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A result on certain coset subalgebras of the $\beta\gamma$-system*

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Abstract: In this paper, for the highest weight module V_4 of sl(2,$\mathbb{C}$) with the highest weight 4, we describe subalgebras $S_\beta(V_4)^{\Theta_+}$ and $S_\gamma(V_4)^{\Theta_+}$ of the $\beta\gamma$-system coset $S(V_4)^{\Theta_+}$ by giving their generators. These coset subalgebras are interesting, new examples of strongly finitely generated vertex algebra.

Key words: vertex algebra coset, $\beta\gamma$-system, ∂-ring, Hilbert series

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1 Introduction

In conformal field theory, coset construction is a method of obtaining a new two-dimensional conformal field theory from two old ones. It appeared originally in the paper [1] of Goddard-Ken-Olive as the quotient of two Wess-Zumino-Novikov-Witten models. The general construction of the coset energy-momentum tensor was first exposed and used to prove the unitarity of minimal model sequence $\mathcal{M}(m,m{+}1)$. Coset construction is expected to provide the framework for the complete classification of all rational conformal field theories.

In coset framework, some character decompositions of coset models were derived by Kac and Wakimoto [2]. Characters and maverick examples of coset conformal field theories were discussed by Dundar and Joshi [3, 4], and modular invariants were described in Ref. [5]. In Ref. [6], Zamolodchikov and Fateev first noticed the relations between the parafermionic theory and some $\widehat{SU}(2)$ coset theory. In terms of some kinds of coset theories, the generalized parafermions were introduced and studied in Ref. [7]. For affine Lie algebras at fractional levels, Kent first noticed the coset descriptions of these non-unitary models [8]. Using a coset approach, Mathieu and Walton described the admissible weights and non-unitary minimal models [9]. A general analysis of non-unitary diagonal coset models was given in Ref. [10]. Altschuler discovered the general method to establish the equivalence of some coset models [11]. Irrational conformal field theories [12] provided a hint to a large generalization of the coset construction.

The $\beta\gamma$-ghost System was introduced by Friendan-Martinec-Shenker in Ref. [13]. As a simple representative of non-unitary conformal field theories, it forms a block of field theory approaches to disorder systems and plays a crucial role in the free-field representation of supergroup WZNW models. In particle physics, a current algebra was introduced in order to describe the strong interactions among articles. In fact, a current algebra reflects some symmetry of strong interactions. For a Lie algebra g and a representation V. The current algebra $\mathcal{O}(g,B)$ can act on the $\beta\gamma$-system $S(V)$ by a certain way [14]. Under the action of the current algebra $\mathcal{O}(g,B)$, we expect to find the symmetry structure preserved by the current algebra in $S(V)$. Describing such symmetry is equivalent to describing the coset subalgebra of $S(V)$ under the action of $\mathcal{O}(g,B)$ on $S(V)$.

Let W be a vertex algebra, and U be its subalgebra. One can construct a subalgebra known as the coset Com(U,W) of U in W. In fact, this construction was introduced by Frenkel I and Zhu in Ref. [15] in mathematics. The construction is analogous to the coset construction in associative algebra theory.

Denote $S(V)^{\Theta_+}$ to be the coset under the action of $\mathcal{O}(g,B)$ on $S(V)$. We expect to explicitly describe $S(V)^{\Theta_+}$ by giving its generators. In Refs. [14, 16],

