Kinematics in Randers-Finsler geometry and secular increase of the astronomical unit *

LI Xin(李昕)[1,3;1)] CHANG Zhe(常哲)[2,3;2)]

[1] Theoretical Physics Center for Science Facilities, Chinese Academy of Sciences, Beijing 100049, China
[2] Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China
[3] Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

Abstract: Kinematics in Finsler space is investigated. It is shown that the result based on the kinematics with a special Finsler structure is in good agreement with the reported value of the secular trend in the astronomical unit, $\mathrm{d}AU/\mathrm{d}t=15\pm4$[m/century]. The space deformation parameter λ in this special structure is very small, with a scale of 10^{-6}, and should be a constant. This is consistent with the reported value of an anomalous secular eccentricity variation of the Moon's orbit.

Key words: ephemerides, celestial mechanics, astronomical unit, moon

PACS: 02.40.-k, 04.50.Kd, 95.10.Eg **DOI:** 10.1088/1674-1137/35/10/005

1 Introduction

Rapid technological progress has made astronomical observations more and more accurate. New physical phenomena have appeared, which cannot be explained by conventional physical mechanisms. The most well-known among them is the accelerated expanding universe [1] and the flat rotational velocity curves of spiral galaxies [2]. The astronomical unit (AU) is the fundamental and standard scale in astronomy. The latest planetary ephemerides [3] presented an accurate value of AU with a tiny error

$$1[AU]=1.495978706960\times10^{11}\pm0.1[\mathrm{m}]. \tag{1}$$

However, recent reports from Krasinsky and Brumberg [4] and also from Standish [5] show a positive secular trend in AU as $\mathrm{d}AU/\mathrm{d}t=15\pm4$ [m/century]. These authors have analyzed all available radiometric measurements on distances between the Earth and the inner planets, including observations of Martian landers and orbiters. This value is about 100 times larger than the current determination error of AU [3]. The theoretical value of the round-trip time of radar signal is given as

$$t_{\mathrm{theo}}=\frac{d_{\mathrm{theo}}[AU]}{c}, \tag{2}$$

where d_{theo} is the interplanetary distance obtained from ephemerides and c is the speed of light. The secular trend was obtained by the following formula

$$t_{\mathrm{theo}}=\frac{d_{\mathrm{theo}}\left[AU+\dfrac{\mathrm{d}AU}{\mathrm{d}t}(t-t_0)\right]}{c}, \tag{3}$$

where t_0 is the initial epoch. Currently, none of the theoretical predictions is consistent with the time-dependent term $\dfrac{\mathrm{d}AU}{\mathrm{d}t}(t-t_0)$. To explain this fact, physicists have made several attempts, such as the effects of cosmic expansion [4, 6, 7], the time variation of the gravitational constant [4], the mass loss of the Sun [4, 8], and the influence of dark matter on light propagation in the solar system [9]. However, none of them seems to be successful. Recently, one sound model has been proposed [10]. It assumes the existence of some tidal interactions that transfer angular momentum from the Sun to the planet's system, and made use of the conservation law of total angular momentum to explain the secular trend in AU. This model needs more work before it can be considered to be viable.

As mentioned above, general relativity also encounters problems. Analysis of the data from Pi-

Received 21 February 2011, Revised 16 March 2011
* Supported by National Natural Science Foundation of China (10525522, 10875129)
1) E-mail: lixin@itp.ac.cn
2) E-mail: changz@ihep.ac.cn

oneer 10 and 11 spacecraft shows that an anomaly acceleration exists in the solar system, which cannot be explained by Newtonian gravity and general relativity [11]. Finsler geometry is a natural generation of Riemann geometry. The gravitational theory based on Finsler geometry provides a reasonable method to solve the problems mentioned above. Gravity in Finsler space has been studied for a long time [12–15]. Considering spacetime to be Finslerian one may solve these anomalous phenomena in cosmology. In a previous paper [16], we proposed a modified Friedmann model based on the Einstein equations in Finsler space, which guarantees an accelerated expanding universe without invoking dark energy. Also, in the framework of Finsler geometry, the flat rotation curves of spiral galaxies can be deduced naturally without invoking dark matter [17]. Special relativity in Randers space (a special kind of Finsler space) [18] was investigated [19]. We found that the anomalous acceleration observed by Pioneers 10 and 11 corresponds to a special structure of Randers space [20].

