, $\theta_1=\theta_\mathrm{r}\theta$, then $\theta_2=(1-2\theta_\mathrm{r})\theta$.

From Fig. 1, we can get the relations:

$$\beta_1=\beta(L_1,-\alpha_2,\beta_2)=\beta_2+2\alpha_2\theta\theta_\mathrm{r}\rho_1+\frac{(1+\alpha_2^2)\theta^2\theta_\mathrm{r}^2\rho_1^2}{\beta_2}, \tag{31}$$

$$\alpha_1=-\alpha(L_1,-\alpha_2,\beta_2)=\alpha_2+\frac{(1+\alpha_2^2)\theta\theta_\mathrm{r}\rho_1}{\beta_2}, \tag{32}$$

$$\beta_3=\frac{L_2(1+\alpha_3^2)}{2\alpha_3}, \tag{33}$$

$$\beta_4=\beta(L_2/2,\alpha_3,\beta_3)=\frac{\theta(1-2\theta_\mathrm{r})\rho_2}{2\alpha_3}, \tag{34}$$

$$\begin{pmatrix}\eta_2\\ \eta_2'\end{pmatrix}=\begin{pmatrix}\eta(L_1,0,0,\rho_1)\\ \eta'(L_1,0,0,\rho_1)\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\theta^2\theta_\mathrm{r}^2\rho_1\\ \theta\theta_\mathrm{r}\end{pmatrix}, \tag{35}$$

$$\begin{aligned}\begin{pmatrix}\eta_3\\ \eta_3'\end{pmatrix} &= \begin{pmatrix}\eta(L_2/2,\eta_4,0,\rho_2)\\ -\eta'(L_2/2,\eta_4,0,\rho_2)\end{pmatrix}\\ &= \begin{pmatrix}\eta_4+\frac{1}{8}\theta^2(1-2\theta_\mathrm{r})^2\rho_1\rho_\mathrm{r}^2\\ -\frac{1}{2}\theta(1-2\theta_\mathrm{r})\end{pmatrix}.\end{aligned} \tag{36}$$

The phase advance from s_2 to s_3 is μ

$$\begin{pmatrix}\eta_3\\ \eta_3'\end{pmatrix}=M(s_3/s_2)\begin{pmatrix}\eta_2\\ \eta_2'\end{pmatrix}, \tag{37}$$

where $M(s_3/s_2)$ is the transfer matrix (9).

Solving formula (37), we obtain

$$\rho_{\mathrm{r}}=\frac{\sqrt{2\alpha_3}\theta_{\mathrm{r}}^2[(\alpha_3-2\beta_{\mathrm{r}}-\alpha_2)\cos\mu+\sin\mu+\alpha_3(2\beta_{\mathrm{r}}+\alpha_2)\sin\mu]}{\sqrt{1+\alpha_3^2}\sqrt{\beta_{\mathrm{r}}\theta_{\mathrm{r}}}(1-2\theta_{\mathrm{r}})^{3/2}}, \tag{38}$$

$$\begin{aligned}\eta_4 = {} & \theta^2\theta_{\mathrm{r}}^3\rho_1\{-(\alpha_3+\alpha_3^3)[1+(\alpha_2+2\beta_{\mathrm{r}})^2]+\{\alpha_2^2\alpha_3(3+\alpha_3^2)+8\beta_{\mathrm{r}}\\ & +4\alpha_2[1+\alpha_3(3+\alpha_3^2)\beta_{\mathrm{r}}]+\alpha_3(3+\alpha_3^2)(4\beta_{\mathrm{r}}^2-1)\}\cos2\mu\\ & +2\{(\alpha_2+2\beta_{\mathrm{r}})[\alpha_2-\alpha_3(3+\alpha_3^2)+2\beta_{\mathrm{r}}]-1\}\sin2\mu\}/[8(1+\alpha_3^2)\beta(2\theta_{\mathrm{r}}-1)].\end{aligned} \tag{39}$$

