Teleportation via thermally entangled states of a two-qubit Heisenberg XXZ chain*

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Abstract We investigate quantum teleportation as a tool to study the thermally entangled state of a two-qubit Heisenberg XXZ chain. Our work is mainly to investigate the characteristics of a Heisenberg XXZ chain and get some analytical results of the fully entangled fraction. We also consider the entanglement teleportation via a two-qubit Heisenberg XXZ chain.

Key words Heisenberg XXZ chain, teleportation, thermal entanglement

PACS 03.65.-w, 03.67.Hk, 05.50+q

1 Introduction

Quantum entanglement, first recognized by Einstein, Podolsky, Rosen[1] and Schrödinger[2], is one of the most astonishing features of quantum mechanics. The quantum inseparability implies the existence of a pure entangled state which produces non-classical phenomena. Although entanglement has been recognized as a remarkable feature of quantum mechanics, it remains only incompletely understood. In recent years, there has been an ongoing effort to characterize qualitatively and quantitatively the entanglement properties and apply them to quantum communication and information. Many schemes are proposed for the generation of two or more particle entanglement.

Entanglement in different quantum systems has been investigated extensively. Recently, interest on the topic of the entanglement in a thermal equilibrium state is growing, because the physical realizable quantum mechanisms do not always occur at zero temperature but often in a thermal equilibrium, e.g. the initial state in NMR based quantum computing is a thermal entanglement.

The quantum entanglement in solid systems such as spin chains is an important emerging field[3—6]. Spin chains are natural candidates for the realization of entanglement compared with other physical systems. In spin chain systems, an unknown state, which is placed on one site, can be transmitted to a distant site with some fidelity by using the dynamics of the spin system[7, 8].

In some recent papers, Lee and Kim considered teleportation of an entangled two-body pure spin-1/2 state[9]. Yeo studied the entanglement teleportation via thermal entangled states of a two-qubit Heisenberg XX chain[10—12]. Zhang investigated the thermal entanglement of a two-qubit spin chain with DM anisotropic interaction and entanglement teleportation via the model[13]. Zhang studied the partial teleportation of entanglement through natural thermal entanglement in a two-qubit XXX model[14], got a fidelity better than 2/3 using arbitrary entangled pure states and also investigated the correlation information in the paper.

In this paper, we investigate Lee and Kim's two-qubit teleportation protocol using two independent thermally entangled states of a two-qubit Heisenberg XXZ chain. The structure of the paper is as follows. In the second part we introduce the model in brief. In the third and the fourth part, we discuss the thermal pairwise entanglement, the full entangled fraction and the teleportation respectively. The paper ends in the fifth part with the conclusion.

Received 8 June 2007, Revised 28 February 2008
* Supported by Natural Science Research Project of Shaanxi Province (2004A15)
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2 The system and its thermal equilibrium state

The Hamiltonian H for a two-qubit Heisenberg XXZ chain in an external magnetic field B along the z axis is

$$H=\frac{J}{2}(\sigma^x\sigma^x+\sigma^y\sigma^y+\Delta\sigma^z\sigma^z)+\frac{B}{2}(\sigma^0\sigma^z+\sigma^z\sigma^0), \quad (1)$$

where σ^0 is the identity matrix and σ^i $(i=x,y,z)$ are the Pauli matrices. J is the real coupling constant for the spin interaction. The chain is said to be antiferromagnetic for $J>0$ and ferromagnetic for $J<0$. The parameter Δ quantifies the anisotropy in the interaction. The eigenvalues and eigenvectors of H are given by $H|00\rangle=(J\Delta/2+B)|00\rangle$, $H|\psi^{\pm}\rangle=(\pm J-J\Delta/2)|\psi^{\pm}\rangle$, $H|11\rangle=(J\Delta/2-B)|11\rangle$. $\psi\pm=\frac{1}{\sqrt{2}}(|01\rangle\pm|10\rangle)$.

The state of a typical condensed-matter system in thermal equilibrium (temperature T) is $\rho=\exp(-\beta H)/Z$, where H is the Hamiltonian, $Z=\mathrm{tr}(\exp(-\beta H))$ is the partition function, and $\beta=1/(kT)$, where k is the Boltzmann's constant. The entanglement associated with the thermal state ρ is referred to as thermal entanglement[15].

