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Dynamics of Multipartite Entanglement in the Heisenberg Model*

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Abstract We investigate dynamics of the multipartite entanglement in the Heisenberg model and give analytical expressions of the average concurrence $\langle C\rangle$ and the multipartite entanglement measure Q. It is found that both $\langle C\rangle$ and Q initially increase with the increase of the scaled time t, and finally reach a plateau, oscillating irregularly around a steady value. And for the case of $N\langle C\rangle$, this steady value is nearly independent of the length of the chain, and only determined by the NNN coupling constant J.

Key words Heisenberg model, multipartite entanglement, next-nearest-neighbor interaction

1 Introduction

Quantum entanglement, first noted by Einstein, Podolsky, and Rosen (EPR) and Schrödinger, is one of the most distinctive features of quantum mechanics. It has attracted much attention in recent years, mainly due to its central roles in quantum protocols such as quantum teleportation[1], superdense coding[2], quantum key distribution[3], and other processes that involve transfer of quantum information[1]. Entanglement has thus been recognized as a resource for various tasks of quantum information processing[4] and quantum computing[5].

Since we can perform so many useful tasks with entangled states, it is desirable to quantify the amount of entanglement these states have, and thus several measures have been introduced so far in the literature[6—12]. Particularly, for a pair of qubits, Hill and Wootters[7, 8] showed that a quantity that they called concurrence was a good measure of entanglement. However, for the general case, the study of entanglement measure is far from completely developed. There is currently no consensus as to the best method to define an entanglement measure for all possible multipartite states. And it seems that the various proposed measures quantify different aspects of entanglement in multipartite states, aspects that need further elucidation.

2 Preliminaries

In this section, we briefly recapitulate the definitions of the different entanglement measures we adopt in this paper. We first recall the definition of the concurrence[7, 8], which is defined as

$$C=\max\{\lambda_1-\lambda_2-\lambda_3-\lambda_4,0\}, \tag{1}$$

where λ_1, λ_2, λ_3, and λ_4 are the square roots of the eigenvalues of the product matrix $R{=}\rho_{ij}\tilde{\rho}_{ij}$, in decreasing order. Here the spin-flipped density matrix $\tilde{\rho}_{ij}$ is given by

$$\tilde{\rho}_{ij}=(\sigma_i^y\otimes\sigma_j^y)\rho_{ij}^*(\sigma_i^y\otimes\sigma_j^y). \tag{2}$$

The symbol ρ_{ij}^* denotes the complex conjugation of the reduced density matrix ρ_{ij} in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

Next let's see the geometric multipartite entanglement measure Q, which was firstly introduced by

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Meyer and Wallach[12], and has been shown to be simply related to one-qubit purities. It is defined as

$$Q=2\left[1-\frac{1}{N}\sum_{n=1}^{N}\mathrm{tr}(\rho_n^2)\right]. \tag{3}$$

From the unit trace of the density matrices, it follows immediately that for qubits $1-\mathrm{tr}(\rho_n^2)=2\det(\rho_n)$, thus we get that

$$Q=\frac{4}{N}\sum_{n=1}^{N}\det(\rho_n). \tag{4}$$

Clearly, if we want to calculate the entanglement measure Q, we only need to know knowledge of the reduced density matrix ρ_n, and this is simple for a given state.

3 Results and discussion

In this paper, we study time-evolution problem in the one-dimensional Heisenberg model governed by the Hamiltonian

$$\hat{H}=\sum_{n=1}^{N}\left(S_n^xS_{n+1}^x+S_n^yS_{n+1}^y\right)+J\sum_{n=1}^{N}\left(S_n^xS_{n+2}^x+S_n^yS_{n+2}^y\right), \tag{5}$$

where N is the length of the chain, and the periodic boundary condition is imposed.

