Spin-1/2 relativistic particle in a magnetic field in NC phase space *

YUAN Yi(袁毅)[1] LI Kang(李康)[2;1)] WANG Jian-Hua(王剑华)[1;2)] CHEN Chi-Yi(陈驰一)[2]

[1] Shaanxi University of Technology, Hanzhong 723001, China

[2] Hangzhou Teachers University, Hangzhou 310036, China

Abstract This work provides an accurate study of the spin-1/2 relativistic particle in a magnetic field in NC phase space. By detailed calculation we find that the Dirac equation of the relativistic particle in a magnetic field in noncommutative space has similar behaviour to what happens in the Landau problem in commutative space even if an exact map does not exist. By solving the Dirac equation in NC phase space, we not only obtain the energy level of the spin-1/2 relativistic particle in a magnetic field in NC phase space but also explicitly offer some additional terms related to the momentum-momentum non-commutativity.

Key words NC phase space, Landau problem, Dirac equation

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1 Introduction

There are many papers devoted to the study of various aspects of quantum mechanics in NC space with usual time coordinate. For example, string theory in nontrivial backgrounds was studied in Ref. [1] and noncommutative field theories related to M-theory compactification was studied in Ref. [2]. Besides these, Ref. [3] supported that inclusion of noncommutativity in quantum field theory can be achieved in two different ways: via Moyal $\star$-product on the space of ordinary functions, or defining the field theory on a coordinate operator space which is intrinsically noncommutative. Refs. [4–10] declared that a simple insight on the role of noncommutativity in field theory can be obtained by studying the one particle sector, which prompted an interest in the study of noncommutative quantum mechanics (NCQM). Refs. [11–20] studied the topological phase and energy level AC effects in NC phase, which paid some attention to two-dimensional NCQM and its relation to the Landau problem. And Ref. [21] showed the equation of motion of a particle in a constant magnetic field and in the lowest Landau level. However, almost no job related to NC phase space has been done until now.

In this paper we will do something related to NC phase space. Especially, we will focus on the spin-1/2 relativistic particle and study it in a magnetic field in NC phase space. To be clear, we plan to organize the work as follows: in the next part we discuss the Dirac equation of the spin-1/2 relativistic particle in a magnetic field in commutative space. In Section 3 we study the Dirac equation in NC space. In Section 4, by solving the Dirac equation we deduce the energy level of the particle in a magnetic field in NC phase space. A summary is given in the last section.

2 Dirac equation of the charged particle in magnetic field

In this section we mainly study the charged particle in magnetic field in relativistic circumstances. As we all know that in the stationary state Dirac equation of the charged particle in commutative space can be defined by the following [22],

$$\left[c\vec{\alpha}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)+\beta mc^2\right]\psi=E\psi, \tag{1}$$

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1) E-mail: kangli@hztc.edu.cn

2) E-mail: jianhua.wang@263.net

where

$$\psi(\vec{r})=\begin{pmatrix}\psi_{\mathrm{a}}\ (\vec{r})\\ \psi_{\mathrm{b}}\ (\vec{r})\end{pmatrix},\ \vec{\alpha}=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix},\ \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}. \quad (2)$$

A straightforward calculation leads to the following two simultaneous equations

$$c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\psi_{\mathrm{b}}(\vec{r})=(E-mc^2)\psi_{\mathrm{a}}(\vec{r}), \quad (3)$$

$$c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\psi_{\mathrm{a}}(\vec{r})=(E+mc^2)\psi_{\mathrm{b}}(\vec{r}). \quad (4)$$

Here, in Eq. (4) $\psi_{\mathrm{b}}(\vec{r})$ is the small component of the wave function, which tends to be zero in the non-relativistic limit. Now inserting Eq. (4) into Eq. (3), we can have

$$\left[c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\right]\left[c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\right]\psi_{\mathrm{a}}(\vec{r})=$$
$$(E^2-m^2c^4)\psi_{\mathrm{a}}(\vec{r}), \quad (5)$$

in which $\vec{\sigma}$ is the Pauli matrices and the large component wave function $\psi_{\mathrm{a}}(\vec{r})$ is a two component spinor, i.e.

