Fast computation of observed cross section for $\psi' \to$ PP decays*

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Abstract: It has been conjectured that the relative phase between strong and electromagnetic amplitudes is universally -90° in charmonium decays. ψ' decaying into a pseudoscalar pair provides a possibility to test this conjecture. However, the experimentally observed cross section for such a process is depicted by the two-fold integral, which takes into account the initial state radiative (ISR) correction and energy spread effect. Using the generalized linear regression approach, a complex energy-dependent factor is approximated by a linear function of energy. Taking advantage of this simplification, the integration of ISR correction can be performed and an analytical expression with accuracy at the level of 1% is obtained. Then, the original two-fold integral is simplified into a one-fold integral, which reduces the total computing time by two orders of magnitude. Such a simplified expression for the observed cross section usually plays an indispensable role in the optimization of scan data taking, the determination of systematic uncertainty, and the analysis of data correlation.

Key words: cross section, narrow resonance, pseudoscalar pair, e^+e^- collider

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1 Introduction

The relative phase between the strong and the electromagnetic amplitudes of the charmonium decays is a basic parameter in understanding decay dynamics. Studies have been carried out for many J/ψ two-body decay modes: 1^-0^- [1, 2], 0^-0^- [3–5], 1^-1^- [5] and $N\overline{N}$ [6]. These analyses reveal that there exists a relative orthogonal phase between the strong and the electromagnetic amplitudes in J/ψ decays [1–7]. As to ψ', there is also a theoretical argument that favors the $\pm 90^\circ$ phase [8]. Experimentally, some analyses [9–11] based on limited 1^-0^- and 0^-0^- data indicate that the large phase is compatible with the data. Moreover, some efforts have been made to extend the phase study to ψ'' decay phenomenologically [12, 13] and experimentally [14].

The great merit of the phase study lies in the fact that it can provide a valuable clue for the relation between the strong and the electromagnetic interactions. Now with the upgraded accelerator and detector, BEPC Ⅱ/BESⅢ, on May 2009, the high luminosity of 3×10^{32} cm^{-2}·s^{-1} had achieved, which is the highest luminosity in τ-charm energy region that ever existed. The 106 M ψ' and 226 M J/ψ events have been collected, even more colossal data are to be collected in the coming years, which gives a great opportunity to determine the phase between the strong and the electromagnetic amplitudes with unprecedented statistical precision.

A favorable way to measure the phase is through the scan experiment, which is the most model-independent approach. However, even with a high luminosity accelerator, the exclusive scan experiment of charmonium decay is fairly difficult due to low statistics at each energy point. Therefore, the optimization study for the data taking strategy is of great importance in order to obtain the most accurate results with limited luminosity (equivalently within the limited data taking time).

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Without losing generality, we focus on the mode of ψ' decays to two pseudoscalars. Because, as will be shown in the next section, this decay mode can accommodate a comparatively simple parametrization form, which is of great benefit to extract the relative phase. To get the optimized data taking scheme, we resort to the sampling simulation technique, which is successfully used in the study of the data taking strategy for a high precision τ mass measurement [15, 16]. For such a kind of method, many fits should be carried out, where a large number of calculations need to be performed to get the theoretically expected observed cross section. Unfortunately, two nested integrations in this calculation take too long to make the actual optimization procedure practical.

This paper aims to simplify calculation of the observed cross section of ψ' decaying to a pseudoscalar pair. Some reasonable assumptions lead us to obtain the analytic expression for the Initial State Radiative (ISR) corrected cross section. That is to say, we transform the two-fold integral into a one-fold integral, which speeds up the calculation by one hundred times.

2 Observed cross section

The process of ψ' decays to Pseudoscalar and Pseudoscalar (PP) final state can be parameterized by merely two amplitudes [5, 17], that is

$$\begin{aligned} A_{\pi^+\pi^-} &= A_{\rm EM}, \\ A_{\rm K^+K^-} &= A_{\rm EM}+A_{\rm S}, \\ A_{\rm K^0_S K^0_L} &= A_{\rm S}, \end{aligned} \tag{1}$$

where $A_{\rm EM}$ denotes the electromagnetic amplitude and $A_{\rm S}$ the $SU(3)$ breaking strong amplitude. Here, the G-parity violating channel $\pi^+\pi^-$ is through the electromagnetic process (the contribution from the isospin-violating part of QCD is expected to be small [18] and is neglected), $\rm K^0_S K^0_L$ through the $SU(3)$ breaking strong process, and $\rm K^+K^-$ through both. For $\rm e^+e^-$ experiment, the actual amplitudes must include the contribution of continuum, which features the electromagnetic process [9, 10, 19],

$$\begin{aligned} A_{\pi^+\pi^-} &= A^{\rm c}_{\rm EM}+A_{\rm EM}, \\ A_{\rm K^+K^-} &= A^{\rm c}_{\rm EM}+A_{\rm EM}+A_{\rm S}, \\ A_{\rm K^0_S K^0_L} &= A_{\rm S}, \end{aligned} \tag{2}$$