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Linshaw and Lian B singled out a certain category $\mathfrak{R}$ of vertex algebras containing $S(V)$ and $\mathcal{O}(g,B)$. They reduced the description of $S(V)^{\Theta_+}$ to the description of the associated graded ∂-ring $\mathrm{gr}(S(V)^{\Theta_+})$. As an example, Linshaw proved $S(\mathrm{sl}(2,\mathbb{C}))^{\mathcal{A}_+}=\Theta$ and gave a Howe pair $(\Theta, S(\mathrm{sl}(2,\mathbb{C}))^{\Theta_+})$ by the properties of Gröbner base [14]. For an abelian Lie algebra g acting diagonally on a vector space V, Linshaw described $S(V)^{\Theta_+}$ and gave some Howe pairs [17]. Let g be one of Lie algebras $\mathrm{sl}(n,\mathbb{C}),\mathrm{so}(n,\mathbb{C}),\mathrm{sp}(2n,\mathbb{C})$, and V be a sum of standard representations of g. By using tools from commutative algebra and algebraic geometry, in particular, the theory of jet schemes, Linshaw and Song Bailin described $S(V)^{\Theta_+}$ [18]. In this paper, for the highest weight representation V_4 of $\mathrm{sl}(2,\mathbb{C})$ with the highest weight 4, based on the theory of ∂-rings [18], we get generators of coset subalgebras $S_\beta(V_4)^{\Theta_+}$ and $S_\gamma(V_4)^{\Theta_+}$. We believe these coset subalgebras all correspond to the new conformal field theories. If we can describe the coset subalgebras in the $\beta\gamma$-system in vertex algebra category, then we can know the symmetry structure preserved by the current algebra in the $\beta\gamma$-system. Moreover, we expect to classify such new conformal field theories depending on the representation V and study their properties.

2 Preliminaries

Let $\{e,f,h\}$ be the standard generators of $\mathrm{sl}(2,\mathbb{C})$ satisfying the following relations

$$[e,f]=h,\ [e,h]=2e,\ [f,h]=-2f.$$

Denote V_4 to be the highest weight module of $\mathrm{sl}(2,\mathbb{C})$ with the highest weight 4, which is realized as the homogeneous subspace of degree 4 of the polynomial algebra $\mathbb{C}[x,y]$ via $\rho:\mathrm{sl}(2,\mathbb{C})\longrightarrow\mathbb{C}[x,y]$, where ρ is given by

$$\rho(e)=x\frac{\partial}{\partial y},\ \rho(f)=y\frac{\partial}{\partial x},\ \rho(h)=x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}.$$

Set $\varphi_i=x^{4-i}y^i,\ i=0,1,\cdots,4$, where $\varphi_0=x^4$ is a highest weight vector. Let V_4^* be the dual space of V_4, and $\xi_i,\ i=0,1,\cdots,4$ be the basis of V_4^* corresponding to the basis $\varphi_i,\ i=0,1,\cdots,4$. There is an induced representation $\rho^*:\mathrm{sl}(2,\mathbb{C})\longrightarrow\mathrm{End}V_4^*$.

Throughout this paper, a commutative circle algebra will be regarded as a vertex algebra [19, 20]. Let V be a n-dimensional g-module via $\rho:g\longrightarrow\mathrm{End}V$, where g is a finite dimensional Lie algebra. Then ρ induces a vertex algebra homomorphism $\widehat{\rho}:\mathcal{O}(g,B)\longrightarrow S(V)$, where $B(u,v)=-\mathrm{Tr}(\rho(u)\rho(v))$ for $u,v\in g$ [14]. In fact, this gives an action of the current algebra $\mathcal{O}(g,B)$ on $S(V)$ [14, 16, 18–20].

Definition 2.1 Let W be a vertex algebra, and U be its a subalgebra. The commutant of U in W, denoted by $\mathrm{Com}(U,\ W)$, is the subalgebra of vertex operators $v(w)\in W$ such that $[u(z),\ v(w)]=0$ for all $u(z)\in U$. Equivalently, $u(z)\circ_n v(w)=0$ for all all $u(z)\in U$ and $n\geqslant 0$.