$$\psi_{\mathrm{a}}(\vec{r})=\begin{pmatrix}u_1\\ u_2\end{pmatrix}. \quad (6)$$

Since

$$A=\frac{\vec{B}\times\vec{r}}{2}, \quad (7)$$

we can define w_1 as

$$w_1=\frac{eB}{2\ mc}. \quad (8)$$

At this point, in two dimensions with some simple rearrangement and use of familiar properties of the spin matrices we have

$$c^2[(p_x^2+p_y^2)+m^2w_1^2(x^2+y^2)+2mw_1L_z+$$
$$\mathrm{i}\,\sigma_z(p_xp_y-p_yp_x)+\mathrm{i}\,\sigma_z m^2w_1^2(xy-yx)+$$
$$2\sigma_z m\hbar w]\begin{pmatrix}u_1\\ u_2\end{pmatrix}=(E^2-m^2c^4)\begin{pmatrix}u_1\\ u_2\end{pmatrix}. \quad (9)$$

Moreover, in commutative space, by straightforward calculation we obtain the following equation for a particle with spin-$\frac{1}{2}$

$$c^2[(p_x^2+p_y^2)+m^2w^2(x^2+y^2)+2mwL_z]u_1=$$
$$(E^2-m^2c^4-2mc^2\hbar w)u_1 \quad (10)$$

and also the equation for an anti-particle with spin-$\frac{1}{2}$

$$c^2[(p_x^2+p_y^2)+m^2w^2(x^2+y^2)+2mwL_z]u_2=$$
$$(E^2-m^2c^4+2mc^2\hbar w)u_2. \quad (11)$$

These equations are similar to what happens in the Landau problem and is equivalent to a two dimensional relativistic oscillator with additional spin-orbit terms. And the equations have a constant of energy with a different sign for particles and antiparticles. Thus, the energy eigenvalues in both Eqs. (10) and (11) are given by

$$E^2_{n_xn_ym_\ell}=\ 2\ mc^2\hbar w_1(n_x+n_y+1)+$$
$$4\ mc^2w_1\left(\frac{m_\ell\hbar}{2}\pm\frac{\hbar}{2}\right)+m^2c^4. \quad (12)$$

3 Energy level of the Dirac equation in NC space

This section provides a study of the Dirac equation in NC space. As is known in NC space the coordinate $\hat{x}_i$ and momentum $\hat{p}_i$ operators satisfy the following commutation relations

$$[\hat{x}_i,\hat{x}_j]=\mathrm{i}\theta_{ij},\ [\hat{p}_i,\hat{p}_j]=0,\ [\hat{x}_i,\hat{p}_j]=\mathrm{i}\hbar\delta_{ij}. \quad (13)$$

By replacing the normal product with a star product, the Schrödinger equation in commuting space changes to the Schrödinger equation in NC space which reads

$$H(p,x)\star\psi(x)=E\psi(x) \quad (14)$$

where the Moyal-Weyl (or star) product between the two functions is

$$(f\star g)(x)=\mathrm{e}^{\frac{\mathrm{i}}{2}\Theta_{ij}\partial_{x_i}\partial_{x_j}}f(x_i)g(x_j)=$$
$$f(x)g(x)+\frac{\mathrm{i}}{2}\Theta_{ij}\partial_i f\partial_j g\big|_{x_i=x_j}+\mathcal{O}(\theta^2), \quad (15)$$

in which $f(x)$ and $g(x)$ are two arbitrary functions. Instead of solving the NC Schrödinger equation by using the star product procedure, a Bopp's shift method will be used in this paper. That is, we replace the star product in Schrödinger equation with the usual product together with a Bopp's shift

$$\hat{x}_i=x_i-\frac{1}{2\hbar}\theta_{ij}p_j,\ \hat{p}_i=p_i. \quad (16)$$

Then the Schrödinger equation can be solved in the commuting space, and the non-commutative properties can be realized by the θ related terms.