Then,

$$L_1/L_2=\frac{(1+\alpha_3^2)\beta_{\mathrm{r}}(1-2\theta_{\mathrm{r}})^2}{2\alpha_3\theta_{\mathrm{r}}^2[(\alpha_3-\alpha_2-2\beta_{\mathrm{r}})\cos\mu+(1+\alpha_2\alpha_3+2\alpha_3\beta_{\mathrm{r}})\sin\mu]^2}. \tag{40}$$

The radiation integral of a TBA is

$$I_5=2\times I_5(\rho_1,L_1,\alpha_1,\beta_1,0,0)+2\times I_5(\rho_2,L_2/2,0,\beta_4,\eta_4,0) \tag{41}$$

$$I_2=2L_1/\rho_1^2+L_2/\rho_2^2, \tag{42}$$

The emittance factor F is

$$\begin{aligned}F(\alpha_2,\alpha_3,\theta_{\mathrm{r}},\beta_{\mathrm{r}},\mu) = {} & \frac{I_5}{I_2}/\theta^3=\{8(8+8\alpha_2^2+25\alpha_2\beta_{\mathrm{r}}+20\beta_{\mathrm{r}}^2)\theta_{\mathrm{r}}^7+\{(1+\alpha_3^2)\beta_{\mathrm{r}}^2(2\theta_{\mathrm{r}}-1)^7\{-(1+\alpha_3^2)(10+\alpha_3^2)[1+(\alpha_2+2\beta_{\mathrm{r}})^2]\\ & +[-5-\alpha_3^2(4+\alpha_3^2)+\alpha_2^2(5+4\alpha_3^2+\alpha_3^4)-4\alpha_3(5+3\alpha_3^2)\beta_{\mathrm{r}}+4(5+4\alpha_3^2+\alpha_3^4)\beta_{\mathrm{r}}^2\\ & +2\alpha_2(-5\alpha_3-3\alpha_3^3+2(5+4\alpha_3^2+\alpha_3^4)\beta_{\mathrm{r}}]\}\}\cos2\mu-\{\alpha_2^2\alpha_3(5+3\alpha_3^2)+20\beta_{\mathrm{r}}\\ & +\alpha_3[-5-3\alpha_3^2+4\alpha_3(4+\alpha_3^2)\beta_{\mathrm{r}}+4(5+3\alpha_3^2)\beta_{\mathrm{r}}^2]+2\alpha_2\{5+\alpha_3[10\beta_{\mathrm{r}}+\alpha_3(4+\alpha_3^2+6\alpha_3\beta_{\mathrm{r}})]\}\}\sin2\mu\}\}\\ & /\{\alpha_3^2[(\alpha_3-\alpha_2-2\beta_{\mathrm{r}})\cos\mu+\sin\mu+\alpha_3(\alpha_2+2\beta_{\mathrm{r}})\sin\mu]^4\}/\{120\beta_{\mathrm{r}}[4\theta_{\mathrm{r}}^4\\ & +(1+\alpha_3^2)\beta_{\mathrm{r}}(1-2\theta_{\mathrm{r}})^4/\{\alpha_3[(\alpha_3-\alpha_2-2\beta_{\mathrm{r}})\cos\mu+\sin\mu+\alpha_3(\alpha_2+2\beta_{\mathrm{r}})\sin\mu]^2\}]\}.\end{aligned} \tag{43}$$

Searching the minimum emittance using the order "Find Minimum" of mathematics [7], we get

$$F=0.00158259=\frac{1}{4\sqrt{15}(2+3^{1/3})^3}, \tag{44}$$

when

$$\begin{aligned}\alpha_2 &= -6.45497=-5\sqrt{\frac{5}{3}},\\ \alpha_3 &= 3.87298=\sqrt{15},\\ \theta_{\mathrm{r}} &= 0.290508=1/(2+3^{1/3}),\\ \beta_{\mathrm{r}} &= 4.13118=\frac{16}{\sqrt{15}},\\ \mu &= 2.22985=\arccos\left(-\sqrt{\frac{3}{8}}\right).\end{aligned}$$

This result fits very well with the result in Section 3.

With a different phase advance μ, we get the emittance curve, which is shown in Fig. 2.

