Critical behavior of a dynamical percolation model*

YU Mei-Ling(喻梅凌)[1;1)] XU Ming-Mei(许明梅)[2]
LIU Zheng-You(刘正猷)[1] LIU Lian-Shou(刘连寿)[2;3;2)]

1 (Department of Physics, Wuhan University, Wuhan 430072, China)
2 (Institute of Particle Physics, Huazhong Normal University, Wuhan 430079, China)
3 (Key Laboratory of Quark & Lepton Physics (Huazhong Normal University), Ministry of Education, Wuhan 430079, China)

Abstract The critical behavior of the dynamical percolation model, which realizes the molecular-aggregation conception and describes the crossover between the hadronic phase and the partonic phase, is studied in detail. The critical percolation distance for this model is obtained by using the probability P_∞ of the appearance of an infinite cluster. Utilizing the finite-size scaling method the critical exponents γ/ν and τ are extracted from the distribution of the average cluster size and cluster number density. The influences of two model related factors, i.e. the maximum bond number and the definition of the infinite cluster, on the critical behavior are found to be small.

Key words percolation, critical exponent, molecule-like aggregation, delocalization

PACS 12.38.Mh, 25.75.Nq, 64.60.Ak

It is predicted that under the condition of high temperature and/or high density the strongly interacting matter will undergo a deconfinement phase transition from the hadronic matter to the quark-gluon plasma — QGP[1]. On the predicted QCD phase diagram, at the high density, low temperature region there is a first order phase transition line, while at the low density, high temperature there is an analytic crossover process predicted by the lattice QCD[2].

Recently, the molecule-like aggregation model (MAM) is proposed[3] to understand the mechanism of crossover and a bond-percolation model is constructed to simulate the cluster formation in the crossover process. However the critical behavior of the model has not been discussed. The aim of the present article is to take up this problem and study, in detail, the finite-size critical behavior of the bond percolation model.

In the lattice version of the usual geometrical bond-percolation procedure, "a cluster is defined as a group of nearest-neighboring occupied sites that are linked by occupied bonds with occupation probability p, which is a model parameter to control phase transition. The cluster size is the number of sites in a cluster"[4]. The occupation probability where the infinite clusters appear is called critical probability p_c, i.e. when $p \leqslant p_c$ there is no percolating infinite cluster while when $p > p_c$ infinite clusters form.

In our dynamical percolation procedure, the percolation bond is given a dynamical meaning which is referred to as quark delocalization[5] and the occupation probability p of the bond is replaced by the maximum delocalization distance S between hadrons, within which the hadrons can be connected by bonds.

Let us now turn to review briefly the bond-percolation model[3]. In the simplified 2-D version of the model the initial system is set to be a hadron gas consisting of 2×197 cells, which are small circles of hard-core radius $r_e = 0.1$ fm distributed randomly in a big circle of radius $R = 7$ fm. In the following, the system size is characterized by $L = R/r_e$. A cell which departs from the center of the big circle farther than $R - r_e$ is called a boundary cell. The percolation

Received 7 October 2008
* Supported by National Natural Science Foundation of China (10775056, 10835005)
1) E-mail: yuml@iopp.ccnu.edu.cn
2) E-mail: liuls@iopp.ccnu.edu.cn

procedure is as follows:

(1) Randomly select a cell α as a mother cell.

(2) Find the candidate cells that can form bonds with the mother cell, which will be referred to as bond-candidate cells, and are defined as those cells with $|r - r_\alpha| \leqslant S$. It is assumed that each cell can be connected by n_b bonds at most. So we randomly select n_b cells from the bond-candidate cells to form n_b bonds connected to the mother cell α. These are referred to as daughters. If the number of candidate cells is less than n_b, then the number of daughters is equal to the candidate number.

(3) Find the bond-candidate cells for the daughters of cell α. For every daughter find her bond-candidate cells from the remaining unbounded cells, and randomly select $n_b - 1$ bond-candidate cells to form bonds. The cells connected to daughters are called granddaughters.

(4) Repeat the procedures to granddaughters and granddaughters' daughters ..., we will get a cluster, which grows until no bond-candidate cell can be found any more.

