Quantum tunnelling of higher-dimensional Kerr-anti-de Sitter black holes beyond semi-classical approximation*

LIU Wei-Wei(刘伟伟)[1)] LUO Zhi-Quan(罗志全) YANG Juan(杨娟) BIAN Gang(边刚)

Institute of Theoretical Physics, China West Normal University, Nanchong 637002, China

Abstract: Based on the theory of Klein-Gordon scalar field particles, the Hawking radiation of a higher-dimensional Kerr-anti-de Sitter black hole with one rotational parameter is investigated using the beyond semi-classical approximation method. The corrections of quantum tunnelling probability, Hawking temperature and Bekenstein-Hawking entropy are also included.

Key words: beyond semi-classical approximation, higher-dimensional Kerr-anti-de Sitter black hole, hawking radiation, quantum tunnelling, modified entropy

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1 Introduction

Since Hawking proved that a black hole emits thermal radiation [1, 2], various methods have been adopted by researchers to study Hawking radiation, and these studies have had a positive impact on the understanding and exploration of the basic properties of black holes. Recently, Kraus, Parikh and Wilczek et al put forward a semi-classical approximate tunnelling method to study the Hawking radiation of black holes [3, 4]. They thought that a virtual particle situated inside the horizon of black hole tunnels to outside and becomes a real particle, then radiates to infinity. The essentials of this method were to use a dynamic mode to deal with the Hawking radiation of black holes. In this method, Hawking radiation was viewed as a tunnelling process, and one can explain the generation mechanism of Hawking radiation via the effect of quantum tunnelling and, then, using this method, the researchers studied a variety of exotic space-time [5–8]. In 2007, Kerner and Mann first adopted the tunnelling method to study the Hawking radiation of 1/2 spin uncharged particles [9, 10]. Since then, many researchers have studied the tunnelling behavior of various types of black holes using this new method. These made a great contribution to the further study of black holes [11–13]. However, because the higher order items of $\hbar$ was neglected, the previous work only achieved an approximate result. In 2008, Banerjee and Majhi extended the case of semi-classical approximation to the case beyond semi-classical approximation in which the higher order correction items are included. Finally, the quantum tunnelling method beyond semi-classical approximation was put forward [14–19]. Furthermore, K. Lin et al improved the method of beyond semi-classical approximation to make it applicable to a wider range. They also made a further study of the tunnelling radiation characteristics of black holes and some significant modified results were obtained [20, 21].

In this paper, we study a higher-dimensional exotic space-time with this improved method beyond semi-classical approximation. Until now, no-one has used this new method to study higher-dimensional black holes. It seems important then to extend this new quantum tunnelling method to higher-dimensional space-time. We use this improved method to investigate the Hawking radiation of higher-dimensional Kerr-anti-de Sitter black holes with one rotational parameter and obtain the corrections of quantum tunnelling probability, Hawking temperature and Bekenstein-Hawking entropy.

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1) E-mail: juliet-0915@163.com

2 Quantum tunnelling in a n-dimensional Kerr-anti-de Sitter black hole

In some modern physics theories, the concept of extra dimensions can help to resolve several theoretical issues, so the theory of higher-dimensional black holes in curved space-time was put forward [22–25]. According to Refs. [26–28], in curved space-time, the metric of a n-dimensional Kerr-anti-de Sitter black hole with one rotational parameter is given by

$$\begin{aligned} ds^2 =& -\frac{\Delta_r}{\rho^2}\left(dt-\frac{a}{\Xi}\sin^2\theta d\phi\right)^2+\frac{\rho^2}{\Delta_r}dr^2+\frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &+\frac{\Delta_\theta\sin^2\theta}{\rho^2}\left[a dt-\frac{r^2+a^2}{\Xi}d\phi\right]^2+r^2\cos^2\theta d\Omega_{n-4}^2, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \rho^2 &= r^2+a^2\cos^2\theta, \Xi=1-a^2l^{-2}, \\ \Delta_r &= (r^2+a^2)(r^2l^{-2}+1)-2Mr^{5-n}, \\ \Delta_\theta &= 1-a^2l^{-2}\cos^2\theta, \end{aligned} \tag{2}$$

in which, M, l, a are the mass, inverse cosmological constant and angular momentum respectively, and $d\Omega_{n-4}^2$ represents the standard metric of the $(n-4)$-dimensional sphere. The event horizon r_+ of this black hole can be valued by the equation $\Delta_r(r_+)=0$, and the non-zero inverse metric of this black hole is written as