We think of the so-called “one-particle” states with one spin pointing up and $N-1$ spins pointing down (i.e., the initial state of the system is $\sigma_1^+|0\rangle^{\otimes N}$). In a recent relevant work[13], we have shown that for this case, the state vector at time t is

$$|\Psi(t)\rangle=\sum_{n=1}^{N}b_n(t)\sigma_n^+|0\rangle^{\otimes N}, \tag{6}$$

where

$$b_n(t) = \frac{1}{N}\sum_{k=1}^{N}\exp\left[\mathrm{i}\frac{2k(n-1)\pi}{N}-\mathrm{i}t\left(\cos\frac{2k\pi}{N}+J\cos\frac{4k\pi}{N}\right)\right]. \tag{7}$$

For Eq. (6), by direct calculations, the concurrence between any two qubits i and j is easily found to be $C_{ij}(t)=2|b_i(t)b_j(t)|$. However, in this study, we are more interested in the gross measure of entanglement. So in the following, we shall study the average pairwise concurrence[14]

$$\langle C\rangle = \frac{2}{N(N-1)}\sum_{i<j}C_{ij} = \frac{2}{N(N-1)}\left[\left(\sum_{n=1}^{N}|b_n(t)|\right)^2-1\right]. \tag{8}$$

Fig. 1 displays the dynamical behaviors of the average concurrence $\langle C\rangle$ as a function of the scaled time t for different next-nearest-neighbor (NNN) coupling constants J, where the length of the chain is chosen to be 350. It is clear that for any fixed NNN coupling constant J, the average concurrence $\langle C\rangle$ first linearly increases with the increase of the scaled time t, and then when a critical point t_c is reached, $\langle C\rangle$ reaches a plateau and oscillates irregularly around a steady value. This steady value increases with the increase of J, while t_c decreases with the increase of J.

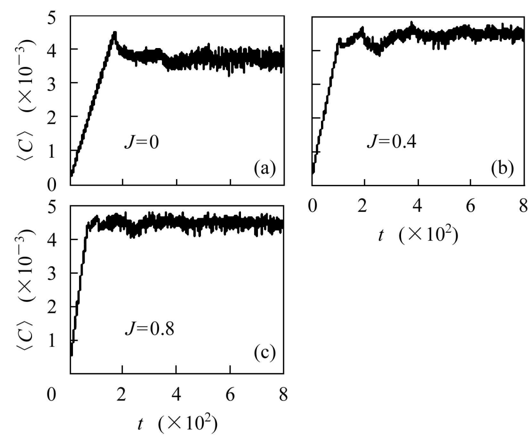

Fig. 1. Dynamics of the average concurrence $\langle C\rangle$ for N=350. (a) J=0; (b) J=0.4; (c) J=0.8.

In Fig. 2 we plot $N\langle C\rangle$ as a function of the scaled time t for J=0, 0.4 and 0.8. It is easy to find that for any fixed NNN coupling constant J, $N\langle C\rangle$ still initially increases linearly with the increase of the scaled time t, and finally arrives at a plateau, oscillating irregularly around a steady value, as one expected. However, this steady value is nearly independent of the length of the chain, and is only determined by the NNN coupling constant J.

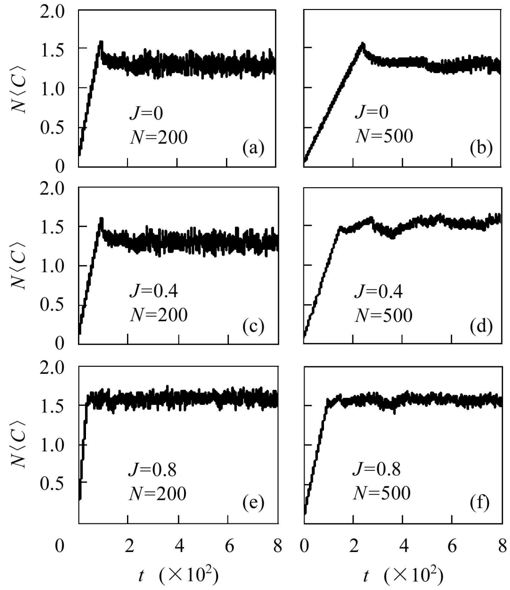

Fig. 2.　Dynamics of $N\langle C\rangle$ for different values of N and J.