In this paper, we discuss the secular trend in AU in the framework of Finsler geometry. We note the difference between metrics in Finsler geometry and Riemann geometry. In Sec. 2, we briefly review the basic notation of Finsler geometry. The length of the unit tangent sphere in the Finsler manifold is investigated. In Sec. 3, the secular trend of AU is described in terms of Finslerian language. The area of the unit tangent sphere in the Finsler manifold is given. The connection between the secular trend of AU and the anomalous secular eccentricity of the Moon's orbit is proposed. In Sec. 4, we discuss the Lorentz violation induced by Finsler structure.

2 The length of the unit tangent sphere in the Finsler manifold

The metric in Riemann geometry is a function of position. However, this is not the case in Finsler geometry, where the metric is a function of both position and velocity. Finsler geometry is based on the so-called Finsler structure F with the property $F(x,\lambda y)=\lambda F(x,y)$, where $x\in M$ represents position $y\in T_xM$ represents velocity, and M is an n-dimensional manifold. The Finslerian metric is given as [21]

$$g_{\mu\nu}\equiv\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}\left(\frac{1}{2}F^2\right). \tag{4}$$

Then, $g_{\mu\nu}$ defines a metric

$$g_{\mu\nu}\mathrm{d}y^\mu\otimes\mathrm{d}y^\nu \tag{5}$$

on the punctured tangent space $T_xM\backslash 0$. This metric is a part of a Sasaki metric [22]. It admits the unit tangent sphere (or indicatrix) $I_xM\equiv\{y\in T_xM: F(y)=1\}$ as a smooth manifold. Topologically, I_xM is diffeomorphic to the unit sphere S^{n-1} in R^n. The volume form of the indicatrix I_xM is

$$\sqrt{g}\sum_{\mu=1}^{n}(-1)^{\mu-1}\frac{y^\mu}{F}\mathrm{d}y^1\wedge\cdots\wedge\mathrm{d}y^{\mu-1}\wedge\mathrm{d}y^{\mu+1}\wedge\cdots\wedge\mathrm{d}y^n, \tag{6}$$

where g denotes the determinant of the metric $g_{\mu\nu}$.

All the trajectories of the planets in the solar system lie almost in the same plane; the eccentricity of the planets (excluding Mercury and Pluto) is very small. Thus, their trajectories can be considered as circular orbits embedded in three-dimensional space. It is well known that in Euclidean space the length of a unit circle equals $2\pi_E$ or the value of $2\times 3.1415926\cdots$. However, in Finsler space it is typically not equal to $2\times 3.1415926\cdots$. Instead, as mentioned in Formula (6), the 2-dimensional indicatrix I_x^2M has length element

$$\mathrm{d}s=\frac{\sqrt{g}}{F}\left(y^1\frac{\mathrm{d}y^2}{\mathrm{d}t}-y^2\frac{\mathrm{d}y^1}{\mathrm{d}t}\right)\mathrm{d}t, \tag{7}$$

where t is a real parameter. Then, the length of the indicatrix I_x^2M is

$$L\equiv\int_{F=1}\mathrm{d}s. \tag{8}$$

Here, we confine the Finsler structure F as Randers type

$$F=\sqrt{(y^1)^2+(y^2)^2}+\lambda y^1, \tag{9}$$

where the parameter λ, in general, is a function of positions. We introduce polar coordinates on I_x^2M, $y^1=r\cos\phi$ and $y^2=r\sin\phi$. The determinant of n-dimensional Randers space is given as [21]

$$\det(g_{\mu\nu})=\left(\frac{F}{\alpha}\right)^{n+1}\det(a_{\mu\nu}), \tag{10}$$