$$\begin{aligned} g^{tt} &= \frac{a^2\sin^2\theta}{\rho^2\Delta_\theta}-\frac{(r^2+a^2)^2}{\rho^2\Delta_r}, \\ g^{\phi\phi} &= \frac{\Xi^2}{\rho^2\Delta_\theta\sin^2\theta}-\frac{\Xi^2a^2}{\rho^2\Delta_r}, \\ g^{t\phi} &= \frac{\Xi a(r^2+a^2)}{\rho^2\Delta_r}-\frac{a\Xi}{\rho^2\Delta_\theta}, \\ g^{rr} &= \frac{\Delta_r}{\rho^2},\ g^{\theta\theta}=\frac{\Delta_\theta}{\rho^2}, \\ g^{\tau\tau} &= r^{-2}\cos^{-2}\theta h^{\tau\tau}. \end{aligned} \tag{3}$$

where, $g^{\tau\tau}$ is expressed as the inverse metric of extra-dimensional terms, and $h^{\tau\tau}$ only depends on the extra-dimensional coordination. Next, the determinant of this metric is obtained as

$$g=-\frac{\rho^4r^2\sin^2\theta\cos^2\theta}{\Xi^2h^{\tau\tau}}. \tag{4}$$

In curved space-time, the Klein-Gordon equation describing the motion of scalar particles with the mass m is given by

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\ g^{\mu\nu}\frac{\partial\Phi}{\partial x^\nu}\right)-\frac{m^2}{\hbar^2}\Phi=0. \tag{5}$$

Inserting Eq. (3) into Eq. (5), at the same time, because the behavior of the particles' radiation is radial, for the purpose of separating the radial equation, one can suppose a spherical waving function as follows

$$\Phi=R(r)Y(\theta)N(\tau)e^{-\frac{i}{\hbar}(\omega t-j\phi)}, \tag{6}$$

here, ω is the energy of radiant particles, and j is the angular momentum corresponding to ϕ. $R(r)$, $Y(\theta)$ and $N(\tau)$ are the terms representing generalized momentum. Putting Eq. (6) into Eq. (5), we have

$$\begin{aligned} &\frac{\omega^2}{\hbar^2}\frac{(r^2+a^2)^2}{\Delta_r}+\frac{2j\omega}{\hbar^2}\frac{\Xi a(r^2+a^2)}{\Delta_r}+\frac{1}{R}\frac{\partial R}{\partial r}\frac{\partial\Delta_r}{\partial r} \\ &+\frac{\Delta_r}{r}\frac{1}{R}\frac{\partial R}{\partial r}+\frac{\Delta_r}{R}\frac{\partial^2R}{\partial r^2}+\frac{j^2}{\hbar^2}\frac{\Xi^2a^2}{\Delta_r} \\ &-\frac{r^2m^2}{\hbar^2}-\frac{\lambda a^2}{r^2}=\eta, \end{aligned} \tag{7}$$

$$\begin{aligned} &\frac{\omega^2}{\hbar^2}\frac{a^2\sin^2\theta}{\Delta_\theta}+\frac{2j\omega}{\hbar^2}\frac{\Xi a}{\Delta_\theta}-\frac{1}{Y}\frac{\partial Y}{\partial\theta}\frac{\partial\Delta_\theta}{\partial\theta}\frac{\partial(\sin\theta\cos\theta)}{\partial\theta} \\ &-\frac{\Delta_\theta}{\sin\theta\cos\theta}\frac{1}{Y}\frac{\partial Y}{\partial r}-\frac{\Delta_\theta}{Y}\frac{\partial^2Y}{\partial\theta^2}+\frac{j^2}{\hbar^2}\frac{\Xi^2}{\Delta_\theta\sin^2\theta} \\ &+\frac{a^2\cos^2\theta m^2}{\hbar^2}+\frac{\lambda}{\cos^2\theta}=\eta, \end{aligned} \tag{8}$$

$$-\hbar^{\tau\tau}\frac{1}{N}\frac{\partial^2N}{\partial\tau^2}=\lambda, \tag{9}$$

in which, λ and η are the constants, and Eqs. (7)–(9) are the radial equation, the angular equation and the extra-dimensional equation of this black hole, respectively. Similarly, Hawking radiation is the radial behavior of black holes, so we are only interested in the radial equation. Simplifying Eq. (7), we get

$$\begin{aligned} &\frac{\Delta_r}{a^2}\frac{\partial^2R}{\partial r^2}+\left(\frac{1}{a^2}\frac{\partial\Delta_r}{\partial r}+\frac{1}{a^2}\frac{\Delta_r}{r}\right)\frac{\partial R}{\partial r} \\ &+\left[\left(\frac{\omega}{a\hbar}\frac{r^2+a^2}{\sqrt{\Delta_r}}+\frac{j}{\hbar}\frac{\Xi}{\sqrt{\Delta_r}}\right)^2\right. \\ &\left.-\frac{r^2m^2}{a^2\hbar^2}-\frac{\lambda}{r^2}-\frac{\eta}{a^2}\right]R=0. \end{aligned} \tag{10}$$