where $A^{\rm c}_{\rm EM}$ is the amplitude of the continuum contribution. In addition to the common part, $A^{\rm c}_{\rm EM}$, $A_{\rm EM}$ and $A_{\rm S}$ can be expressed explicitly as

$$\begin{aligned} A^{\rm c}_{\rm EM} &\propto \frac{1}{s}, \\ A_{\rm EM} &\propto \frac{1}{s}B(s), \\ A_{\rm S} &\propto \mathcal{C}{\rm e}^{{\rm i}\phi}\cdot\frac{1}{s}B(s), \end{aligned} \tag{3}$$

where the real parameters ϕ and $\mathcal{C}$ are the relative phase and the relative strength between the strong and the electromagnetic amplitudes, and $B(s)$ is defined as [9]

$$B(s)=\frac{3\sqrt{s}\Gamma_{\rm ee}/\alpha}{s-M^2_{\psi'}+{\rm i}M_{\psi'}\Gamma_{\rm t}}. \tag{4}$$

Here, $\sqrt{s}$ is the center of mass energy, α is the QED fine structure constant; $M_{\psi'}$ and $\Gamma_{\rm t}$ are the mass and the total width of ψ'; $\Gamma_{\rm ee}$ is the partial width to $\rm e^+e^-$.

The Born order cross sections for the three channels read

$$\begin{aligned} \sigma^{\pi^+\pi^-}_{\rm Born}(s)=& \frac{4\pi\alpha^2}{s^{3/2}}\left[1+2\Re B(s)+|B(s)|^2\right] \\ &\times|\mathcal{F}_{\pi^+\pi^-}(s)|^2\mathcal{P}_{\pi^+\pi^-}(s), \end{aligned} \tag{5}$$

$$\begin{aligned} \sigma^{\rm K^+K^-}_{\rm Born}(s)=& \frac{4\pi\alpha^2}{s^{3/2}}\left[1+2\Re(C_\phi B(s))+|C_\phi B(s)|^2\right] \\ &\times|\mathcal{F}_{\rm K^+K^-}(s)|^2\mathcal{P}_{\rm K^+K^-}(s), \end{aligned} \tag{6}$$

$$\sigma^{\rm K^0_S K^0_L}_{\rm Born}(s)=\frac{4\pi\alpha^2}{s^{3/2}}\mathcal{C}^2|B(s)|^2|\mathcal{F}_{\rm K^0_S K^0_L}(s)|^2\mathcal{P}_{\rm K^0_S K^0_L}(s), \tag{7}$$

where $C_\phi=1+\mathcal{C}{\rm e}^{{\rm i}\phi}$; $\mathcal{F}_{\rm f.s.}(s)=f_{\rm f.s.}/s$, with $f_{\rm f.s.}$ being an energy independent constant, and f.s. $=\pi^+\pi^-,\rm K^+K^-,K^0_S K^0_L$; $\mathcal{P}_{\rm f.s.}(s)=2q^3_{\rm f.s.}/3s$, with $q^2_{\rm f.s.}=E^2_{\rm f.s.}-m^2_{\rm f.s.}=s/4-m^2_{\rm f.s.}$.

It is obvious that in Eq. (6), if $C_\phi=1$, $\sigma^{\rm Born}_{\rm K^+K^-}(s)$ is identical to $\sigma^{\rm Born}_{\pi^+\pi^-}(s)$ while if $C_\phi=\mathcal{C}{\rm e}^{{\rm i}\phi}$, $\sigma^{\rm Born}_{\rm K^+K^-}(s)$ is identical to $\sigma^{\rm Born}_{\rm K^0_S K^0_L}(s)$. From a mathematical point of view, the cross section expression of $\sigma^{\rm Born}_{\rm K^+K^-}(s)$ is more general with the expressions of $\sigma^{\rm Born}_{\pi^+\pi^-}(s)$ and $\sigma^{\rm Born}_{\rm K^0_S K^0_L}(s)$ as its special cases. Therefore, in the following study, only tackled is the formula for $\rm K^+K^-$ final state and f.s. is simply denoted as K.

In $\rm e^+e^-$ collision, the Born order cross section is modified by the ISR in the way [20]

$$\sigma_{\rm r.c.}(s)=\int_0^{X_{\rm f}}{\rm d}xF(x,s)\frac{\sigma_{\rm Born}(s(1-x))}{|1-\Pi(s(1-x))|^2}, \tag{8}$$

where $X_{\rm f}=1-s'/s$. $F(x,s)$ has been calculated to an accuracy of 0.1% [20–22] and $\Pi(s)$ is the vacuum polarization factor. In the upper limit of the integration, $\sqrt{s'}$ is the experimentally required minimum invariant mass of the final particles. In this work, $X_{\rm f}=0.15$

is used, which corresponds to invariant mass cut of 3.4 GeV/c^2.