(5) Then choose another cell β from the left unbounded cells as another mother cell, and repeat the procedure starting from step (2).

In this way, every cell is assigned to a cluster. In every cluster, find the boundary cells if any, calculate the distance between every two boundary cells, and denote the maximum distance by d. A cluster with $d > d_0$ is called an infinite cluster, where d_0 is a parameter, and the appearance of the infinite cluster is taken as the formation of a new phase of matter—QGP.

It needs to be noted that in the above percolation procedure, the maximum number n_b of bonds that a cell can be connected with and the value of d_0 in the definition of the infinite cluster are adjustable. At present we choose $n_b = 3$ and $d_0 = \sqrt{2}R$. Later we will study the effect of these two factors on the critical behavior.

In the percolation theory, the typical quantities usually studied are the critical percolation probability, the average cluster size, the cluster number density and the critical exponents.

In the molecule-like aggregation scheme the crossover from hadronic gas to QGP is controlled by the appearance of an infinite cluster. The probability P_∞ for the appearance of an infinite cluster is defined as:

$$P_\infty = \frac{\mathscr{N}_\infty}{\mathscr{N}}, \tag{1}$$

where $\mathscr{N}_\infty$ is the number of events with an infinite cluster, $\mathscr{N}$ is the total number of events in the sample. P_∞ as defined in Eq. (1) is referred to by some authors[6] as the percolation cumulant, is analogous to the Binder cumulant[7], which is a powerful tool to extract the critical point for the finite size system[6]. If we calculate P_∞ as a function of S for different system size, all curves for the not too small system size will cross at the critical point S_c for percolation transition, cf. Fig. 1, where the dependence of P_∞ on S is plotted for a system size of $L = 70 \sim 350$ with the same particle number density. We see that at a certain value of S, P_∞ starts to increase from zero and gradually arrives at the saturation value 1 as the increasing S. For an infinite system P_∞ must increase from 0 abruptly to 1 at $S = S_c$. We determine the critical point S_c by the cross point of the curves in Fig. 1. Since the finite size effect is large for small L, we only consider the cross point of the curves for $L > 70$. The calculation resolution for S is 0.001 as shown in the smaller panel and the evaluated critical point from this figure is $S_c = 0.696 \pm 0.003$ fm.

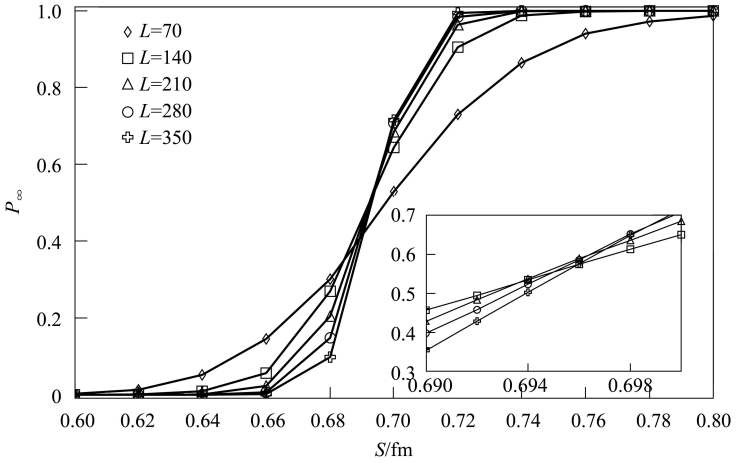

Fig. 1. The dependence of P_∞ on percolation distance S for different system size. The calculation resolution in the smaller panel is 0.001.

Cluster size is the number of sites in a cluster. The average cluster size χ is defined as

$$\chi = \frac{1}{\sum_{k=1}^{\prime\infty}(n_k k)} \sum_{k=1}^{\prime\infty} k\cdot(n_k k)\,, \qquad (2)$$

where n_k is the number of clusters of size k and the prime on the summation symbol means that the infinite cluster is excluded. This variable indicates how big on average the finite clusters are. For infinite system size, the average cluster size diverges for $S \to S_{\rm c}$ as a power law with critical exponent $-\gamma$ in terms of the distance of S from $S_{\rm c}$, that is,

$$\chi(S) \propto |S - S_{\rm c}|^{-\gamma} \quad \text{for} \quad S \to S_{\rm c}\,.$$