In order to simplify Eq. (10), the tortoise coordinate transformation is adopted as follows

$$\frac{\partial}{\partial r}=\frac{a^2}{\Delta_r}\frac{\partial}{\partial r_*}, \tag{11}$$

$$\frac{\partial^2}{\partial r^2}=-\frac{a^2}{\Delta_r^2}\frac{\partial\Delta_r}{\partial r}\frac{\partial}{\partial r_*}+\frac{a^4}{\Delta_r^2}\frac{\partial}{\partial r_*^2}. \tag{12}$$

Putting Eqs. (11)–(12) into Eq. (10), for $\Delta_r\rightarrow0$ near the event horizon, we find

$$\left[\frac{K^2}{\hbar^2}-\Delta_r\frac{\partial}{\partial r}\left(\Delta_r\frac{\partial}{\partial r}\right)\right]R=0, \tag{13}$$

where

$$K=(a^2+r^2)\omega+a\Xi j=(a^2+r^2)(\omega-A_{\rm t}j). \quad (14)$$

In the above equation, $A_{\rm t}=-a\Xi/(a^2+r^2)$ is the angular velocity of this black hole. On the other hand, the radial wave function of the emission spherical shell should be written as

$$R\sim {\rm e}^{\frac{\rm i}{\hbar}S(r)}, \quad (15)$$

From Ref. [14], we have $S(r,t)=S(r)+Kt$, K is the part relating to the energy of the particle. Through WKB approximation, expanding the $S(r)$ and K in powers of $\hbar$, we get

$$S(r) = S_0(r)+\hbar S_1(r)+\hbar^2 S_2(r)+\cdots, \quad (16)$$

$$K = K_0+\hbar K_1+\hbar^2 K_2+\cdots, \quad (17)$$

where, $S_0(r)$ is the semi-classical part of $S(r)$, and K_0 is the semi-classical part of K. From Eq. (19) in Ref. [14], we can also find that our work differs from Ref. [14] where the action $S(r,t)$ is expanded, but the essence is the same. This is because the time variable part of the action $S(r,t)$ is calculated earlier [14, 21]. Then substitute Eqs. (15)–(17) into Eq. (13) and, according to the different powers of $\hbar$, it can be separated into

$$\hbar^0: \frac{K_0^2}{\Delta_r}-\Delta_r\left(\frac{\partial S_0}{\partial r}\right)^2=0, \quad (18)$$

$$\hbar^1: \frac{2K_0K_1}{\Delta_r}+{\rm i}\frac{\partial}{\partial r}\left(\Delta_r\frac{\partial S_0}{\partial r}\right)-2\Delta_r\frac{\partial S_0}{\partial r}\frac{\partial S_1}{\partial r}=0. \quad (19)$$

Simplifying Eq. (18), we obtain

$$\frac{\partial S_0}{\partial r}=\pm\frac{K_0}{\Delta_r}. \quad (20)$$

Reforming Eq. (19) and taking Eq. (18) into account, we have

$$\frac{\partial S_1}{\partial r}=\pm\frac{K_1}{\Delta_r}. \quad (21)$$

Similarly, we can obtain all relationships about S_i and S_0. We then note that any S_i is always proportional to S_0. Thus the action $S(r)$ beyond semi-classical approximation can be rewritten as (in units of $G=c=k_B=1$)

$$\begin{aligned} S(r) &= S_0(r)+\sum_i\beta_i\frac{\hbar^{i-1}}{S^i_{\rm BH}}S_0(r) \\ &= \left(1+\sum_i\beta_i\frac{\hbar^{i-1}}{S^i_{\rm BH}}\right)S_0(r), \end{aligned} \quad (22)$$

here, β_i is an undetermined parameter. It is clear that we only need to resolve semi-classical approximate action S_0 to get action S. From Eq. (20), the imaginary parts of action S_0 are

$$\begin{aligned} {\rm Im}S_{0\pm} &= \pm{\rm Im}\int\frac{K_0}{\Delta'_r(r-r_+)}{\rm d}r=\pm\frac{\pi K_0}{\Delta'_r(r_+)} \\ &= \pm\frac{\pi(\omega_0-A_{\rm t}(r_+)j_0)(a^2+r_+^2)}{\Delta'_r(r_+)}, \end{aligned} \quad (23)$$