Fig. 2 indicates the relationship between $F_{\mathrm{TBA}}/\{1/[4\sqrt{15}(2+3^{1/3})^3]\}$ and phase advance μ, from this figure; we can get the attaining minimum emittance with a different phase advance. There are sharps in $\mu=0$ since we calculate the case $\mu=0$ separately.

We discuss a few special cases as follows; table 1 gives the minimum emittance and Fig. 3 indicates the comparison between the attained minimum emittance with a different phase advance.

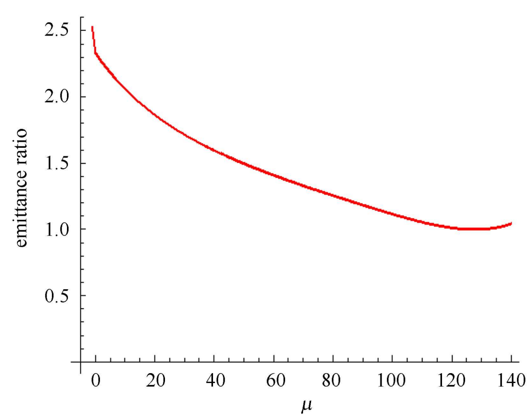


Fig. 2. (color online) Conditions of reaching minimum emittance with different μ.

Table 1. The minimum emittance in some concrete cases.

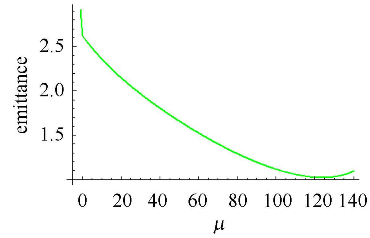
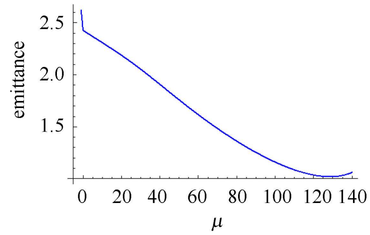
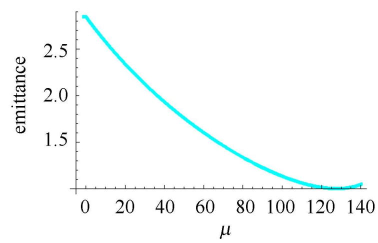
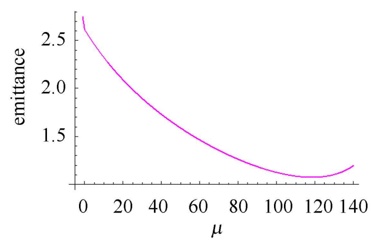
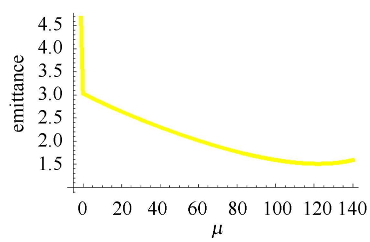
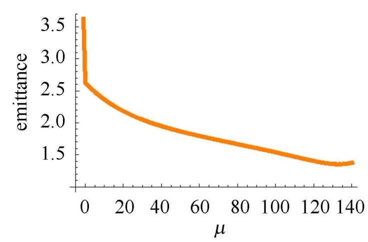
restriction	the attain minimum emittance F_{TBA}	conditions	attaining minimum emittance with different phase advance $F_{\mathrm{TBA}}/\{1/[4\sqrt{15}(2+3^{1/3})^3]\}$
$L_2=L_1$	0.00162033	$\alpha_2=-5.99511,\ \alpha_3=4.21945,$ $\beta_{\mathrm{r}}=3.69409,\ \mu=2.16096,$ $\theta_{\mathrm{r}}=0.290395$	
$L_1=L_2/2$	0.00161569	$\alpha_2=-7.05659,\alpha_3=3.65581,$ $\beta_{\mathrm{r}}=4.61861,\mu=2.24161,$ $\theta_{\mathrm{r}}=0.291167$	
$\rho_1=\rho_2$	0.00158259	$\alpha_2=-6.45497,\ \alpha_3=3.87298,$ $\beta_{\mathrm{r}}=4.13118,\mu=2.22985,$ $\theta_{\mathrm{r}}=0.290508$	
$\rho_1=2\rho_2$	0.00180386	$\alpha_2=-6.49114,\alpha_3=3.87305,$ $\beta_{\mathrm{r}}=4.06587,\mu=2.25833,$ $\theta_{\mathrm{r}}=0.275443$	
$L_1=L_2,\ \rho_1=\rho_2$	0.00208453	$\alpha_2=-6.01503,\alpha_3=9.77783,$ $\beta_{\mathrm{r}}=3.71767,\mu=2.37634,$ $\theta_{\mathrm{r}}=1/3$	
$L_1=L_2,\ \rho_1=2\rho_2$	0.00214232	$\alpha_2=-7.39219,\alpha_3=3.82692,$ $\beta_{\mathrm{r}}=4.87853,\mu=2.32599,$ $\theta_{\mathrm{r}}=1/4$	