Set $\varphi:V_4\longrightarrow V_4^*$ to be a linear map by

$$\varphi_0\longmapsto\xi_4,\quad \varphi_1\longmapsto-\frac{1}{4}\xi_3,\quad \varphi_2\longmapsto\frac{1}{6}\xi_2,$$
$$\varphi_3\longmapsto-\frac{1}{4}\xi_1,\quad \varphi_4\longmapsto\xi_0.$$

Then the map φ is an $\mathrm{sl}(2,\mathbb{C})$-module isomorphism. There is a bilinear symmetric $\mathrm{sl}(2,\mathbb{C})$-invariant form $B'(\cdot,\cdot)$ on V_4 defined by $B'(\varphi_i,\varphi_j)=\langle\varphi(\varphi_i),\varphi_j\rangle$, where $\langle,\rangle$ is the pair of V_4 and V_4^*. For the representation V_4 and B', there is a Lie algebra homomorphism ψ from $\mathrm{sl}(2,\mathbb{C})$ to $D(V_4)^{\mathrm{sl}(2,\mathbb{C})}$ [14]. According to Lemma 2.15 in Ref. [14], the homomorphism ψ induces a vertex algebra homomorphism $\widehat{\psi}:\mathcal{O}(\mathrm{sl}(2,\mathbb{C}),-\frac{5}{8}K)\longrightarrow S(V_4)^{\Theta_+}$.

Note that $S(V_4)$ contain abelian vertex subalgebras $S_\beta(V_4),S_\gamma(V_4)$, where $S_\beta(V_4)$ is generated by $\{\beta^{\varphi_i}(z)\mid i=0,1,\cdots,4\}$ and $S_\gamma(V_4)$ is generated by $\{\gamma^{\xi_i}(z)\mid i=0,1,\cdots,4\}$. These subalgebras both preserve the action of $\mathcal{O}(\mathrm{sl}(2,\mathbb{C}),-5K)$. Hence they give rise to canonical subalgebras $S_\gamma(V_4)^{\Theta_+},S_\beta(V_4)^{\Theta_+}$ of the commutant $S(V_4)^{\Theta_+}$.

Denote $P=\mathrm{gr}(S(V_4))$, and the images of $\partial^k\beta^{\varphi_i}(z),\partial^k\gamma^{\xi_i}(z)$ in P by $\beta_k^{\varphi_i}$ and $\gamma_k^{\xi_i}$, respectively. According to Ref. [18], there is $P=\mathbb{C}[\beta_k^{\varphi_i},\gamma_k^{\xi_i}|i=0,1,\cdots,4]$. For each $u\in\mathrm{sl}(2,\mathbb{C})$, $\widehat{u}(n)_{\mathrm{Der}}$ is a derivation of degree 0 and $\mathrm{gr}(S(V_4)$ is a $\mathrm{sl}(2,\mathbb{C})\otimes\mathbb{C}[t]-$module [14]. The actions of $\widehat{u}(n)_{\mathrm{Der}}$ on the generators of P are given by

$$\widehat{u}(n)_{\mathrm{Der}}(\beta_k^{\varphi_i})=C_k^n\beta_{k-n}^{\rho(u)(\varphi_i)},$$
$$\widehat{u}(n)_{\mathrm{Der}}(\gamma_k^{\xi_i})=C_{k-n}^n\gamma^{\rho^*(u)(\xi_i)},\qquad(1)$$

where $C_k^n=k(k-1)\cdots(k-n+1)$ for $n,\ k\geqslant 0$, $C_k^0=1,\ C_k^n=0$ for $n>k$. Denote by $P_\beta=\mathbb{C}[\beta^{\varphi_i}\mid i=0,1,\cdots,4]$ and $P_\gamma=\mathbb{C}[\gamma^{\xi_i}\mid i=0,1,\cdots,4]$. Then P_β,P_γ are both subalgebras of P preserving the action (1).