where $\alpha=\sqrt{(y^1)^2+(y^2)^2}$ and $a_{\mu\nu}$ is the metric of α. One should notice that F equals constant 1 in the indicatrix I_x^2M; the determinant g in polar coordinates is

$$g=\left(\frac{1}{r}\right)^3. \tag{11}$$

Thus, the length L is of the form

$$\begin{aligned}L&=\int_0^{2\pi_E}\frac{1}{\sqrt{1+\lambda\cos\phi}}\mathrm{d}\phi\\&=\frac{4}{\sqrt{1+\lambda}}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-\dfrac{2\lambda}{1+\lambda}\sin^2\theta}},\end{aligned} \tag{12}$$

where $\theta = \phi/2$. In fact, the second integral in the above equation is a complete elliptic integral of the first kind. Here, we can see from the integral (12) that the length L equals $2\pi_{\rm E}$ only if λ takes the value 0. Since the modification on Newtonian gravity is tiny, we suppose that λ is very small. Hence, to second order in λ, the length can be derived as

$$\begin{aligned} L &= \int_0^{2\pi_{\rm E}} \left(1-\frac{\lambda}{2}\cos\phi+\frac{3\lambda^2}{8}\cos^2\phi\right) \\ &= 2\pi_{\rm E}\left(1+\frac{3\lambda^2}{16}\right). \end{aligned} \tag{13}$$

If one sets $\lambda = 0.001$, the numerical result of $L/2\pi - 1$ is 1.875×10^{-7}. This result is consistent with the approximate expression (13). The equation (13) tells us that the value of π in Finsler geometry is

$$\pi_{\rm F} = \pi_{\rm E}\left(1+\frac{3\lambda^2}{16}\right). \tag{14}$$

At the end of this section, we must point out that the length

$$L = \int_a^b \left(\sqrt{g_{\mu\nu}\frac{{\rm d}y^\mu}{{\rm d}t}\frac{{\rm d}y^\nu}{{\rm d}t}}\right){\rm d}t$$

of the indicatrix I_xM is different from

$$\int_a^b F\left(x, \frac{{\rm d}x}{{\rm d}t}\right){\rm d}t$$

defined in Finsler space M. However, the value of π derived from the length element $\sqrt{g_{\mu\nu}{\rm d}y^\mu{\rm d}y^\nu}$ is still suitable for a Finsler space of Landsberg type. The Gauss-Bonnet theorem [23] in a Finsler space of Landsberg type reads:

$$\frac{1}{L}\int_M K\sqrt{g}{\rm d}x^1\wedge{\rm d}x^2 = \chi(M), \tag{15}$$

where (M,F) is a compact connected Landsberg surface, K is the Gaussian curvature of the Finsler surface, $\chi(M)$ is the Euler characteristic of M and L is the length of indicatrix I_xM. The Euler characteristic $\chi(M)$ of S^2 equals 2; by making use of the Gauss-Bonnet theorem (15), we obtain that

$$\int_{S^2} K\sqrt{g}{\rm d}x^1\wedge{\rm d}x^2 = 2L = 4\pi_{\rm F}. \tag{16}$$

Therefore, the volume of the surface of a unit sphere on a Landsberg surface is $4\pi_{\rm F}$. This means that the $\pi_{\rm F}$ derived from indicatrix I_xM is still vaild on a Landsberg surface. The Finsler structure $F = \sqrt{(y^1)^2+(y^2)^2}+\lambda y^1$ we used is of Landsberg type, with λ constant. Ref. [21] gave a definition of the Landsberg space. A Finsler structure $F = \sqrt{(y^1)^2+(y^2)^2}+\lambda y^1$ is of Landsberg type, if λ is a constant.

We give a detailed discussion about the volume on a Finsler space in the appendix.