Table 1. (continued)

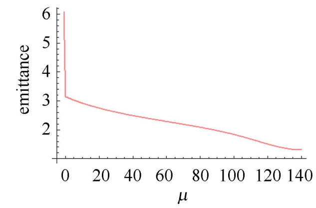
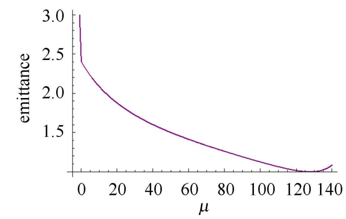
restriction	the attain minimum emittance F_{TBA}	conditions	attaining minimum emittance with different phase advance $F_{\mathrm{TBA}}/\{1/[4\sqrt{15}(2+3^{1/3})^3]\}$
$L_1=L_2/2,\ \rho_1=\rho_2$	0.00207506	$\alpha_2=-11.4593,\ \alpha_3=3.65911,$ $\beta_{\mathrm{r}}=7.85698,\ \mu=2.44258,$ $\theta_{\mathrm{r}}=1/4$	
$\alpha_3=\sqrt{15}$	0.00158259	$\alpha_2=-6.45497,\ \alpha_3=3.87298,$ $\beta_{\mathrm{r}}=4.13118,\ \mu=2.22985,$ $\theta_{\mathrm{r}}=0.290508$	

Table 2. Minimum emittance when $\mu=0$.

restriction	the attain minimum emittance F	conditions
none length and radius restriction	0.00399202	$\alpha_2=-5.67335,\ \alpha_3=3.61097,$ $\beta_{\mathrm{r}}=3.44141,\ \theta_{\mathrm{r}}=0.37067$
$L_2=L_1$	0.00460814	$\alpha_2=-5.24004,\ \alpha_3=3.32108,$ $\beta_{\mathrm{r}}=1.81109,\ \theta_{\mathrm{r}}=0.211549$
$L_1=L_2/2$	0.0041498	$\alpha_2=-6.6176,\ \alpha_3=3.77813,$ $\beta_{\mathrm{r}}=4.04281,\ \theta_{\mathrm{r}}=0.388892$
$\rho_1=\rho_2$	0.00450154	$\alpha_2=-4.54775,\ \alpha_3=2.65895,$ $\beta_{\mathrm{r}}=2.12105,\ \theta_{\mathrm{r}}=0.294316$
$\rho_1=2\rho_2$	0.0043502	$\alpha_2=-5.3333,\ \alpha_3=3.06262,$ $\beta_{\mathrm{r}}=1.95122,\ \theta_{\mathrm{r}}=0.232401$
$L_1=L_2$ $\rho_1=\rho_2$	0.00738164	$\alpha_2=-1.92336,\ \alpha_3=1.08711,$ $\beta_{\mathrm{r}}=1.00349,\ \theta_{\mathrm{r}}=1/3$
$L_1=L_2$ $\rho_1=2\rho_2$	0.00572654	$\alpha_2=-2.97106,\ \alpha_3=1.94029,$ $\beta_{\mathrm{r}}=1.22784,\ \theta_{\mathrm{r}}=1/4$
$L_1=L_2/2$ $\rho_1=\rho_2$	0.00960369	$\alpha_2=-7.70103,\ \alpha_3=1.84391,$ $\beta_{\mathrm{r}}=2.38624,\ \theta_{\mathrm{r}}=1/4$
$\alpha_3=\sqrt{15}$	0.00474258	$\alpha_2=-6.98036,\ \beta_{\mathrm{r}}=2.32828,$ $\theta_{\mathrm{r}}=0.214522$