where, $\Delta'_r=\dfrac{\partial\Delta_r}{\partial r}$, the signs $(+,-)$ denote the outgoing and ingoing solutions of semi-classical approximate action respectively. We can get the quantum tunnelling probability beyond semi-classical approximation at the event horizon of this black hole as

$$\begin{aligned} \Gamma_{\rm h} &\propto \exp(-2{\rm Im}S) \\ &= \exp\left[\left(1+\sum_i\beta_i\frac{\hbar^{i-1}}{S^i_{\rm BH}}\right)\frac{-4\pi K_0}{\Delta'_r(r_+)}\right] \\ &= \exp\left[\left(1+\sum_i\beta_i\frac{\hbar^{i-1}}{S^i_{\rm BH}}\right)\frac{-4\pi(\omega_0-A_{\rm t}(r_+)j_0)}{\Delta'_r(r_+)/(a^2+r_+^2)}\right]. \end{aligned} \quad (24)$$

The modified Hawking temperature is

$$\begin{aligned} T_{\rm h} &= \frac{\Delta'_r}{4\pi(a^2+r_+^2)}\left(1+\sum_i\beta_i\frac{\hbar^{i-1}}{S^i_{\rm BH}}\right)^{-1} \\ &= T_{\rm H}\left(1+\sum_i\beta_i\frac{\hbar^{i-1}}{S^i_{\rm BH}}\right)^{-1}, \end{aligned} \quad (25)$$

in which, $T_{\rm H}=\Delta'_r/4\pi(a^2+r_+^2)$ is the usual semi-classical Hawking temperature. Next, the correctional entropy of this black hole will be investigated via the first law of black hole thermodynamics. The form of the first law of black hole thermodynamics is stated as

$${\rm d}M=T_{\rm h}{\rm d}S_{\rm bh}+\Omega{\rm d}J+V_0{\rm d}Q, \quad (26)$$

where, Ω, J, V_0 and Q are the angular momentum, the angular velocity, the electromagnetic potential and the electric charge of black holes, respectively. The correctional entropy of the higher-dimensional Kerr-anti-de Sitter black hole with one rotational parameter in differential forms is given by

$${\rm d}S_{\rm bh}=\frac{1}{T_{\rm h}}({\rm d}M-\Omega{\rm d}J), \quad (27)$$

Integrating the equation above at the event horizon of this black hole, the correctional entropy becomes

$$\begin{aligned} S_{\rm bh} &= \int{\rm d}S_{\rm bh}=\int\frac{1}{T_{\rm h}}({\rm d}M-\Omega{\rm d}J) \\ &= S_{\rm BH}+\beta_1\ln S_{\rm BH}+{\rm const.} \end{aligned} \quad (28)$$

where, $S_{\rm BH}$ is the usual semi-classical Bekenstein-Hawking entropy, and the other terms are the correction terms. In Eq. (28), the leading correction term is the logarithmic correction and it contains a undetermined parameter β_1 which can be ascertained by the theory of loop quantum gravity. Because the parameter β_1 expanded with $\hbar$ is equal to one determined by one loop quantum gravity, they should have the same related parameter β_1, that is $\beta_1 = -n/2(n-2)$ [20, 29]. From this point of view, this improved method beyond semi-classical approximation is reliable. After neglecting all of the correction items, only $S_{\rm BH}$ in Eq. (28) is left, so the resulted entropy returns to the case of semi-classical approximation. The whole presentation of Bekenstein-Hawking entropy is Eq. (28), which is the modified entropy.

3 Conclusions

In this paper, using the Klein-Gordon equation to describe the movement of scalar particles and the improved method beyond semi-classical approximation, the corrections of quantum tunnelling probability, Hawking temperature and Bekenstein-Hawking entropy from the higher-dimensional Kerr-anti-de Sitter black hole with one rotational parameter are obtained. The method of beyond semi-classical approximation that takes the higher order items of $\hbar$ into account is a new method of studying quantum tunnelling more accurately. When all higher-order correction terms are neglected, we can obtain the semi-classical approximate quantum tunnelling probability, the Hawking temperature and the Bekenstein-Hawking entropy. The improved method of beyond semi-classical approximation is also successful on a larger scale and we can continue to study quantum tunnelling behavior from higher-dimensional charged black holes and other higher-dimensional non-stationary black holes using this method. Therefore the whole presentation of quantum tunnelling probability, Hawking temperature and Bekenstein-Hawking entropy of these black holes can be derived. Finally, the undetermined parameter β_1 can be ascertained not only by the theory of loop quantum gravity, but also by the trace anomaly. This is because the undetermined parameter is related to the trace anomaly. What is more, other more advanced theories and methods are required with the aim of obtaining a more accurate parameter β_1.

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