By convention, Γ_{ee} has the QED vacuum polarization in its definition [23, 24]. Here, it is natural to extend this convention to the partial widths of other pure electromagnetic decays, that is

$$\Gamma_K = 2\tilde{\Gamma}_{ee}\left(\frac{q_K}{M_{\psi'}}\right)^3 \left|\mathcal{F}(M^2_{\psi'})\right|^2, \tag{9}$$

where

$$\tilde{\Gamma}_{ee} \equiv \frac{\Gamma_{ee}}{|1-\Pi(m^2_{\psi'})|^2},$$

with the vacuum polarization effect included.

The e^+e^- colliders have finite energy resolution, which is much wider than the intrinsic width of ψ'. Such energy resolution is usually a Gaussian distribution [25, 26],

$$G(W,W') = \frac{1}{\sqrt{2\pi}\Delta} e^{-\frac{(W-W')^2}{2\Delta^2}},$$

where $W = \sqrt{s}$ and Δ, a function of the energy, is the standard deviation of the Gaussian distribution. The experimentally observed cross section is the radiative corrected cross section folded with the energy resolution function,

$$\sigma_{obs}(W) = \int_0^\infty dW' \sigma_{r.c.}(W') G(W',W). \tag{10}$$

For briefness, the variables $\tilde{\Gamma}_{ee}$, $M_{\psi'}$, and Γ_t are respectively written as Γ_{ee}, M, and Γ hereafter.

3 Simplification of ISR correction

In this section, we focus on the simplification of ISR correction of the observed cross section. In the energy region concerned (3.67– 3.71 GeV), the vacuum polarization factor could be concerned as constant and absorbed into $\tilde{\Gamma}_{ee}$, as in Eq. (9). So we could begin with this expression,

$$\sigma_{r.c.}(s) = \int_0^{X_f} dx F(x,s) \sigma_{Born}(s(1-x)) \ , \tag{11}$$

where $F(x,s)$ is the structure function, which can be expressed as

$$F(x,s) = x^{t-1}\cdot B_1(t) + x^t \cdot B_2(t) + x^{t+1}\cdot B_3(t) + O(x^{t+1}t^2), \tag{12}$$

where

$$B_1(t) = t\cdot\left[1+\frac{\alpha}{\pi}\left(\frac{\pi^2}{3}-\frac{1}{2}\right)+\frac{3}{4}t+t^2\left(\frac{9}{32}-\frac{\pi^2}{12}\right)\right],$$

$$B_2(t) = -t-\frac{t^2}{4},$$

$$B_3(t) = \frac{t}{2}-\frac{3}{8}t^2, \tag{13}$$

with

$$t = \frac{2\alpha}{\pi}\left(\ln\frac{s}{m_e^2}-1\right).$$

Based on Eq. (6), the whole expression of the observed cross section is subdivided into three terms: the continuum, the resonance, and the interference terms. The simplification of each term will be discussed separately.

3.1 Continuum term

In the light of Eq. (6), the Born order expression for the continuum is written explicitly as

$$\sigma^C_{Born} = \frac{8\pi\alpha^2 f_K^2}{3}\cdot\frac{(s/4-m_K^2)^{3/2}}{s^{9/2}} \ . \tag{14}$$

In the above equation, the most crucial part is the factor

$$l_{9/2}(s) = \frac{(s/4-m_K^2)^{3/2}}{s^{9/2}}.$$

For the study of charmonium physics, s is much greater than m_K^2, therefore the factor $l_{9/2}(s)$ varies almost linearly in the vicinity of ψ' peak, as shown in Fig. 1. With this observation, it is natural to approximate the factor $l_{9/2}(s)$ with a linear function, viz.

$$\bar{l}_{9/2}(s) \approx \lambda_{9/2}\cdot s + \zeta_{9/2}.$$

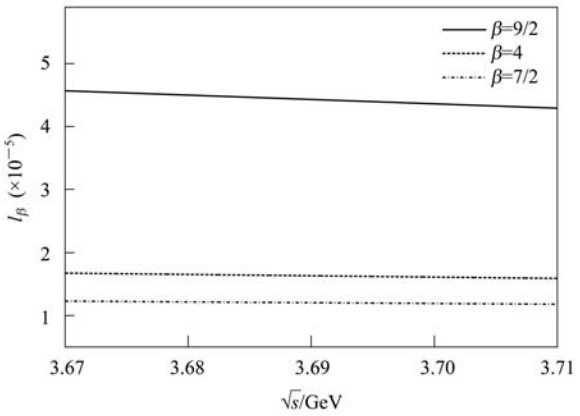

Fig. 1. Variations in factor $l_\beta(s)$ against center-of-mass energy ($\sqrt{s}$) in the vicinity of ψ' resonance peak for $\beta = 9/2,\ 4,\ 7/2$.