3 Main results

Since SL(2) is a connected, linear reductive algebraic group with Lie algebra $\mathrm{sl}(2,\mathbb{C})$, $P^{\mathrm{sl}(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is generated by $P_0^{\mathrm{sl}(2,\mathbb{C})}=\mathbb{C}[\beta_0^{\varphi_i},\gamma_0^{\xi_i}|i=0,1,\cdots,4]^{\mathrm{sl}(2,\mathbb{C})}$, $P_\beta^{\mathrm{sl}(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is generated by $P_{\beta,0}^{\mathrm{sl}(2,\mathbb{C})}=\mathbb{C}[\beta_0^{\varphi_i}|i=0,1,\cdots,4]^{\mathrm{sl}(2,\mathbb{C})}$, and $P_\gamma^{\mathrm{sl}(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is generated by $P_{\gamma,0}^{\mathrm{sl}(2,\mathbb{C})}=\mathbb{C}[\gamma_0^{\xi_i}|i=0,1,\cdots,4]^{\mathrm{sl}(2,\mathbb{C})}$ as ∂- rings [18]. By Hilbert theorem [21], $P_0^{\mathrm{sl}(2,\mathbb{C})}$, $P_{\beta,0}^{\mathrm{sl}(2,\mathbb{C})}$ and $P_{\gamma,0}^{\mathrm{sl}(2,\mathbb{C})}$

are all finitely generated. So $P^{\mathrm{sl}(2,\mathbb{C})\otimes\mathbb{C}[t]}$, $P_\beta^{\mathrm{sl}(2,\mathbb{C})\otimes\mathbb{C}[t]}$ and $P_\gamma^{\mathrm{sl}(2,\mathbb{C})\otimes\mathbb{C}[t]}$ are finitely generated as ∂-rings.

According to the results [21], Hilbert series of $P_{\gamma,0}^{\mathrm{sl}(2,\mathbb{C})}$ can be expressed

$$P(t)=\frac{1}{(1-t^2)(1-t^3)}. \tag{2}$$

Using the action relations of $n=0$ in (1), we find

$$g_{2,\xi} = \gamma_0^{\xi_0}\gamma_0^{\xi_4}-\frac{1}{4}\gamma_0^{\xi_1}\gamma_0^{\xi_3}+\frac{1}{12}\gamma_0^{\xi_2}\gamma_0^{\xi_2}, \tag{3}$$

$$\begin{aligned} g_{3,\xi} = {} & 8\gamma_0^{\xi_0}\gamma_0^{\xi_2}\gamma_0^{\xi_4}-3\gamma_0^{\xi_0}\gamma_0^{\xi_3}\gamma_0^{\xi_3}-3\gamma_0^{\xi_1}\gamma_0^{\xi_1}\gamma_0^{\xi_4} \\ & +\gamma_0^{\xi_1}\gamma_0^{\xi_2}\gamma_0^{\xi_3}-\frac{2}{9}\gamma_0^{\xi_2}\gamma_0^{\xi_2}\gamma_0^{\xi_2} \end{aligned} \tag{4}$$

are two generators of $P_{\gamma,0}^{\mathrm{sl}(2,\mathbb{C})}$. Obviously, these two generators are algebraic independent. Therefore, the invariant ring $P_{\gamma,0}^{\mathrm{sl}(2,\mathbb{C})}$ is generated by two elements $g_{2,\xi}, g_{3,\xi}$. Moreover, the invariant ring $P_\gamma^{\Theta_+} = P_\gamma^{\mathrm{sl}(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is generated by the finite set $\{g_{2,\xi}, g_{3,\xi}\}$ as a ∂-ring.

Let $g_{2,\xi}(z), g_{3,\xi}(z)$ be the corresponding vertex operators of polynomials $g_{2,\xi}, g_{3,\xi}$. They are obtained from polynomials $g_{2,\xi}, g_{3,\xi}$ by replacing each γ_0 with $\gamma(z)$, and by replacing ordinary multiplication with the Wick product. We shall also use the usual convention for iterated Wick products [14]. From the homomorphism $\widehat{\psi}$ [14], we find $g_{2,\xi}(z)$ belongs to $S_\gamma(V_4)^{\Theta_+}$.

The relation (2.1) in Ref. [14] suggests that the above conditions are equivalent to the OPE relations $u(z)v(w) \sim 0$. Note that Θ is generated by $\widehat{e}(z)$, $\widehat{f}(z)$, $\widehat{h}(z)$. If the OPE relations of $v(z)$ with $\widehat{e}(z)$, $\widehat{f}(z)$, $\widehat{h}(z)$ are all 0, then there is $v(z)\in S(V)^{\Theta_+}$.