It is known that in noncommutative space the Dirac equation can be described by the following equation

$$\left[c\vec{\alpha}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)+\beta mc^2\right]\star\psi=E\psi. \quad (17)$$

And in NC space the large component wave function

$\psi_{\mathrm{a}}(\vec{r})$ in the Dirac Eq. (5) can satisfy the following

$$\left[c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\right]\left[c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\right]\star\psi_{\mathrm{a}}(\vec{r})=(E^2-m^2c^4)\psi_{\mathrm{a}}(\vec{r}), \tag{18}$$

i.e.

$$c^2[(p_x^2+p_y^2)+m^2w_1^2(x^2+y^2)+2mw_1L_z+\mathrm{i}\sigma_z m^2w_1^2(xy-yx)+2\sigma_z m\hbar w]\star\psi_{\mathrm{a}}(\vec{r})=(E^2-m^2c^4)\psi_{\mathrm{a}}(\vec{r}). \tag{19}$$

Instead of solving the NC Dirac equation by using the star product, an equivalent method will be used in this paper. In other words, we replace the star product in Dirac equation with the usual product by shifting NC coordinates with a Bopp's shift , i.e.,

$$c^2[(\hat{p}_x^2+\hat{p}_y^2)+m^2w_1^2(\hat{x}^2+\hat{y}^2)+2mw_1\hat{L}_z+\mathrm{i}\sigma_z m^2w_1^2(\hat{x}\hat{y}-\hat{y}\hat{x})+2\sigma_z m\hbar w]\begin{pmatrix}u_1\\u_2\end{pmatrix}=(E^2-m^2c^4)\begin{pmatrix}u_1\\u_2\end{pmatrix}. \tag{20}$$

Thus, in the two dimensions (2+1 dimensional space-time) Eq. (16) changes to

$$\hat{x}=x-\frac{1}{2\hbar}\theta p_y,\ \hat{y}=y+\frac{1}{2\hbar}\theta p_x.\ \hat{p_x}=p_x,\ \hat{p_y}=p_y. \tag{21}$$

Now inserting Eq. (21) into Eq. (20), we have

$$c^2\left\{(p_x^2+p_y^2)+m^2\omega_1^2\left[\left(x-\frac{1}{2\hbar}\theta p_y\right)^2+\left(y+\frac{1}{2\hbar}\theta p_x\right)^2\right]+2m\omega_1\left[\left(x-\frac{1}{2\hbar}\theta p_y\right)p_y-\left(y+\frac{1}{2\hbar}\theta p_x\right)p_x\right]+\mathrm{i}\sigma_z m^2\omega_1^2(\hat{x}\hat{y}-\hat{y}\hat{x})+2\sigma_z m\hbar\omega_1]\right\}\begin{pmatrix}u_1\\u_2\end{pmatrix}=(E^2-m^2c^4)\begin{pmatrix}u_1\\u_2\end{pmatrix}. \tag{22}$$

After a straightforward calculation, we can get not only the Dirac equation in a constant magnetic field in NC space for a particle with spin-$\frac{1}{2}$

$$c^2\left[\left(1-\frac{m\omega_1\theta}{2\hbar}\right)^2(p_x^2+p_y^2)+m^2\omega_1^2(x^2+y^2)-\frac{m^2\omega_1^2\theta}{\hbar}(L_z+\hbar)+2m\omega_1(L_z+\hbar)\right]u_1=(E^2-m^2c^4)u_1 \tag{23}$$

but also the equation for an anti-particle with spin-$\frac{1}{2}$

$$c^2\left[\left(1-\frac{m\omega_1\theta}{2\hbar}\right)^2(p_x^2+p_y^2)+m^2\omega_1^2(x^2+y^2)-\frac{m^2\omega_1^2\theta}{\hbar}(L_z-\hbar)+2m\omega_1(L_z-\hbar)\right]u_2=(E^2-m^2c^4)u_2. \tag{24}$$

Thus, the energy eigenvalues in Eq. (23) and Eq. (24) are given by

$$E^2_{n_xn_ym_\ell}=2\ mc^2\hbar\tilde{\omega}_1(n_x+n_y+1)-m^2c^2\omega_1^2\theta(m_\ell\pm\hbar)+4\ mc^2\omega_1\left(\frac{m_\ell\hbar}{2}\pm\frac{\hbar}{2}\right)+m^2c^4 \tag{25}$$

where

$$\tilde{\omega}_1=\omega_1\left(1-\frac{m\omega_1\theta}{2\hbar}\right). \tag{26}$$

Here the energy level $E^2_{n_xn_ym_\ell}$ represents the non-commutativity for space. Now we can see when $\theta=0$, $E^2_{n_xn_ym_\ell}$ (in Eq. (25)) in NC space will be $E^2_{n_xn_ym_\ell}$ (in Eq. (12)) in commutative space.