As a matter of fact, the similar factors appear in the resonance and interference terms as well. So, generally, we define

$$l_\beta(s) = \frac{(s/4-m_K^2)^{3/2}}{s^\beta}, \tag{15}$$

and utilizing the approximation

$$\bar{l}_\beta(s)\approx\lambda_\beta\cdot s+\zeta_\beta. \tag{16}$$

Here the coefficients λ_β and ζ_β can be determined analytically. The details are degraded into the appendix[1)].

With the linearization of the factor $l_{9/2}(s)$, the x-concerned ISR integral for the continuum term actually has the form

$$\rho_0=\int_0^{X_{\rm f}}x^\mu{\rm d}x, \tag{17}$$

which can be integrated easily. So the ISR corrected cross section of the continuum is expressed analytically as follows,

$$\sigma^{\rm C}_{\rm r.c.}=\frac{8\pi\alpha^2f_{\rm k}^2}{3}\cdot[(\lambda_{9/2}\cdot s+\zeta_{9/2})\cdot H_0(s)-\lambda_{9/2}\cdot s\cdot H_1(s)], \tag{18}$$

with

$$H_\mu(s)\equiv\int_0^{X_{\rm f}}x^\mu F(x,s){\rm d}x=\sum_{\nu=1}^{3}\frac{X_{\rm f}^{t+\mu+\nu-1}}{t+\mu+\nu-1}\cdot B_\nu(t)\ .$$

3.2 Resonance term

In the light of Eq. (6), the Born order expression for the resonance is written explicitly as

$$\sigma^{\rm R}_{\rm Born}=\frac{8\pi\alpha^2f_{\rm k}^2}{3}\cdot\frac{A_1}{(s-M^2)^2+M^2\Gamma^2}\frac{(s/4-m_{\rm K}^2)^{3/2}}{s^{7/2}}, \tag{19}$$

where

$$A_1=9\Gamma_{\rm ee}^2/\alpha^2\cdot(1+\mathcal{C}^2+2\mathcal{C}\cos\phi)\ .$$

As far as the factor

$$l_{7/2}(s)=\frac{(s/4-m_{\rm K}^2)^{3/2}}{s^{7/2}}$$

is concerned, the similar approximation as the previous section is adopted, viz.

$$\bar{l}_{7/2}(s)\approx\lambda_{7/2}\cdot s+\zeta_{7/2}.$$

The x-concerned ISR integral for the resonance term then reads

$$\rho(s,t)=\int_0^{X_{\rm f}}\frac{x^{t-1}{\rm d}x}{(s(1-x)-M^2)^2+M^2\Gamma^2}, \tag{20}$$

which can be integrated analytically [27, 28]

$$\begin{aligned}\rho(s,t)\ =\ &\frac{1}{ts^2}\cdot a^{t-2}\frac{\pi t\sin[\theta(1-t)]}{\sin\theta\sin\pi t}\\&+\frac{1}{s^2}\cdot\left[\frac{1}{t-2}\cdot X_{\rm f}^{t-2}+\frac{2(s-M^2)}{(t-3)s}\cdot X_{\rm f}^{t-3}\right.\\&\left.+\frac{3(s-M^2)^2-M^2\Gamma^2}{(t-4)s^2}\cdot X_{\rm f}^{t-4}\right],\end{aligned} \tag{21}$$

where

$$\begin{aligned}a^2\ &=\ \left(1-\frac{M^2}{s}\right)^2+\frac{M^2\Gamma^2}{s^2}\ (a>0),\\ \cos\theta\ &=\ \frac{1}{a}\cdot\left(\frac{M^2}{s}-1\right).\end{aligned}$$

With the expression of $\rho(s,t)$, the ISR corrected cross section of the resonance is re-cast as

$$\sigma^{\rm R}_{\rm r.c.}=\frac{8\pi\alpha^2f_{\rm K}^2}{3}\cdot A_1\cdot[(\lambda_{7/2}\cdot s+\zeta_{7/2})\cdot G_0(s)-\lambda_{7/2}\cdot s\cdot G_1(s)], \tag{22}$$

with

$$\begin{aligned}G_\mu(s)\ &=\ \int_0^{X_{\rm f}}\frac{x^\mu\cdot F(x,s){\rm d}x}{(s(1-x)-M^2)^2+M^2\Gamma^2}\\&=\ \sum_{\nu=1}^{3}\rho(s,t+\mu+(\nu-1))\cdot B_\nu(t)\ .\end{aligned} \tag{23}$$