Realistically, the percolation models used are of finite size determined by the size of the colliding nuclei. The finite-size scaling method[9], investigating the scaling of quantities as a function of lattice size at $S = S_{\rm c}$, is adopted to extract values for the critical exponents. From the dependence of χ on system size L at $S = S_{\rm c}$,

$$\chi(L; S_{\rm c}) \propto L^{\gamma/\nu} \quad \text{for} \quad 1 \ll L \ll \xi\,, \qquad (3)$$

the critical index can be extracted, where ξ is the correlation length and its divergence at $S = S_{\rm c}$ is characterized by the critical exponent ν with

$$\xi(S) \propto |S - S_{\rm c}|^{-\nu} \quad \text{for} \quad S \to S_{\rm c}.$$

At $S = S_{\rm c}$, the correlation length is infinite, which guarantees that $L \ll \xi$ in Eq. (3). Fig. 2 shows the distribution of average cluster size χ as a function of percolation range S for different system size L. We see that $\chi(S)$ develops a peak when $S \to S_{\rm c}$. The height of the peak increases with the increase of system size. The dependence of χ on system size L when $S = S_{\rm c}$ is shown in Fig. 3 as open circles. Eq. (3) is used to fit this distribution and we obtain $\gamma/\nu = 1.603 \pm 0.005$. The fitting result is shown in Fig. 3 as a solid curve. For percolation with cells located on the sites of a two dimensional lattice, theory gives $\gamma/\nu = 43/24 \approx 1.79$ independent of the lattice details[8].

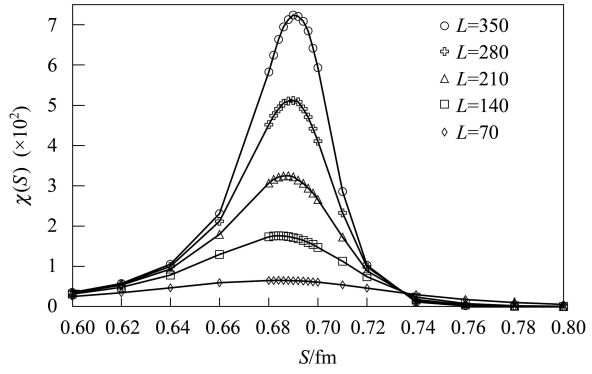

Fig. 2. The distribution of average cluster size χ with respect to the percolation distance S for different system size.

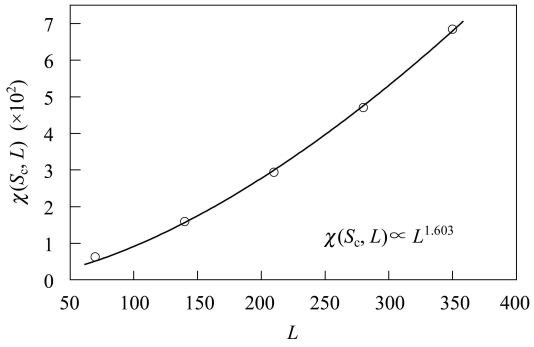

Fig. 3. The average cluster size at critical percolation distance $S_{\rm c}$ as a function of system size L. The solid line is a fitting to the power law function in Eq. (3).

The cluster number density is defined as

$$n(k, S) = \frac{N(k, S; L)}{N_{\rm tot}}\,, \qquad (4)$$

where $N(k, S; L)$ is the cluster size frequency, $N_{\rm tot}$ is the total number of cells in the system. There are two distinct behaviors of $n(k, S)$, depending on whether $S \neq S_{\rm c}$ or $S = S_{\rm c}$. When $S \neq S_{\rm c}$ we have

$$n(k, S) \propto \begin{cases} k^{-\tau} & \text{for} \quad 1 \leqslant k \leqslant k_\xi \\ \text{decay rapidly} & \text{for} \quad k \geqslant k_\xi \end{cases}\,. \qquad (5)$$