4 Energy level of the Dirac equation for a particle in a magnetic field in NC phase space

Now we are in the position to discuss the energy level of the Dirac equation mainly in NC phase space. In fact, when both space-space and momentum-momentum non-commutating are considered, we mean we are studying the problem in NC phase space just as what the Bose-Einstein statistics in non-commutative quantum mechanics required. As known in NC phase space the commutation relations in Eq. (13) should be replaced with

$$[\hat{x}_i,\hat{x}_j]=\mathrm{i}\theta_{ij},\ [\hat{p}_i,\hat{p}_j]=\mathrm{i}\bar{\theta}_{ij},\ [\hat{x}_i,\hat{p}_j]=\mathrm{i}\hbar\delta_{ij}. \tag{27}$$

And in NC phase space the large component wave function $\psi_{\mathrm{a}}(\vec{r})$ in the Dirac Eq. (5) should satisfy the following

$$\left[c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\right]\left[c\vec{\sigma}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)\right]*\psi_{\rm a}(\vec{r})=$$

$$(E^2-m^2c^4)\psi_{\rm a}(\vec{r}) \tag{28}$$

where the star product reads the following [18]

$$(f*g)(x,p)={\rm e}^{\frac{\rm i}{2\alpha^2}\theta_{ij}\partial_i^x\partial_j^x+\frac{\rm i}{2\alpha^2}\bar{\theta}_{ij}\partial_i^p\partial_j^p}f(x,p)g(x,p)=$$

$$f(x,p)g(x,p)+\frac{\rm i}{2\alpha^2}\theta_{ij}\partial_i^x f\partial_j^x g\big|_{x_i=x_j}+$$

$$\frac{\rm i}{2\alpha^2}\bar{\theta}_{ij}\partial_i^p f\partial_j^p g\big|_{p_i=p_j}+\mathcal{O}(\theta^2), \tag{29}$$

in which $\mathcal{O}(\theta^2)$ stands for the second and higher order terms of θ and $\bar{\theta}$. In order to replace the star product in Schrödinger equation and Dirac equation in NC phase space we need a generalized Bopp's shift [19]

$$x_\mu\to\alpha x_i-\frac{1}{2\alpha\hbar}\theta_{\mu\nu}p_\nu\,,\ p_\mu\to\alpha p_\mu+\frac{1}{2\alpha\hbar}\bar{\theta}_{\mu\nu}x_\nu\,, \tag{30}$$

which is the partner of shift in Eq. (16) in NC space. At this point, we can see the Dirac equation after the shift is similar to that in NC space, but the star product and the shifts are defined in Eq. (29) and Eq. (30) individually.

Furthermore, in the two dimensions(2+1 dimensional space-time) Eq. (30) changes to

$$\hat{x}\ =\ \alpha x-\frac{1}{2\alpha\hbar}\theta p_y,\ \hat{y}=\alpha y+\frac{1}{2\alpha\hbar}\theta p_x\,,$$

$$\hat{p_x}\ =\ \alpha p_x+\frac{1}{2\alpha\hbar}\bar{\theta}y,\ \hat{p_y}=\alpha p_y-\frac{1}{2\alpha\hbar}\bar{\theta}x\,. \tag{31}$$

And in NC phase space the Dirac equation reads

$$c^2[(\hat{p}_x^2+\hat{p}_y^2)+m^2w_1^2(\hat{x}^2+\hat{y}^2)+2mw_1\hat{L}_z+$$

$${\rm i}\sigma_z(\hat{p}_x\hat{p}_y-\hat{p}_y\hat{p}_x)+{\rm i}\sigma_z m^2w_1^2(\hat{x}\hat{y}-\hat{y}\hat{x})+$$

$$2\sigma_z m\hbar w]\begin{pmatrix}u_1\\u_2\end{pmatrix}=(E^2-m^2c^4)\begin{pmatrix}u_1\\u_2\end{pmatrix}. \tag{32}$$