For the system (1) in equilibrium at temperature T, the density operator is

$$\rho=\frac{1}{Z}(\mathrm{e}^{-\beta(\frac{J\Delta}{2}+B)}|00\rangle\langle 00|+\mathrm{e}^{-\beta(J-\frac{J\Delta}{2})}|\psi^+\rangle\langle\psi^+|+ \mathrm{e}^{-\beta(-J-\frac{J\Delta}{2})}|\psi^-\rangle\langle\psi^-|+\mathrm{e}^{-\beta(\frac{J\Delta}{2}-B)}|11\rangle\langle 11|), \quad (2)$$

where the partition function

$$Z=\sum\exp(-\beta E_i)=2\exp(J\Delta\beta/2)\cosh(\beta J)+ 2\exp(-J\Delta\beta/2)\cosh(\beta B),$$

the Boltzmann's constant $k\equiv 1$ from hereon, and $\beta=1/T$.

3 Concurrence and fully entangled fraction

To quantify the amount of entanglement associated with ρ, we consider the concurrence[16], which is defined as $C=\max[0,2\max(\lambda_i)-\sum\lambda_i]$, where λ_i are the square roots of the eigenvalues of the matrix $R=\rho S\rho^* S$, in which ρ is the density matrix, $S=\sigma^y\otimes\sigma^y$, and the asterisk stands for the complex conjugate. After some straightforward algebra, we obtain

$$C[\rho]=\frac{2}{Z}\max(|\mathrm{e}^{\frac{J\Delta\beta}{2}}\sinh(J\beta)|-\mathrm{e}^{-\frac{J\Delta\beta}{2}},0). \quad (3)$$

The concurrence C=0 indicates vanishing entanglement. The critical temperature T_{c} above which the concurrence is zero is determined by the above nonlinear equation. This result is in accord with the conclusion in Wang's work[17].

$$\begin{cases}\sinh\left(\dfrac{J}{T}\right)=\mathrm{e}^{\frac{-J\Delta}{T}}, & J>0\\[2ex] \sinh\left(\dfrac{|J|}{T}\right)=\mathrm{e}^{\frac{-|J|\Delta}{T}}. & J<0\end{cases} \quad (4)$$

In the standard teleportation protocol P_0, the maximal teleportation fidelity $\Phi_{\max}[\Lambda^{\rho,P_0}]$ achievable is given by[18, 19].

$$\Phi_{\max}[\Lambda^{\rho,P_0}]=(2F[\rho]+1)/3\ , \quad (5)$$

where the fully entangled fraction

$$F[\rho]=\max_{i=0,1,2,3}\{\langle\psi^i_{\mathrm{Bell}}|\rho|\psi^i_{\mathrm{Bell}}\rangle\}. \quad (6)$$

After some straightforward algebra, we obtain

$$F[\rho]= \max\left\{\frac{1}{Z}\mathrm{e}^{\frac{J(\Delta-2)}{2T}},\ \frac{1}{Z}\mathrm{e}^{\frac{J(\Delta+2)}{2T}},\ \frac{1}{Z}\mathrm{e}^{\frac{-J\Delta}{2T}}\cosh\left(\frac{B}{T}\right)\right\}. \quad (7)$$

For antiferromagnetic case $(J>0)$ the concurrences are given by Eq. (3), and only for $\Delta>-1$ can the entanglement exist. To be more simple, we consider the case $J=1$. The concurrence is invariant under the substitution $B\to -B$. We restrict our considerations to $B>0$, $\Delta>-1$.

$$F=\begin{cases}\mathrm{e}^{\frac{\Delta+2}{2T}}\ , & \Delta>M-1\\[2ex] \mathrm{e}^{\frac{-\Delta}{2T}}\cosh\left(\dfrac{B}{T}\right), & \Delta<M-1\end{cases} \quad (8)$$

where $M=\ln(\cosh(B\beta))/\beta$. Eq. (8) reduces to the following possibilities in the zero-temperature limit. That is, for $\beta\to\infty$, the system is in its ground state in this case. There exist three possibilities for $\Delta>M$–1. $F=0$, for B–$\Delta>1$; $F=1$, for B–$\Delta>1$ and $F=1/2$, for $B-\Delta=1$. There also exist three possibilities for $\Delta<M$–1. $F=0$, for Δ+1 $>B$, $F=1/4$, for Δ+1 $=B$; $F=1/2$, for $\Delta+1<B$. For the ferromagnetic case $(J<0)$ the concurrence exists only if $\Delta<1$. One can get the similar results for the case $J<0$.