Using Wick's formula and Taylor's formula [20], we calculate OPE relations of $\widehat{e}(z)$, $\widehat{f}(z)$, $\widehat{h}(z)$ with $g_{3,\xi}(z)$, and get $\widehat{u}(z)g_{3,\xi}(w)\sim 0$ for $i=1, \cdots, 4$, $u=e, f, h$. Then $g_{3,\xi}(z)$ belongs to $S_\gamma(V_4)^{\Theta_+}$.

According to the above results, we derive that $\Gamma : \mathrm{gr}(S_\gamma(V_4)^{\Theta_+}) \hookrightarrow \mathrm{gr}(S_\gamma(V_4))^{\Theta_+}$ is surjective. By the Reconstruction Property ([14]), we give our main result.

Theorem 3.1 The commutant $S_\gamma(V_4)^{\Theta_+}$ is strongly generated by the finite set

$$\{g_{2,\xi}(z), g_{3,\xi}(z)\}.$$

In another case, we have

$$r_{2,\varphi} = 4\beta_0^{\varphi_1}\beta_0^{\varphi_3}-\beta_0^{\varphi_0}\beta_0^{\varphi_4}-3\beta_0^{\varphi_2}\beta_0^{\varphi_2}, \tag{5}$$

$$\begin{aligned} r_{3,\varphi} = {} & \beta_0^{\varphi_0}\beta_0^{\varphi_2}\beta_0^{\varphi_4}-\beta_0^{\varphi_0}\beta_0^{\varphi_3}\beta_0^{\varphi_3}-\beta_0^{\varphi_1}\beta_0^{\varphi_1}\beta_0^{\varphi_4} \\ & +2\beta_0^{\varphi_1}\beta_0^{\varphi_2}\beta_0^{\varphi_3}-\beta_0^{\varphi_2}\beta_0^{\varphi_2}\beta_0^{\varphi_2} \end{aligned} \tag{6}$$

are two generators of $P_{\beta,0}^{\mathrm{sl}(2,\mathbb{C})}$. Obviously, these two generators are algebraic independent. Therefore, the invariant ring $P_{\beta,0}^{\mathrm{sl}(2,\mathbb{C})}$ is generated by the finite set $r_{2,\varphi}, r_{3,\varphi}$. Moreover, the invariant ring $P_\beta^{\Theta_+} = P_\beta^{\mathrm{sl}(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is generated by the finite set $\{r_{2,\varphi}, r_{3,\varphi}\}$ as a ∂-ring.

Let $r_{2,\varphi}(z), r_{3,\varphi}(z)$ be the corresponding vertex operators of polynomials $r_{2,\varphi}, r_{3,\varphi}$. They are obtained from polynomials $r_{2,\varphi}, r_{3,\varphi}$ by replacing each β_0 with $\beta(z)$, and by replacing ordinary multiplication with the Wick product. From the homomorphism $\widehat{\psi}$ [14], we find $r_{2,\varphi}(z)$ belongs to $S_\beta(V_4)^{\Theta_+}$.

Using Wick's formula and Taylor's formula [20], we calculate OPE relations of $\widehat{e}(z), \widehat{f}(z), \widehat{h}(z)$ with $r_{3,\varphi}(z)$, and get $\widehat{u}(z)r_{3,\varphi}(w)\sim 0$ for $i=1,\cdots,4$, $u=e,f,h$. Then $r_{3,\varphi}(z)$ belongs to $S_\beta(V_4)^{\Theta_+}$.

Similar to the case of $S_\gamma(V_4)^{\Theta_+}$, there is the result.

Theorem 3.2 The commutant $S_\beta(V_4)^{\Theta_+}$ is strongly generated by the finite set

$$\{r_{2,\varphi}(z), r_{3,\varphi}(z)\}.$$

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