An Unbiased Estimator of Poisson Statistics*

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Abstract In the case that the mean value of Poisson distribution is a function of an unknown parameter to estimate, the commonly adopted maximum likelihood estimate of the parameter based on a single measurement is generally biased. With the aid of moment expressions, an unbiased estimator is proposed for the Poisson distribution.

Key words Poisson distribution, moment, unbiased estimator

1 Introduction

The Poisson distribution is widely used as a model for the counting experiment. The probability of finding exactly k events in a single measurement is given by

$$p(k)=\frac{\lambda^k}{k!}\,\mathrm{e}^{-\lambda}. \tag{1}$$

The distribution is determined by its mean value λ, which is mostly in turn a function of another parameter, μ, to be determined by the measurement:

$$\lambda=f(\mu). \tag{2}$$

Once k events are observed in a single measurement, the purpose of the experiment is to give an estimate of the parameter μ based on the observation.

To estimate a parameter, a function of observations which is called the estimator is chosen in statistics. The numerical value of the estimator for a particular set of observations is the estimate of the parameter. For our case here, as k is the only experimental information observed, one may expect that the estimator should be a function of k.

The most popular estimator used is based on maximum likelihood (ML) estimation. For the Poisson distribution, when k events are observed, ML implies that the estimate maximizes Eq. (2). The solution is

$$k=f(\mu). \tag{3}$$

So the ML estimator can be expressed as

$$\mu(k)=f^{-1}(k), \tag{4}$$

where f^{-1} is the inverse function of f.

Though it is simple and widely adopted in practice, the ML estimator given in Eq. (4) suffers from the fact that it usually gives biased estimate of the parameter μ. This fact can be seen from, unless f^{-1} is a linear function,

$$\overline{\mu(k)}\equiv\sum_{k=0}^{\infty}\mu(k)p(k)\neq\mu\ . \tag{5}$$

If the number of expected events λ is large, the ML estimator becomes asymptotic unbiased. However it is somehow inadequate to use the ML estimator for cases when only small number of events are present. In this work, we will propose an unbiased estimator for the Poisson distribution. The article is organized as follows: after this introduction, we recall some basic properties of the Poisson distribution in Section 2.

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A simple example is introduced in Section 3, where we discuss the problem and possible solutions in an intuitive way. The procedure for constructing an unbiased estimator is formally presented in Section 4. The last section is devoted to some discussions and a brief conclusion.

2 Notations

We collect some basic notations and formulae relevant for further discussions of the Poisson distribution in this section . The details on how they are derived can be found, for example, from the statistical text book[1].

The expectation of any function g of the random variable k is defined to be

$$\boldsymbol{E}[g(k)]=\sum_{k=0}^{\infty} g(k)p(k) \ . \tag{6}$$

And the n-th moment of the variable k is defined by

$$\overline{k^n}=\boldsymbol{E}[k^n] \ . \tag{7}$$

For Poisson distribution, one can easily verify that the n-th moment can be expressed as a polynomial of λ of order n

$$\overline{k^n}=\sum_{l=0}^{n} \xi_l \lambda^l \ . \tag{8}$$

In practice, the coefficients $\{\xi_l\}$ can be deduced from the generating function defined as

$$\phi(t)=\boldsymbol{E}[t^k]=\mathrm{e}^{\lambda(t-1)} \ . \tag{9}$$

We list the expression of moments up to the 4th order here for further use:

$$\overline{k} = \lambda, \tag{10}$$

$$\overline{k^2} = \lambda+\lambda^2, \tag{11}$$

$$\overline{k^3} = \lambda+3\lambda^2+\lambda^3, \tag{12}$$

$$\overline{k^4} = \lambda+7\lambda^2+6\lambda^3+\lambda^4. \tag{13}$$

Inversely, the n-th power of λ can be expressed as the linear combination of the moments

$$\lambda^n=\sum_{l=0}^{n} \eta_l \overline{k^l} \ , \tag{14}$$

where $\{\eta_l\}$ are known coefficients once $\{\xi_l\}$ have been obtained. The first few such expressions, for example, are:

$$\lambda = \overline{k}, \tag{15}$$

$$\lambda^2 = -\overline{k}+\overline{k^2}, \tag{16}$$

$$\lambda^3 = 2\overline{k}-3\overline{k^2}+\overline{k^3}, \tag{17}$$

$$\lambda^4 = -6\overline{k}+11\overline{k^2}-6\overline{k^3}+\overline{k^4}. \tag{18}$$

3 An example

For illustration, we introduce a simple case in this section. The relation between λ and μ is given by

$$\lambda=\sqrt{\mu}-1 \ . \tag{19}$$

The ML estimator implies $\lambda=k$. When applied to the case under study, we find

$$\mu(k)=(1+k)^2 \ . \tag{20}$$

With a little algebra, one can verify that

$$\begin{aligned}\overline{\mu} &= \boldsymbol{E}[\mu(k)]=1+2\overline{k}+\overline{k^2}= \\ &(1+\lambda)^2+\overline{k}=\mu+\overline{k}.\end{aligned} \tag{21}$$

The extra $\overline{k}$ term in the last equation shows clearly that the ML estimator is biased. It also suggests that a new estimator,

$$\widetilde{\mu}=(1+k)^2-k \tag{22}$$

is statistically unbiased because of

$$\overline{\widetilde{\mu}}=\boldsymbol{E}[\widetilde{\mu}(k)]=\mu \ . \tag{23}$$

Going a little further, one can examine the efficiency of each estimator given above. We calculate the variance of the estimator, which is regarded as the measure of the efficiency. In general, the one with smaller variance will be regarded as more efficient.