3.3 Interference term

The Born order expression for the interference can be acquired readily from Eq. (6). However, for clearness, the expression of the interference is further divided into two sub-terms, as follows,

$$\sigma^{\rm I_1}_{\rm Born}=\frac{8\pi\alpha^2f_{\rm K}^2}{3}\cdot\frac{A_2\cdot(s-M^2)}{(s-M^2)^2+M^2\Gamma^2}\cdot\frac{(s/4-m_{\rm k}^2)^{3/2}}{s^4}, \tag{24}$$

and

$$\sigma^{\rm I_2}_{\rm Born}=\frac{8\pi\alpha^2f_{\rm k}^2}{3}\cdot\frac{A_3}{(s-M^2)^2+M^2\Gamma^2}\cdot\frac{(s/4-m_{\rm k}^2)^{3/2}}{s^4}, \tag{25}$$

where

$$A_2=6\,(\Gamma_{\rm ee}/\alpha)\cdot(1+\mathcal{C}\cos\phi),$$

$$A_3=6\,(\Gamma_{\rm ee}/\alpha)\cdot\mathcal{C}M\Gamma\sin\phi.$$

The simplification strategy is the same as those used for the continuum and resonance. First, the factor

$$l_4(s)=\frac{(s/4-m_{\rm k}^2)^{3/2}}{s^4}$$

is approximated as

$$\bar{l}_4(s)\approx\lambda_4\cdot s+\zeta_4;$$

second, the x-concerned ISR integrals for the interference terms have the same forms as those in Eqs. (17) and (20), which can be integrated out directly or by

1) The determination of linear coefficients λ_β and ζ_β is similar to that of linear regression, where the optimization is used. However, for linear regression, a linear function is used to fit a set of separated data while for our problem, a linear function is used to approximate another non-linear function. Such an idea of linearization is referred to as the generalized linear regression.

formula (21). Finally, the ISR corrected cross section of the interference is obtained,

$$\sigma_{\rm r.c.}^{\rm I_1} = \frac{8\pi\alpha^2 f_{\rm k}^2}{3}\cdot A_2\cdot\{(\lambda_4\cdot s+\zeta_4)(s-M^2)\cdot G_0 -[2\lambda_4\cdot s^2+(\zeta_4-\lambda_4 M^2)s]\cdot G_1(s) +\lambda_4 s^2\cdot G_2(s)\}, \quad (26)$$

$$\sigma_{\rm r.c.}^{\rm I_2} = \frac{8\pi\alpha^2 f_{\rm k}^2}{3}\cdot A_3\cdot[(\lambda_4 s+\zeta_4)\cdot G_0(s) -\lambda_4 s\cdot G_1(s)], \quad (27)$$

where $G_\mu(s)$ is given by formula (23).

In summary, the ISR corrected cross section formula is

$$\sigma_{\rm r.c.}(s)=\sigma_{\rm r.c.}^{\rm C}(s)+\sigma_{\rm r.c.}^{\rm R}(s)+\sigma_{\rm r.c.}^{\rm I_1}(s)+\sigma_{\rm r.c.}^{\rm I_2}(s), \quad (28)$$

with expressions of the cross section for each term given in Eqs. (18), (22), (26), and (27), respectively.

4 Possible simplification of energy spread integral

As indicated in Eq. (10), the experimentally observed cross section is the $\sigma_{\rm r.c.}$ convoluted $G(W',\, W)$, which might be simplified further. Two methods, the Taylor Expansion (TE) method and the Fast Fourier Transformation (FFT) method, have been considered for such a simplification.

For the TE method, we begin from Eq. (10), and Taylor expand the $\sigma_{\rm r.c.}$ at W, viz.

$$\sigma_{\rm r.c.}(W')=\sum_{n=0}^{\infty}\frac{\sigma_{\rm r.c.}^{(n)}(W)}{n!}\cdot(W'-W)^n,$$

where $\sigma_{\rm r.c.}^{(n)}(W)$ denotes the n-th derivative of function $\sigma_{\rm r.c.}$ at value W. Replacing the Taylor expansion of $\sigma_{\rm r.c.}$ into Eq. (10), the integral to be calculated has the following form

$$\int_{-\infty}^{\infty} x^n {\rm e}^{-x^2}{\rm d}x,$$

which can be precalculated. However, in order to achieve a reasonable precision, we need to calculate hundreds, or even thousands, of terms in Taylor expansion. This means that the fairly high order derivatives of $\sigma_{\rm r.c.}$ have to be calculated, and too much time is consumed, which is not acceptable.