3 The Sun-planet system and Earth-Moon system

In Newtonian gravity, the orbital angular momentum of a planet is conserved. In other words, Kepler's second law is valid. The line joining a planet and the Sun sweeps out equal areas over equal intervals of time,

$$\frac{{\rm d}A}{{\rm d}t} = J/2m.$$

Where A is the areas, J is the orbital angular momentum and m is the mass of the planet. However, a report from Krasinsky and Brumberg [4] implies that over equal intervals of time the area swept out by the line joining a planet and the Sun increases. Unlike the explanation of Miura [10], we attribute this phenomenon to the different value of π in Finsler geometry. In a Finsler space of Randers type, the area S of a disk with boundary $(F=R)$ is given as [23]

$$S = \int_{F=R}\sqrt{g}{\rm d}x^1\wedge{\rm d}x^2. \tag{17}$$

The above section shows that the value of π should be modified in Finsler spacetime. Since the deformation parameter λ is small enough, the formula (17) can have the following form

$$S \approx \pi_{\rm F}R^2. \tag{18}$$

Then, the difference between the areas of disks in Riemann geometry and Finsler geometry is

$$\delta A \equiv \pi_{\rm F}R^2 - \pi_{\rm E}R^2 = \frac{3\lambda^2}{16}\pi_{\rm E}R^2. \tag{19}$$

On the other hand, the result of Krasinsky and Brumberg means Kepler's second law is violated. Since the anomaly effect is very small, the angular momentum J can be taken approximately as $J \approx m\sqrt{GM_\odot R}$ (derived in Newtonian gravity). Then, we get

$$\frac{{\rm d}^2A}{{\rm d}t^2} = \frac{\dot{J}}{2m} = \frac{\sqrt{GM_\odot/R}}{4}\dot{R}, \tag{20}$$

where the dot denotes derivative respect to time and $M_\odot$ is the mass of the Sun. Hence, the increased area of the disk in one orbital period of a planet is

$$\delta A = \frac{1}{2}\frac{{\rm d}^2A}{{\rm d}t^2}T^2 = \left(\frac{\pi_{\rm E}\dot{R}}{2\sqrt{GM_\odot/R}}\right)\pi_{\rm E}R^2, \tag{21}$$

where

$$T = 2\pi\sqrt{\frac{R^3}{GM_\odot}}$$

is the orbital period of the planet and we used the Kepler's third law to get the second equality of (21).

The secular trend of AU means the area swept out by the line joining a planet and the Sun is increasing over equal intervals of time. The length difference between Finsler space and Riemannian space may relate to this phenomenon. If an astronomical measurement is carried out in the framework of Finsler geometry, this anomaly should vanish. Combining the equations of (19) and (21), we obtain

$$\lambda = \sqrt{\frac{8\pi\dot{R}}{3}\sqrt{\frac{R}{GM_\odot}}} = \sqrt{\frac{4\dot{R}}{3R}T}. \tag{22}$$

In the second equality of the above equation, we used Kepler's third law as an approximation. Here, by taking the average value of $\mathrm{d}AU/\mathrm{d}t$, we list the values of the λ for each planet of the solar system in Table 1.

Table 1. Values of the semi-major axis of planetary orbits $a_{\rm PL}$ and orbital periods of planets $T_{\rm PL}$ given in Ref. [3]. The space deformation parameter λ, referring to the inner planets, is listed.

planets	$a_{\rm PL}(AU)$	$T_{\rm PL}$/years	$\lambda(10^{-6})$
Mercury	0.38709893	0.240840253	0.910799787
Venus	0.72333199	0.615171854	1.064875317
Earth	1.00000011	1.0	1.154700475
Mars	1.52366231	1.880815968	1.282915811

The values of λ given in Table 1 are very close for the inner planets. One should note that the analyzed data of distances [4] are in the range of the inner planets and Martian landers and orbiters. This fact implies that the space deformation parameter λ should be a constant in the solar system. Calculating the average value of λ listed in Table 1, we set it as the value of the constant parameter $\lambda = 1.10\times 10^{-6}$.