5 The case of $\mu=0$

$$\begin{aligned}
&F_{\mu=0}(\alpha_2,\alpha_3,\theta_{\mathrm{r}},\beta_{\mathrm{r}})\\
=&\ \{8(8+8\alpha_2^2+25\alpha_2\beta_{\mathrm{r}}+20\beta_{\mathrm{r}}^2)\theta_{\mathrm{r}}^7\\
&-\{(1+\alpha_3^2)\beta_{\mathrm{r}}^2[2\alpha_3^4+\alpha_2^2(5+7\alpha_3^2)+20\alpha_3\beta_{\mathrm{r}}\\
&+12\alpha_3^2\beta_{\mathrm{r}}+2\alpha_2(5\alpha_3+3\alpha_3^3+10\beta_{\mathrm{r}}+14\alpha_3^2\beta_{\mathrm{r}})\\
&+5(3+4\beta_{\mathrm{r}}^2)+\alpha_3^2(15+28\beta_{\mathrm{r}}^2)](2\theta_{\mathrm{r}}-1)^7\}\\
&/\{\alpha_3^2(\alpha_3-\alpha_2-2\beta_{\mathrm{r}})^4\}\}/\{120\beta_{\mathrm{r}}[4\theta_{\mathrm{r}}^4+(1+\alpha_3^2)\\
&\times\beta_{\mathrm{r}}(1-2\theta_{\mathrm{r}})^4/\{\alpha_3(\alpha_3-\alpha_2-2\beta_{\mathrm{r}})^2\}]\}. \qquad (45)
\end{aligned}$$

Searching the minimum emittance directly with (45), we find when reaching the minimum emittance, $\alpha_3\to\infty$. This solution is absolutely not reasonable. Because, when $\mu=0$, the transfer matrix (9) transfers from s_2 to s_3 becomes

$$M(s_3/s_2)=\begin{pmatrix}\sqrt{\dfrac{\beta_3}{\beta_2}} & 0\\ -\dfrac{(-\alpha_2+\alpha_3)}{\sqrt{\beta_2\beta_3}} & \sqrt{\dfrac{\beta_2}{\beta_3}}\end{pmatrix}. \qquad (46)$$

The trace of (46) satisfies the transfer stability condition only if $\sqrt{\beta_2/\beta_3}=1$, we need add a restriction (47) in calculation.

$$\beta_2=\beta_3. \qquad (47)$$

The results of attaining the minimum emittance when $\mu=0$ are listed in Table 2.

6 Conclusions

This paper derived the minimum emittance in TBA and MBA. The necessary condition for reaching the minimum emittance of TBA and MBA is that the bending angle of inner dipoles has a factor of $3^{1/3}$ bigger than that of the outer dipoles. We gave the TBA achieving minimum emittance in some special cases with restrictions on dipole length and radius; these results may be of benefit in TBA lattice design. In this paper, we do not discuss the emittance reduction using insertion devices or special dipoles.

References

1 Sands M. The Physics of Electron Storage Ring, SLAC-121, 1970
2 Sommer M. Internal Report DCI/NI/20/81, 1981
3 Teng L. FNAL Report No. FNAL/TM-1269, 1984; ANL Report No. LS-17, 1985
4 Trbojevic D, Courant E. Proceedings of the 4^{th} EPAC. London, 1994, 1000
5 Lee S Y. Phys. Rev. E, 1996, **54**: 1940
6 Lee S Y. Accelerator Physics, World Scientific, 1999. 477
7 http://www.wolfram.com