As for the FFT method[1)], we could easily find that the observed cross section $\sigma_{\rm obs}(W)$ is a convolution of the radiative corrected cross section and a gauss function. Consider the Convolution Theorem in Fourier Transformation,

$$\mathfrak{F}(g\otimes h)=\mathfrak{F}(g)\cdot\mathfrak{F}(h)\ ,$$

where $\mathfrak{F}$ represents Fourier Transformation, $\otimes$ represents convolution. To calculate convolution efficiently, we use Fast Fourier Transformation. First, $\sigma_{\rm r.c.}$ and G should be sampled in the energy region. After that, we get two series of numbers. Then DFT (Discrete Fourier Transformation) should be performed on both series, and the resulting series should be multiplied to generate one final series. Finally, IDFT (Inverse Discrete Fourier Transformation) should be performed on this series and what we get is the distribution of $\sigma_{\rm obs}$ in the energy region on which $\sigma_{\rm r.c.}$ and G are sampled. This process is very fast, and we could get the result on the whole energy region at the same time rather than calculating the integral one by one. To get an accurate result, the sample number should be very large (512 or 1024), which means a large number of cross sections should be calculated. In a real energy scan, the number of data taking points is usually not large (less than 20). The total integration time in a small number of energy points is less than the time cost by sampling a large number of cross sections and perform DFT and IDFT on it. So this method does not fit our purpose.

5 Investigation of simplified formula

5.1 Precision

The accurate observed cross section ($\sigma_{\rm obs}$) is calculated by Eq. (10) while the simplification one (denoted by $\sigma_{\rm obs}^{\rm s}$) is also calculated by Eq. (10) but with ($\sigma_{\rm r.c.}$) replaced by the expression (28). The relative error of two observed cross sections is defined as

$$R_\sigma=\frac{\sigma_{\rm obs}^{\rm s}-\sigma_{\rm obs}}{\sigma_{\rm obs}}. \quad (29)$$

In the calculation of the observed cross section, all parameters of resonances are taken from PDG08 [29], $\Delta = 1.3$ MeV is used. Two real undetermined parameters are the relative phase (ϕ) and the relative strength ($\mathcal{C}$) between the strong and the electromagnetic amplitudes. The dependences of R_σ on ϕ and $\mathcal{C}$ are shown in Figs. 2 and 3, respectively.

The variations in R_σ against the center-of-mass energy ($\sqrt{s}$) in the vicinity of ψ' resonance peak for $\phi = 0°$, $90°$, $180°$, and $270°$ are displayed in Fig. 2, according to which we notice that firstly, the absolute value of R_σ is less than one percent in the en-

1) http://en.wikipedia.org/wiki/Fast_Fourier_transform

ergy region concerned; secondly, the difference between two cross sections fades away at the resonance peak; thirdly, the differences in off-resonance region are larger than those in the on-resonance region. The similar dependence of R_σ on $\mathcal{C}$ can be seen from Fig. 3, where displayed are the variations of R_σ against $\sqrt{s}$ in the vicinity of the ψ' resonance peak for $\mathcal{C}$ =1, 5, and 10. It is obvious that the difference due to the variation in $\mathcal{C}$ is even smaller, which is at the level of a few per mille.

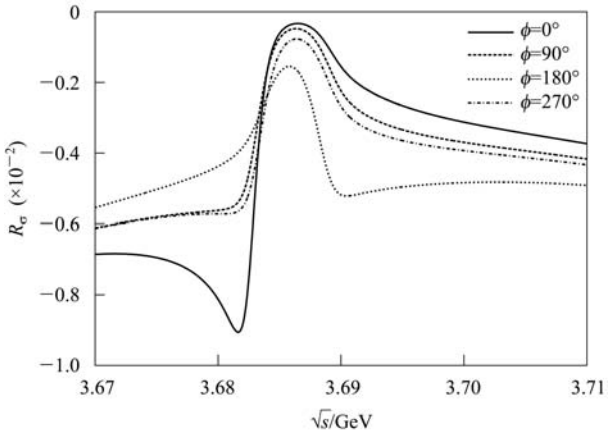

Fig. 2. Variations in R_σ against $\sqrt{s}$ in the vicinity of ψ' resonance peak for $\phi = 0°$, 90°, 180°, and 270°. In the calculation of the observed cross section, $\mathcal{C}$ is fixed at 2.5.

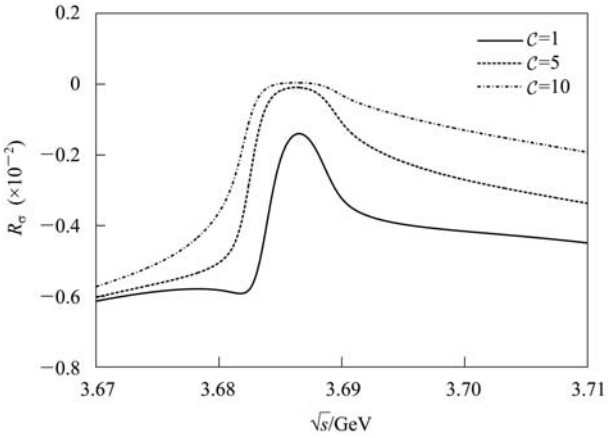

Fig. 3. Variations in R_σ against $\sqrt{s}$ in the vicinity of ψ' resonance peak for $\mathcal{C}$ =1, 5, and 10. In the calculation of the observed cross section, ϕ is fixed at 90°.