Using the results given in Eqs. (10—13), one can obtain:

$$\sigma^2=\boldsymbol{E}[(\mu(k)-\overline{\mu})^2]=9\lambda+14\lambda^2+4\lambda^3 \ , \tag{24}$$

and

$$\widetilde{\sigma}^2=\boldsymbol{E}[(\widetilde{\mu}(k)-\overline{\widetilde{\mu}})^2]=4\lambda+10\lambda^2+4\lambda^3 \ . \tag{25}$$

So that

$$\widetilde{\sigma}^2-\sigma^2=-(5\lambda+4\lambda^2)<0 \ . \tag{26}$$

Although it is very simple, the example discussed here suggests some interesting points to which we must pay attention. First of all, it shows that the

ML estimator is indeed biased even for such a simple case; Secondly, one can find an unbiased estimator by some modifications applied to the ML estimator; Last but not the least, the unbiased estimator so obtained is also more efficient than the ML estimator.

4 The unbiased estimator

Guided by the experience got from the simple example, we start the procedure to obtain an unbiased estimator in this section. Suppose $\widetilde{\mu}(k)$ is an unbiased estimator of the parameter μ under consideration, it should by definition follow

$$\overline{\widetilde{\mu}} \equiv \boldsymbol{E}[\widetilde{\mu}(k)] = \mu = f^{-1}(\lambda) \ . \tag{27}$$

Unless f^{-1} is a linear function, the ML estimator given in Eq. (4) does not satisfy such a requirement:

$$\boldsymbol{E}[\mu(k)] \neq f^{-1}(\boldsymbol{E}[k]) = f^{-1}(\lambda) \ . \tag{28}$$

Though it shows that the ML estimator does not have the unbiased nature required, Eq. (28) is quite intuitive. It actually teaches us how an unbiased estimator could be constructed by some modifications applied to the ML estimator. Such a modification can be achieved by the aid of the moments.

Assuming f^{-1} can be expanded as a Taylor's series,

$$\mu = \sum_m \frac{f^{-1,(m)}(0)}{m!} \lambda^m \ , \tag{29}$$

where

$$f^{-1,(m)}(0) \equiv \frac{\mathrm{d}^m f^{-1}(\lambda)}{\mathrm{d}\lambda^m} |_{\lambda=0} \ ,$$

the ML estimator is simply

$$\mu(k) = \sum_m \frac{f^{-1,(m)}(0)}{m!} k^m \ . \tag{30}$$

Since $\boldsymbol{E}[k^m] \neq \overline{k}^m$ for $m \geqslant 2$, one cannot recover Eq. (29) by taking the expectation value of $\mu(k)$. This is the reason why the ML estimator is biased in general.

Replacing λ^m's in Eq. (29) by the moments, as given in Eq. (14), μ can be re-expressed as

$$\mu = \sum_m \frac{f^{-1,(m)}(0)}{m!} \sum_{l=0}^{m} \eta_l \overline{k^l} \ . \tag{31}$$

At this point, we can easily get an unbiased estimator of μ. It is

$$\widetilde{\mu}(k) = \sum_m \frac{f^{-1,(m)}(0)}{m!} \sum_{l=0}^{m} \eta_l k^l \ . \tag{32}$$

The proof of the unbiasedness of $\widetilde{\mu}(k)$ is rather simple:

$$\begin{aligned}\boldsymbol{E}[\widetilde{\mu}(k)] \ &= \sum_m \frac{f^{-1,(m)}(0)}{m!} \sum_{l=0}^{m} \eta_l \overline{k^l} = \\ &\sum_m \frac{f^{-1,(m)}(0)}{m!} \lambda^m = \mu.\end{aligned} \tag{33}$$

Applying the above procedure to the example given in Section 3, using Eqs. (15, 16) we have

$$\mu = 1 + 2\lambda + \lambda^2 = 1 + 2\overline{k} + \overline{k^2} - \overline{k} \ . \tag{34}$$

The unbiased estimator

$$\widetilde{\mu}(k) = 1 + k + k^2$$

is simply the one given in Eq. (22). For this simple case, we have shown that the unbiased estimator so obtained is also more efficient than the ML estimator. However, it might be difficult to prove that such a statement could hold for more general cases. We would leave this question for future study.

5 Discussions and conclusion

With the aid of the moments, we have shown that an unbiased estimator of Poisson statistics does exist, as given in Eq. (32). We also realize that Eq. (29) is somehow a strong assumption. One may from time to time meet with the cases that f^{-1} does not have such a nice feature. This is clearly the weakness of the method given in this article. However, we argue that within the experimental precision, one can usually approximately replace f^{-1} by a finite order polynomial. So in practice the task for an unbiased estimation would not be so much different than the simple example given in Section 3.

To conclude, we have proposed an estimator to replace the commonly used maximum likelihood estimator for Poisson statistics, and shown such an estimator is statistically unbiased.

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References

1 Eadie W T et al. Statistical Methods in Experimental Physics. Netherlands: North-Holland Publishing Company, 1971

泊松统计的无偏估计*

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摘要　当泊松分布的平均值依赖于被估参数时, 从单次实验结果给出的关于参数的最大似然估计一般是统计学意义上有偏的. 借助矩的表达形式, 提出了一种关于被估参数的无偏估计方法.

关键词　泊松分布　矩　无偏估计

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