Inserting Eq. (31) into Eq. (32), we have

$$c^2\left\{\left(\alpha p_x+\frac{1}{2\alpha\hbar}\bar{\theta}y\right)^2+\left(\alpha p_y-\frac{1}{2\alpha\hbar}\bar{\theta}x\right)^2+m^2\omega_1^2\left[\left(\alpha x-\frac{1}{2\alpha\hbar}\theta p_y\right)^2+\left(\alpha y+\frac{1}{2\alpha\hbar}\theta p_x\right)^2\right]+\right.$$

$$2m\omega_1\left[\left(\alpha x-\frac{1}{2\alpha\hbar}\theta p_y\right)\left(\alpha p_y-\frac{1}{2\alpha\hbar}\bar{\theta}x\right)-\left(\alpha y+\frac{1}{2\alpha\hbar}\theta p_x\right)\left(\alpha p_x+\frac{1}{2\alpha\hbar}\bar{\theta}y\right)\right]-$$

$$\left.\sigma_z\bar{\theta}-\sigma_z m^2\omega_1^2\theta+2\sigma_z m\omega_1\hbar\right\}\begin{pmatrix}u_1\\u_2\end{pmatrix}=(E^2-m^2c^4)\begin{pmatrix}u_1\\u_2\end{pmatrix}. \tag{33}$$

After a straightforward calculation, we get both the Dirac equation in a constant magnetic field in NC phase space for a particle with spin-$\frac{1}{2}$

$$c^2\left[\left(\alpha-\frac{m\omega_1\theta}{2\alpha\hbar}\right)^2(p_x^2+p_y^2)+\left(\alpha-\frac{\bar{\theta}}{2\alpha\hbar m\omega_1}\right)^2\times\right.$$

$$m^2\omega_1^2(x^2+y^2)-\frac{\bar{\theta}+m^2\omega_1^2\theta}{\hbar}(L_z+\hbar)+$$

$$\left.2m\omega_1(L_z+\hbar)\right]u_1=(E^2-m^2c^4)u_1\,, \tag{34}$$

and the equation for an anti-particle with spin-$\frac{1}{2}$

$$c^2\left[\left(\alpha-\frac{m\omega_1\theta}{2\alpha\hbar}\right)^2(p_x^2+p_y^2)+\left(\alpha-\frac{\bar{\theta}}{2\alpha\hbar m\omega_1}\right)^2\times\right.$$

$$m^2\omega_1^2(x^2+y^2)-\frac{\bar{\theta}+m^2\omega_1^2\theta}{\hbar}(L_z-\hbar)+$$

$$\left.2m\omega_1(L_z-\hbar)\right]u_1=(E^2-m^2c^4)u_1. \tag{35}$$

Thus, the energy eigenvalues are given by

$$E^2_{n_xn_ym_\ell}=\ 2\,mc^2\hbar\Omega_1(n_x+n_y+1)-$$

$$(c^2\bar{\theta}+m^2c^2\omega_1^2\theta)(m_\ell\pm\hbar)+$$

$$4mc^2\omega_1\left(\frac{m_\ell\hbar}{2}\pm\frac{\hbar}{2}\right)+m^2c^4 \tag{36}$$

where

$$\Omega_1=\omega_1\left(1-\frac{m\omega_1\theta}{2\hbar}-\frac{\bar{\theta}}{2\hbar m\omega_1}\right). \tag{37}$$

Here the energy level $E^2_{n_xn_ym_\ell}$ represents the non-commutativity for both space and momentum. In the two dimensional non-commutative plane, $\bar{\theta}_{ij}=\bar{\theta}\epsilon_{ij}$, and the two NC parameters θ and $\bar{\theta}$ are related by $\bar{\theta}=4\alpha^2\hbar^2(1-\alpha^2)/\theta$ [6]. When $\alpha=1$, $\bar{\theta}_{ij}=0$, and the $E^2_{n_xn_ym_\ell}$ (in Eq. (36)) in NC phase space will change to $E^2_{n_xn_ym_\ell}$ (in Eq. (25)) in NC space.

5 Summary

To be brief, we have provided a detailed but not tedious study of the energy level of the Dirac equation for a particle in a constant magnetic field in NC phase space. With full and accurate calculation we conclude that the known similarity between an oscillator in a noncommutative phase space and a particle in a constant magnetic field can be extended to a relativistic motion. The method employed in this paper can also be applied to other quantum mechanics problems in NC space, which will be reported in our further studies.

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