A recent orbital analysis of Lunar Laser Ranging (LLR) [24] shows an anomalous secular eccentricity variation of the Moon's orbit $(0.9\pm 0.3)\times 10^{-11}$/yr, equivalent to an extra 3.5 mm/yr in perigee and apogee distance [25]. By supposing the variation of the distance from the center to the focus and the semi-major axis is the same for the Moon's orbit, namely $\delta a = \delta c$, we obtain

$$\delta a = \frac{a\delta e}{1-e}. \tag{23}$$

Here, a denotes the semi-major axis and e denotes the eccentricity. By making use of equation (23) and the observation data of LLR, we obtain the secular variation of Moon's orbital semi-major axis as

$$\delta a = 3.62\pm 1.20 \text{ mm/yr}. \tag{24}$$

Under the premise that λ is constant, and by making use of Equation (22), we obtain the secular variation of Moon's orbital radius $\dot{R}_{\rm M} = 4.66$ mm/yr. This result is consistent with Formula (24). Thus, our hypothesis that the parameter λ is constant is supported by the observation of LLR.

The uniform space deformation means that the secular trend of planetary orbits is $\dot{R}_{\rm Pl} \propto R^{-1/2}$. We list the values of $\dot{R}_{\rm Pl}$ for each planet of the solar system in Table 2. By making use of Equation (23), we list the secular eccentricity variation of each planet in Table 2. We hope this can be tested in future astronomical observations.

Table 2. For the case of uniform space deformation ($\lambda = 1.10\times 10^{-6}$), the secular trend of each planet orbit is listed. The secular eccentricity variation of each planet is also listed.

planets	$\dot{R}_{\rm PL}$/(m/century)	$e_{\rm PL}$	$\delta e_{\rm PL}[10^{-11}/\text{yr}]$
Mercury	21.9	0.206	0.299
Venus	16.0	0.007	0.146
Earth	13.6	0.017	0.089
Mars	11.0	0.093	0.043
Jupiter	5.97	0.048	0.007
Saturn	4.41	0.056	0.003
Uranus	3.11	0.047	0.001
Neptune	2.48	0.009	0.0005
Pluto	2.17	0.250	0.0003

4 The Lorentz violation in Finsler space

Rotational symmetry in Randers space is broken. This is consistent with the phenomena of the secular trend of AU. In recent years, the issue of Lorentz violation (LV) has been reconsidered in light of several different "quantum gravity" (QG) scenarios leading to Lorentz violation [26]. In most QG models, the LV is described in terms of the modified dispersion relations. These modified dispersion relations (MDR) for elementary particles can be cast in the general form

$$\begin{aligned} E^2 &= m^2 + p^2 + D(p,\mu,M) \\ &= m^2 + p^2 + \sum_{n=1}^{\infty}\alpha_n(\mu,M)p^n, \end{aligned} \tag{25}$$

where $p = \sqrt{|\vec{p}|^2}$, μ is a particle mass-scale, α_n are dimensional coefficients and M denotes the relevant QG scale. Girelli, Liberati and Sindoni [27] showed that the MDR can be incorporated into the framework of Finsler geometry. The symmetry of the MDR was described in the Hamiltonian formalism. We have shown that the symmetry of Randers space gives rise to a modified dispersion relation with characteristics of Lorentz Invariance violation [19].

However, studying modifications of particle dispersion relations suffers from some drawbacks [28]. One is that MDR can only describe changes in the free propagation of particles. The other one is certain choices of MDR which may be unphysical. The MDR $p^\mu p_\mu = m^2 + a_\mu p^\mu$ gives Lorentz-violating properties that depend on the preferred vector a_μ, but in fact they are unobservable because a_μ can be eliminated via a physically irrelevant redefinition of energy and momentum. Also, in the application of the effective theory a_μ can be eliminated by introducing a field redefinition by a phase $\exp(\mathrm{i}a\dot{x})$ [29].

We must point out that the MDR given in the above paragraph is the same as the MDR [19] in Randers space. However, the LV effect of Randers space is indicated in this paper. This is similar to the famous Aharonov–Bohm effect: a unphysical electromagnetic four-potential at the classical level has observable effect at the quantum level. The unphysical MDR involved in Randers space may have an observable effect. It may be the source of the secular trend of AU. Our approach approximately explains the secular trend of AU and the anomalous secular eccentricity variation of the Moon's orbit.