Figure 4 showns the cluster number density distribution for different percolation distance S at $L = 350$. When $S = S_{\rm c}$, the characteristic cluster size k_ξ is infinite, so that for any finite cluster, $k \ll k_\xi$. Therefore, the cluster number density decays as a power law for large cluster sizes

$$n(k, S_{\rm c}) \propto k^{-\tau} \quad \text{for} \quad k \gg 1. \qquad (6)$$

To get the critical exponent τ, the system size is set to $L = 350$ to satisfy the condition $k \gg 1$. The distribution of $n(k, S_{\rm c})$ is plotted in Fig. 4 with a solid line and fitted by a power law function in the range of $2 < k < 1000$. From the fitting, we get $\tau = 1.84 \pm 0.0007$ and the theoretical prediction of τ for the percolation with cells on the sites of the two-dimensional lattice is $187/91 \approx 2.05$[10].

In our present percolation procedure, the maximum number of bonds for a cell is set to be $n_b = 3$. Actually this value could be varied and this problem in percolation theory is called percolation with restri-

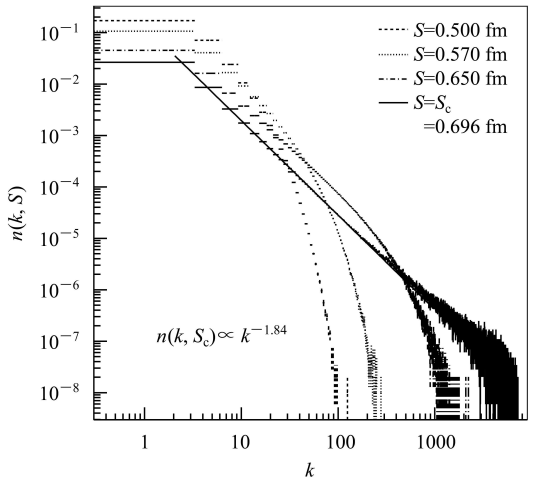

Fig. 4. Cluster number density distribution at different percolation distance for system size $L=350$. The cluster number density distribution at $S=S_c$ is fitted by a power law function in the range $2<k<1000$.

cted valence[11]. We study the influence of the allowed maximum number of bonds on the critical behavior. Fig. 6(a) shows the dependence of P_∞ on percolation distance for various maximum bond numbers. It can be seen that the case with $n_b=2$ is different from the case with $n_b>2$. The critical percolation distance S_c will not be affected by the bond number n_b provided $n_b>2$, which is in accordance with the published results[11, 12].

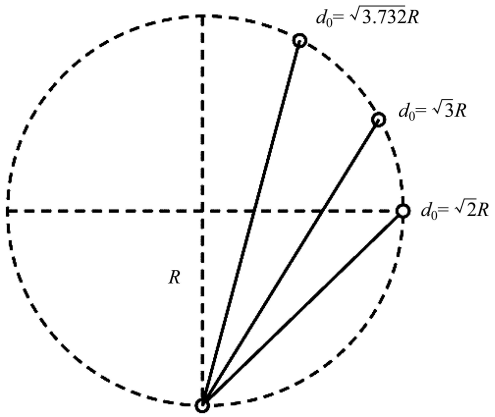

Fig. 5. A sketch of the three different definitions of the infinite cluster.