4 Teleportation via the thermal equilibrium state

Teleportation of a quantum state using a mixed entangled state has been theoretically studied recently. In Ref. [11], the teleportation of an entangled state through Werner states as noisy quantum channels is considered. Now we look at Lee and Kim's two-qubit teleportation protocol using two copies of the above two-qubit thermal state, $\rho_{\rm AB}\otimes\rho_{\rm A'B'}$, as a resource. We consider as input two qubits in the Werner state[20], and this is different compared with Ref. [14].

$$\rho_{\rm w}=\{\sigma^0\sigma^0-[(2\varphi+1)/3](\sigma^x\sigma^x+\sigma^y\sigma^y+\sigma^z\sigma^z)\},\quad (-1\leqslant\varphi\leqslant 1) \tag{9}$$

if φ=1, $\rho_{\rm w}=|\psi^-\rangle\langle\psi^-|$ is a maximally entangled pure state, if $0<\varphi<1$, $\rho_{\rm w}$ is an entangled mixed state. Lastly, if $-1\leqslant\varphi\leqslant 0$, $\rho_{\rm w}$ is a separable mixed state. Since our concern is the entanglement teleportation, we focus on $0<\varphi\leqslant -1$. The output state is then given by[21]

$$\rho_{\rm out}=\sum_{i,j}p_{ij}(\sigma_i\sigma_j)\rho_{\rm w}(\sigma_i\sigma_j), \tag{10}$$

where σ^i $(i=0,\ x,\ y,\ z)$ signifies the unit matrix I and three components of the Pauli spin matrix σ, respectively, $P_{ij}={\rm tr}[E_i\rho(T)]{\rm tr}[E_j\rho(T)]$, $\sum P_{ij}=1$. Here $E^0=|\psi^-\rangle\langle\psi^-|$, $E^1=|\Phi^-\rangle\langle\Phi^-|$, $E^2=|\Phi^+\rangle\langle\Phi^+|$, $E^3=|\psi^+\rangle\langle\psi^+|$, and $\phi\pm=\frac{1}{\sqrt{2}}(|00\rangle\pm|11\rangle)$.

We first calculate the measure of entanglement for the teleported state $\rho_{\rm out}$ to be

$$C[\rho_{\rm out}]=2\max\left(\frac{(2\varphi+1)(-1+{\rm e}^{2J\beta})^2{\rm e}^{2\beta B+J\Delta\beta}-{\rm e}^{2\beta(B+J)}m_1}{6(m_2)^2},0\right), \tag{11}$$

$$\begin{aligned}m_1=&(\varphi-1)({\rm e}^{J(2+\Delta)\beta}+\cosh(\beta\Delta J)+\\&2{\rm e}^{-\beta\Delta J}\sinh(2B))-2(\varphi+2)(\cosh(\beta(B-J)+\\&\cosh(\beta(B+J))+{\rm e}^{J(\Delta-2)}\varphi,\\m_2=&{\rm e}^{\frac{\beta(4B-J\Delta+2J)}{2}}+{\rm e}^{\frac{\beta(4J+J\Delta+2B)}{2}}+\\&{\rm e}^{\frac{-J(\Delta-2)}{2}}+{\rm e}^{\frac{\beta(2B-\Delta J)}{2}}.\end{aligned} \tag{12}$$

To characterize the quality of the teleported state $\rho_{\rm out}$, it is often quite useful to look at the fidelity between $\rho_{\rm w}$ and $\rho_{\rm out}$, defined by[22]

$$F=\{{\rm tr}[\sqrt{(\rho_{\rm w})^{1/2}\rho_{\rm out}(\rho_{\rm w})^{1/2}}]\}^2\ . \tag{13}$$