One should note that the present experiment constraint on LV is carried out in a small area (or small scale). Large-scale observations of indicating or constraining the LV are very few. The secular trend of AU and the anomalous secular eccentricity variation of the Moon's orbit are two large-scale constraints on LV, and can be explained in the framework of Finsler geometry.

The direct and most efficient way to describe the LV is to investigate the isometry group of the Randers metric. A detailed investigation on the symmetry of Randers space and the LV is given in our work [30].

We would like to thank Prof. Z. Shen C. J. Zhu, H. Y. Guo and C. G. Huang for useful discussions.

Appendix A

Volume on a Finsler space

In this paper, we have used the modified angle π_{F} to calculate the volume of a unit disc. One may ask why the volume of physical space can be described by the modified angle π_{F}, which is deduced from the unit tangent sphere or the indicatrix. In this appendix, we present a brief introduction to the concept of volume in a Finsler space. This can help to solve the confusion mentioned above.

First of all, there are two canonical volume forms on a Finsler space. Both reduce to the Riemannian volume form when the Finsler metric becomes Riemannian [31]. The first one is the Hausdorff volume. It is of the form

$$\mathrm{d}V_{\mathrm{F}} \equiv \sigma_{\mathrm{F}}(x)\theta^1\wedge\cdots\wedge\theta^n, \tag{A1}$$

where

$$\sigma_{\mathrm{F}}(x) \equiv \frac{\mathrm{Vol}(\mathbb{B}^n)}{\mathrm{Vol}(B^n_x)}$$

and θ is the basis for cotangent space T^*_xM. Here, "Vol" denotes the Euclidean volume, $\mathbb{B}$ is an n-dimensional unit sphere and B^n_x is a bounded open strongly convex open subset in $\mathbb{R}^n$. To study the volume on a hypersurface, one should involve a co-area formula to set a relation between the induced volume on the hypersurface and the volume on the manifold (for a strict mathematical description, see, for example, Ref. [31]). While the Finsler structure is of Randers type (9), the unit circle in the Finsler manifold M can be described as

$$(1-\lambda^2)^2\left(y^1+\frac{\lambda}{1-\lambda^2}\right)^2+(1-\lambda^2)(y^2)^2=1. \tag{A2}$$

Reparameterizing the equation for the unit circle, we get

$$y^1=\frac{1}{1-\lambda^2}\cos\phi-\frac{\lambda}{1-\lambda^2},\quad y^2=\frac{1}{\sqrt{1-\lambda^2}}\sin\phi. \tag{A3}$$

Then, $F\left(\frac{\mathrm{d}y^1}{\mathrm{d}t},\frac{\mathrm{d}y^2}{\mathrm{d}t}\right)$ is of the form

$$F\left(\frac{\mathrm{d}y^1}{\mathrm{d}t},\frac{\mathrm{d}y^2}{\mathrm{d}t}\right) = \sqrt{\left(\frac{\lambda}{1-\lambda^2}\right)^2\sin^2\phi+\frac{1}{1-\lambda^2}} -\frac{\lambda}{1-\lambda^2}\sin\phi. \tag{A4}$$

Thus, the volume of the indicatrix of the unit circle in Randers space (9) induced by the Hausdorff volume form is given as

$$\begin{aligned}\mathrm{Vol}_{\mathrm{F}}(M) &= \int_0^{2\pi_{\mathrm{E}}}\sqrt{\left(\frac{\lambda}{1-\lambda^2}\right)^2\sin^2\phi+\frac{1}{1-\lambda^2}}\,\mathrm{d}\phi\\ &\simeq 2\pi_{\mathrm{E}}\left(1+\frac{3}{4}\lambda^2\right).\end{aligned} \tag{A5}$$

Here, we have used the small λ approximation to get the second equation of (A5).