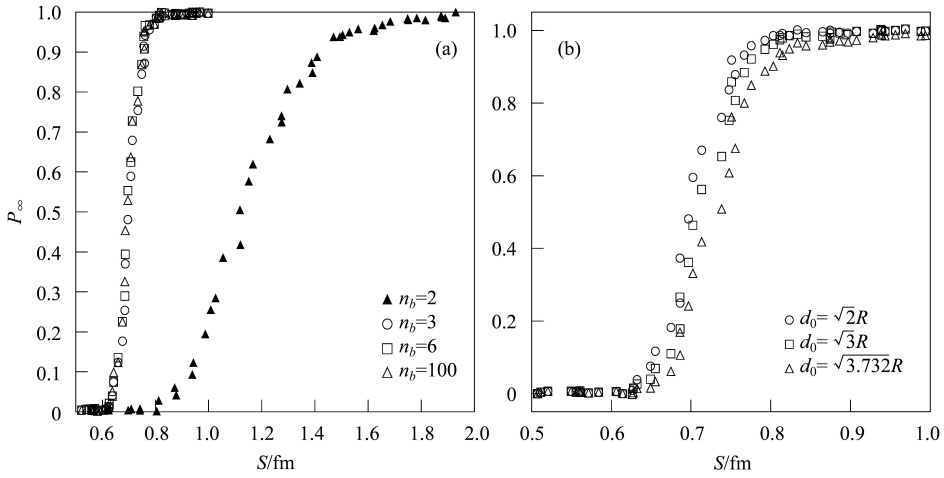

Fig. 6. The dependence of critical percolation distance on two adjustable model parameters when system size is $L=70$. (a) $P_\infty(S)$ for a varied maximum number of bond n_b, (b) $P_\infty(S)$ for different definitions of infinite clusters.

The other factor which may affect the critical behavior is the definition of an infinite cluster. In our present definition, a cluster with $d>d_0$, $d_0=\sqrt{2}R$ is called an infinite cluster, which spans from one half of the circle to the other half. This definition is rather loose and it is interesting to see what happens for tighter definition. Fig. 5 is a sketch to depict different definitions of an infinite cluster. In Fig. 6(b) we show $P_\infty(S)$ for the above three different definitions. It can be seen that the dependence of P_∞ on S is slightly shifted to larger S for tighter definitions of an infinite cluster. This tendency is easy to understand since a tighter definition will result in less formation probability of an infinite cluster. However, the effect coming from the definition of an infinite cluster is small and it could be taken as a systematic error for the determination of S_c. For these two factors, we also calculate the cluster number density and the corresponding critical exponent τ. It turns out that the effect of these two factors is very small

and the corresponding critical exponents τ have only about 0.5% difference.

To summarize, we investigate the critical behaviors of the finite-size bond-percolation model which is used to realize the molecule-aggregation model for crossover from the hadronic phase to the partonic phase. We obtain the critical percolation distance by a finite-size scaling method for this model and extract critical exponents γ/ν and τ which are within 10% difference with that of the percolation with cells on the sites of a two-dimensional lattice. From our calculation we find that when the maximum number of bonds for a cell to be connected with is greater than 2, it has negligible influence on the critical behavior of the bond percolation model. On the other hand, the definition of an infinite cluster has only a small effect on the model, which can be taken as a systematic error when determining S_c.

References

1 Lee T D, Wick G C. Physics. Rev. D, 1974, **9**: 2291; Lee T D. Rev. Mod. Phys., 1975, **47**: 267; Collins J C, Perry M J. Phys. Rev. Lett., 1975, **34**: 1353; Shuryak E V. Phys. Rept., 1980, **61**: 71

2 Fodor Z, Katz S D. JHEP, 2004, **04**: 050

3 XU Ming-Mei, YU Mei-Ling, LIU Lian-Shou. Phys. Rev. Lett., 2008, **100**: 092301

4 Christensen K, Moloney N R. Complexity and Criticality. London: Imperical College Press, 2005. 3

5 WANG Fan et al. Phys. Rev. Lett., 1992, **69**: 2901

6 Fortunato S, Satz H. Phys. Letters. B, 2001, **509**: 189; Nucl. Phys. B, 2001, **598**: 601

7 Binder K, Heermann D W. Monte Carlo Simulations in Statistical Physics. Springer-Verlag, 1988. 40—41

8 Isichenko M B. Rev. Mod. Phys., 1992, **64**: 961

9 Barber M. Finite Size Scaling. In: Domb C, Lebowitz J, ed. Phase Transitions and Critical Phenomena, Volume 8. London: Academic Press, 1983; Cardy J L. ed. Finite-Size Scaling. Amsterdam: North Holland, 1988; Privman V. ed. Finite Size Scaling and Numerical Simulations of Statistical Systems. Singapore: World Scientific, 1990

10 Christensen K, Moloney N R. Complexity and Criticality. London: Imperical College Press, 2005. 81

11 Gaunt D S, Guttmann A J, Whittington S G. J. Phys. A: Math. Gen., 1979, **12**: 75-9

12 Gaunt D S et al. J. Phys. A: Math. Gen., 1980, **13**: 1791-7; Whittington S G, Torrie G M, Gaunt D S. J. Phys. A: Math. Gen., 1979, **12**: L119-23