5.2 Computation time

The symbol $T^{\rm s}$ (T^0) denotes the computation time when $\sigma^{\rm s}_{\rm obs}$ ($\sigma_{\rm obs}$) is used for the cross section calculation. The comparison of $T^{\rm s}$ (denoted by the dashed line) and T^0 (denoted by the solid line) at both resonance and off-resonance regions are shown in Fig. 4.

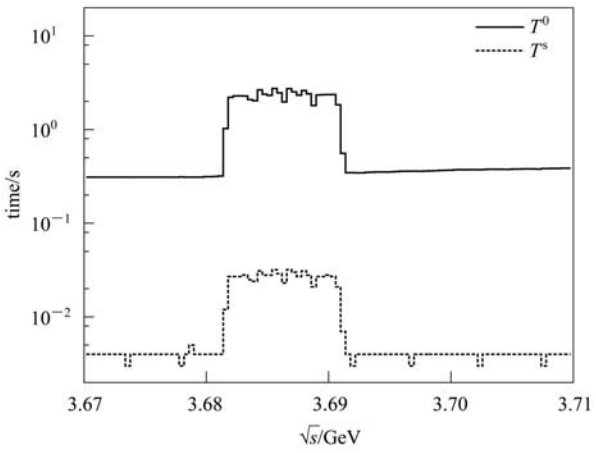

Fig. 4. Comparison of $T^{\rm s}$ and T^0 at both resonance and off-resonance regions.

From a comparison, it can be seen that about one-hundred-time reduction of computation time is achieved by our simplification algorithm. Although only one-fold integral is simplified by analytical expression, the computation time is less than 0.1 second for each energy point, which is fast enough for our scan simulation study.

5.3 Application

As we mentioned in the introduction, the speed of calculation of the observed cross section is the crucial issue of a data taking optimization study of the scan experiment. Without reasonably simplified formula, it will be too long a time to perform the optimization fit, and the detailed scan optimization is impractical.

Besides the application in scan optimization, simplified cross section formulas can also be used for the uncertainty study [30] and correlation study [31]. Since for both of these studies the sampling-and-fitting method is also adopted, the fast computation of the cross section is needed as well.

6 Summary

The complete expressions for $\psi' \to$ PP decays are presented, including the relative phase between the strong and the electromagnetic amplitudes. After linearizing one non-linear kinematic factor, the integrand with the initial state radiation is integrated analytically. Such a simplification of two-fold integral into a one-fold integral reduces the total computing time by about one hundred times.

The possible approaches for the simplification of the energy spread integral are also discussed.

The simplified formulas of the observed cross sections obtained in this paper provide a practical tool for the further optimization study of the scan data taking, which is of great importance for the study of the relative phase between the strong and the electromagnetic amplitudes.

Appendices A

As we have noted in Subsection 3.1, the factor

$$l_{9/2}(s)=\frac{(s/4-m_{\rm K}^2)^{3/2}}{s^{9/2}}$$

varied almost linearly in the vicinity of ψ' peak, and its variation against s is shown in Fig. 1. Therefore, for the factor

$$l_\beta(s)=\frac{(s/4-m_{\rm K}^2)^{3/2}}{s^\beta}, \tag{A1}$$

a linear function (it refers to Eq. (16)),

$$\bar{l}_\beta(s)\approx\lambda_\beta\cdot s+\zeta_\beta \tag{A2}$$

is utilized to approximate it in the vicinity of resonance peak. The coefficients λ_β and ζ_β are determined by the generalized linear regression method. As the first step, we define the integration

$$I=\int_{s_1}^{s_2}{\rm d}s\left[(\lambda_\beta\cdot s+\zeta_\beta)-\frac{(s/4-m_{\rm K}^2)^{3/2}}{s^\beta}\right]^2. \tag{A3}$$

The needed values of coefficients λ_β and ζ_β are obtained by the minimization of the integration I, that is

$$\frac{\partial I}{\partial\lambda_\beta}=0 \ \text{ and } \ \frac{\partial I}{\partial\zeta_\beta}=0. \tag{A4}$$

From the above requirements, we acquire a set of linear equations of λ_β and ζ_β. By solving it, we obtain

$$\lambda_\beta=\frac{\delta_1C_1-\delta_2C_2}{\delta_1\delta_3-\delta_2^2} \ \text{ and } \ \zeta_\beta=\frac{\delta_3C_2-\delta_2C_1}{\delta_1\delta_3-\delta_2^2}, \tag{A5}$$

where

$$\begin{aligned}\delta_i &= \int_{s_1}^{s_2}s^{i-1}{\rm d}s=\frac{s_2^i-s_1^i}{i},\\ C_1 &= \int_{s_1}^{s_2}{\rm d}s\frac{(s/4-m_{\rm K}^2)^{3/2}}{s^{\beta-1}}=\frac{1}{8}D(\beta-1),\\ C_2 &= \int_{s_1}^{s_2}{\rm d}s\frac{(s/4-m_{\rm K}^2)^{3/2}}{s^{\beta}}=\frac{1}{8}D(\beta).\end{aligned}$$