The second volume form is the one we introduced in section 2. One can find that the result for two different volume forms are almost the same. If we use the Hausdorff volume to describe the physics discussed above, we will obtain a similar result. Therefore, we can reasonably suggest that the secular trend of planetary orbits and secular eccentricity variation of the Moon's orbit both have a Finslerian origin. The only difference is that the value of geometrical parameter λ used is one half of ours.

In mathematics, the second volume form is used to give a definition of angles on a metric space. Thus, we adopt the modified $\pi = \pi_{\rm F}$ to describe the difference between the Riemannian volume and Finslerian volume. Using of $\pi_{\rm F} R^2$ to describe the area of a disc is acceptable, since we have already supposed that the deviation of Finsler space from Riemann space is very small.

References

1 Riess A G et al. Astrophys J., 1999, **117**: 707; Perlmutter S et al. Astrophys J., 1999, **517**: 565; Bennett C L et al. Astrophys J., 2003, **148**(Suppl.): 1

2 Trimble V T. Annu. Rev. Astron. Astrophys., 1987, **25**: 425

3 Pitjeva E V. Solar System Research, 2005, **39**: 176

4 Krasinsky G A, Brumberg V A. Celest. Mech. Dyn. Astrn., 2004, **90**: 267

5 Standish E M. Proc. IAU Colloq., 2005, **196**: 163

6 Mashhoon B, Mobed N, Singh D. Class. Quant. Grav., 2007, **24**: 5031

7 Arakida H. New Astron., 2009, **14**: 275

8 Noerdlinger P D. arXiv:astro-ph/0801.3807

9 Arakida H. arXiv:astro-ph/0810.2827

10 Miura T et al. arXiv:astro-ph.EP/0905.3008

11 Anderson J D et al. Phys. Rev. Lett., 1998, **81**: 2858; Anderson J D et al. Phys. Rev. D, 2002, **65**: 082004; Anderson J D et al. Mod. Phys. Lett. A, 2002, **17**: 875

12 Roxburgh I W, Tavakol R. Mon. Not. Roy. Acad. Sc., 1975, **170**: 599

13 Roxburgh I W, Tavakol R K. Gen. Rel. Grav., 1979, **10**: 307

14 Tavakol R K, van den Bergh N. Phys. Lett. A, 1985, **112**: 23

15 Tavakol R K, van den Bergh N. Gen. Rel. Grav., 1986, **18**: 849

16 CHANG Zhe, LI Xin. Phys. Lett. B, 2009, **676**: 173

17 CHANG Zhe, LI Xin. Phys. Lett. B, 2008, **668**: 453

18 Randers G. Phys. Rev., 1941, **59**: 195

19 CHANG Zhe, LI Xin. Phys. Lett. B, 2008, **663**: 103

20 LI Xin, CHANG Zhe. Phys. Lett. B, 2010, **692**: 1

21 BAO D, Chern S S, SHEN Z. An Introduction to Riemann–Finsler Geometry, Graduate Texts in Mathematics **200**, New York: Springer, 2000

22 BAO D, SHEN Z. Results in Math., 1994, **26**: 1

23 BAO D, Chern S S. Ann. Math., 1996, **143**: 233

24 Williams J G, Boggs D H. in Proceedings of 16th International Workshop on Laser Ranging ed. S. Schillak, (Space Research Centre, Polish Academy of Sciences), 2009

25 Anderson J D, Nieto M M. arXiv:gr-qc/0907.2469

26 Liberati S, Maccione L. arXiv:astro-ph.HE/0906.0681

27 Girelli F, Liberati S, Sindoni L. Phys. Rev. D, 2007, **75**: 064015

28 Kostelecky V A. arXiv:gr-qc/0802.0581

29 Colladay D, Kostelecky V A. Phys. Rev. D, 1997, **55**: 6760; Phys. Rev. D, 1998, **58**: 116002

30 LI Xin, CHANG Zhe, MO Xiao-Huan. arXiv:hep-th/1001.2667

31 SHEN Z. Lectures on Finsler geometry, Singapore: World Scientific, 2001