Next we consider the dynamical behavior of the entanglement measure Q. It is straightforward to find that the single qubit reduced density matrix ρ_n which is

$$\rho_n=\begin{pmatrix} \frac{1}{2}+\langle S_n^z\rangle & \langle S_n^-\rangle \\ \langle S_n^+\rangle & \frac{1}{2}-\langle S_n^z\rangle \end{pmatrix}, \tag{9}$$

where the first element is $\langle 0|\rho_n|0\rangle$, etc., and the angular brackets are expectation values corresponding to the full pure state $|\Psi(t)\rangle$ we are interested in.

For Eq. (6), it is easy to obtain that

$$\langle S_n^z\rangle=\frac{1}{2}-|b_n(t)|^2,\quad \langle S_n^+\rangle=\langle S_n^-\rangle=0\ . \tag{10}$$

So we have

$$\rho_n=\begin{pmatrix} 1-|b_n(t)|^2 & 0 \\ 0 & |b_n(t)|^2 \end{pmatrix}. \tag{11}$$

Substituting Eq. (11) to Eq. (4) leads to

$$Q=\frac{4}{N}\left(1-\sum_{n=1}^{N}|b_n(t)|^4\right), \tag{12}$$

where we have used the normalization condition of Eq. (6).

In Fig. 3 we plot the multipartite entanglement measure Q as a function of the scaled time t for N=350, and J=0, 0.4 and 0.8, respectively. Apparently, with the increasing value of t, Q exhibits a rapid initial increase, and then reaches a plateau, oscillating irregularly around a steady value $4/N$. From the inset of Fig. 3, we also observe that with the increase of t, the entanglement measure Q exhibits a "zigzag chain" shape. We define it the main oscillation for the convenience of representation. It is clear that when J=0, the period of the main oscillation is about 180, and its amplitudes decreases with the increase of t. However, when $J\neq 0$, the regular main oscillation were destroyed.

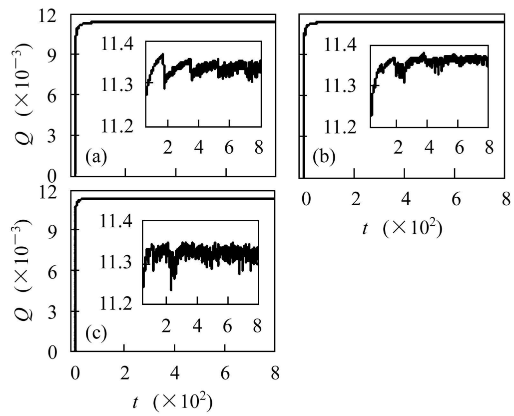

Fig. 3.　Dynamics of the multipartite entanglement measure Q for N=350. (a) J=0; (b) J=0.4; (c) J=0.8. The insets show the dependence of Q at larger values of the scaled time t.

4　Conclusion

In conclusion, we have studied the dynamics of the entanglement sharing in the so-called "one-particle" states governed by the Heisenberg model. We have studied the average of the concurrence $\langle C\rangle$ and the multipartite entanglement measure Q. Our results show that with the increase of the scaled time t, both $\langle C\rangle$ and Q first linearly increase, and then reach a plateau, oscillating irregularly around a steady value. And interestingly, for the case of $N\langle C\rangle$, the steady value is nearly independent of the length of the chain, and is only determined by the NNN coupling constant J.

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多量子位Heisenberg模型中纠缠的时间演化特性*

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摘要　研究了多量子位Heisenberg模型中纠缠的时间演化特性, 并给出了平均纠缠度$\langle C\rangle$和多体纠缠度Q的解析表达式. 结果发现无论是对$\langle C\rangle$还是对Q, 随着时间t的不断增长, 它们均先线性的增大, 而后达到一近似稳定状态, 并绕一平衡值做无规则的上下震荡. 若进一步考察$N\langle C\rangle$则还可以发现, 纠缠上下震荡的平衡值与Heisenberg链的长度几乎无关, 而仅由它们的次近邻耦合常数J决定.

关键词　Heisenberg模型　多体纠缠　次近邻相互作用

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