We can obtain the result

$$\begin{aligned}F=&\frac{1}{36}\left(\frac{2}{Z_1}\sqrt{(\varphi-1)\left[(\varphi-1)M_1-2(2+\varphi)M_2+2(\varphi+1){\rm e}^{-\beta(2B+2J-j\Delta)}-2(2-\varphi){\rm e}^{-3\beta(B+J)}\right]}+\right.\\&\frac{\sqrt{3}}{Z_2}\sqrt{(\varphi+1)\left[3(\varphi+1)M_3+2(1-\varphi)M_4+(\varphi+2)M_5+2{\rm e}^{-2B\beta}(1-\varphi{\rm e}^{-\beta(4B-J\Delta+6J)})\right]}+\\&\left.\frac{1}{Z_2}\sqrt{(\varphi-1)\left[(\varphi-1)M_6-(2+\varphi)M_7+(\varphi-2)M_8-(1+\varphi){\rm e}^{-2B\beta}-M_9\right]}\right)^2\end{aligned} \tag{14}$$

where

$$\begin{aligned}Z_1=&{\rm e}^{\frac{-\beta}{2}J(2+\Delta)}+{\rm e}^{\frac{-\beta}{2}(4B-J\Delta+2J)}{\rm e}^{\frac{-\beta}{2}(2B-J\Delta)}+{\rm e}^{\frac{-\beta}{2}(2B+J\Delta+4J)},\\Z_2=&{\rm e}^{\frac{-\beta}{2}(6J+J\Delta+4B)}+{\rm e}^{\frac{-\beta}{2}(6J-J\Delta+8B)}+{\rm e}^{\frac{-\beta}{2}(4J-J\Delta+6B)}+{\rm e}^{\frac{-\beta}{2}(8J+J\Delta+6B)},\end{aligned} \tag{15}$$

$$\begin{aligned}M_1=&{\rm e}^{2B+J\Delta\beta}+{\rm e}^{-\beta(2J+J\Delta+4B)}+{\rm e}^{-\beta(2J+J\Delta+2B)}+{\rm e}^{-J(2+\Delta)\beta}+{\rm e}^{-\beta(4J-J\Delta+2B)},\\M_2=&{\rm e}^{-\beta(3B+J)}+{\rm e}^{-\beta(B+J)}+{\rm e}^{-\beta(B+3J)},\\M_3=&{\rm e}^{-2\beta(B-J)}+{\rm e}^{-2\beta(B+J)},\\M_4=&{\rm e}^{-\beta(3B-J+J\Delta)}+{\rm e}^{-\beta(B-J+J\Delta)}+{\rm e}^{-\beta(3B+J+J\Delta)}+{\rm e}^{-\beta(B+J+J\Delta)},\\M_5=&2{\rm e}^{-2\beta(B+J\Delta)}+{\rm e}^{-2\beta(2B+J\Delta)}+{\rm e}^{-2\beta J\Delta},\\M_6=&{\rm e}^{-2\beta(B-J)}+2{\rm e}^{-\beta(3B-J+J\Delta)}+2{\rm e}^{-\beta(B+J+J\Delta)}+2{\rm e}^{-\beta(B-J+J\Delta)}+{\rm e}^{-2\beta(B+J)},\\M_7=&{\rm e}^{-2\beta(B+J\Delta)}+{\rm e}^{-2J\Delta\beta},\\M_8=&2{\rm e}^{-\beta(3B+J+J\Delta)}-{\rm e}^{-2(2B+J\Delta)\beta},\\M_9=&2{\rm e}^{-\beta(3B-J+J\Delta)}(1-\varphi{\rm e}^{-\beta(4B-J\Delta+6J)}).\end{aligned} \tag{16}$$

In particular, in the infinite temperature limit, $\beta \rightarrow 0$, when there is zero thermal entanglement in the channels, we have

$$F \rightarrow \frac{1}{4}[(2-\varphi)+\sqrt{3(1-\varphi)^2}], \tag{17}$$

which increases as $\varphi \rightarrow 0$ in accordance with Ref. [10].

5 Conclusions

In conclusion, we have studied the thermal equilibrium state of a two-qubit anisotropic Heisenberg XXZ chain. We analytically computed the concurrence and the fully entangled fraction. Finally we have given the mathematic expressions according to the quantum entanglement in a Werner state teleported via two separate, thermally entangled two-qubit Heisenberg XXZ chains. The two-qubit teleportation together with one–qubit unitary operations are sufficient to implement the universal gates for quantum computation. It is also expected that our work will be of some help in the process of realizing quantum teleportation.

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