Both C_1 and C_2 contain integral

$$D(\beta)=\int_{s_1}^{s_2}{\rm d}x\frac{(x-u)^{3/2}}{x^\beta}, \tag{A6}$$

where

$$\beta=2,\ \frac{5}{2},\ 3,\ \frac{7}{2},\ 4,\ \frac{9}{2},\quad u=4m_{\rm K}^2.$$

For different β, we can calculate the integral analytically[1)]. For $\beta=2$,

$$D(2)=\left[\sqrt{x-u}\left(\frac{u}{x}+2\right)-3\sqrt{u}\tan^{-1}\left(\frac{\sqrt{x-u}}{\sqrt{u}}\right)\right]\Bigg|_{s_1}^{s_2}; \tag{A7}$$

For $\beta=\dfrac{5}{2}$,

$$\begin{aligned}D\left(\frac{5}{2}\right) &= \Big[2\lg\left(2\left(\sqrt{x-u}+\sqrt{x}\right)\right)\\ &\quad+\frac{2}{3}\left(\frac{u}{x^{\frac{3}{2}}}-\frac{4}{\sqrt{x}}\right)\sqrt{x-u}\Big]\Bigg|_{s_1}^{s_2};\end{aligned} \tag{A8}$$

For $\beta=3$,

$$\begin{aligned}D(3) &= \left[\frac{3}{4\sqrt{u}}\tan^{-1}\left(\frac{\sqrt{x-u}}{\sqrt{u}}\right)\right.\\ &\quad\left.+\frac{1}{4}\left(\frac{2u}{x^2}-\frac{5}{x}\right)\sqrt{x-u}\right]\Bigg|_{s_1}^{s_2};\end{aligned} \tag{A9}$$

For $\beta=\dfrac{7}{2}$,

$$D\left(\frac{7}{2}\right)=\frac{2(x-u)^{\frac{5}{2}}}{5ux^{\frac{5}{2}}}\Bigg|_{s_1}^{s_2}; \tag{A10}$$

For $\beta=4$,

$$\begin{aligned}D(4) &= \left[\frac{\tan^{-1}\left(\frac{\sqrt{x-u}}{\sqrt{u}}\right)}{8u^{\frac{3}{2}}}+\sqrt{x-u}\left(\frac{u}{3x^3}\right.\right.\\ &\quad\left.\left.+\frac{1}{8ux}-\frac{7}{12x^2}\right)\right]\Bigg|_{s_1}^{s_2};\end{aligned} \tag{A11}$$

For $\beta=\dfrac{9}{2}$,

$$D\left(\frac{9}{2}\right)=\frac{2(x-u)^{\frac{5}{2}}(5u+2x)}{35u^2x^{\frac{7}{2}}}\Bigg|_{s_1}^{s_2}. \tag{A12}$$

1) The following integrals are obtained by using Mathematica and are checked by hand.

It could be easily checked that for the coefficients λ_β and ζ_β, we obtain

$$\frac{\partial^2 I}{\partial \lambda_\beta^2} = 2\int_{s_1}^{s_2} s^2 \mathrm{d}s = \frac{2}{3}\left(s_2^3 - s_1^3\right) > 0, \tag{A13}$$

$$\frac{\partial^2 I}{\partial \zeta_\beta^2} = 2\int_{s_1}^{s_2} \mathrm{d}s = 2(s_2 - s_1) > 0. \tag{A14}$$

This means that what we get is the minimum of I, not the maximum.

The relative error between the linearized formula and the original formula is defined as

$$R_{\mathrm{l}} = \frac{|\bar{l}_\beta - l_\beta|}{l_\beta}. \tag{A15}$$

When $\beta = 9/2,\ 4,\ 7/2$, the variations in R_{l} against the center-of-mass energy ($\sqrt{s}$) are shown in Fig. A1.

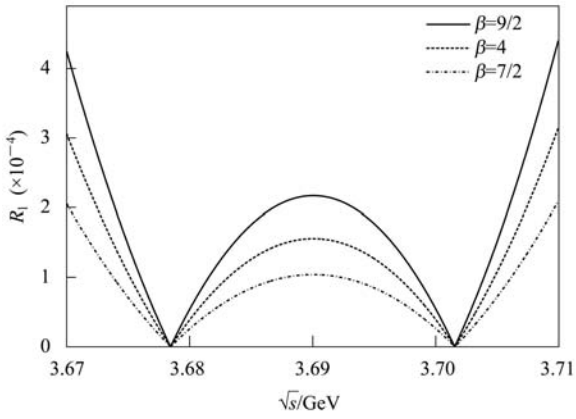

Fig. A1. The variations in R_{l} against $\sqrt{s}$ for $\beta = 9/2,\ 4,\ 7/2$.

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