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由因式化 L 算子所构成的经典可积动力学体系

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摘要 用因式化的 L 算子构造了一类在非周期边界条件下的可积模型. 对体系的 transfer 矩阵取三角和标度极限情况下,得到了 n 维体系(n 为奇数)的经典哈密顿量的具体形式. 结果表明,这类可积体系与 Calogero 等人所发现的一系列可积体系是相类似的.

关键词 因式化的 L 算子 非周期边界条件 transfer 矩阵

1 引言

n 维可积体系是具有 n 个独立的守恒量,而且它们彼此之间泊松括号为零的体系. 近来,Calogero 等人[1—4] 找到了一系列可积体系,这些体系的特点是具有长程的相互作用. 他们用 Lax 对的方法证明了这些体系的可积性. 人们发现,其中的一些体系可以与精确可解的统计模型,与 Ruijsenaars-Macdonald 算子等联系起来[5,6]. 在 Belavin[7] 的统计模型中,可以由差分算子构成因式化 L 算子[5,8],文献[5]中 Hasegawa 将 L 算子与 Ruijsenaars-Macdonald 算子联系起来.

用此因式化的 L 算子来构造在非周期边界条件下 Z_n 对称的 Belavin 模型中一组可以相互对易的 transfer 矩阵集合,并计算在取三角和标度极限下该可积体系哈密顿量的具体形式. 结果表明,该可积体系与 Calogero 等人所发现的一系列可积体系是相类似的.利用这种方法,原则上只要能找到关于 Belavin 模型可积反射边界条件的反射方程一个解,就可以得到一个可积的经典动力学体系.

2 Z_n 对称的 Belavin R 矩阵

Z_n 对称的 Belavin R 矩阵为[7]

$$R_{12}(z)=\frac{1}{n}\sum_{\alpha\in Z_n^2}W_\alpha(z)I_\alpha\otimes I_\alpha^{-1}. \tag{1}$$

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其中 $\alpha=(\alpha_1,\alpha_2)\in Z_n^2$, I_α 是 $n\times n$ 矩阵,

$$W_\alpha(z)=\frac{\sigma_\alpha(z+\eta)}{\sigma_\alpha(\eta)},\ I_\alpha=g^{\alpha_2}h^{\alpha_1},\ h_{ij}=\delta_{i+1,j},$$

$$g_{ij}=\omega^i\delta_{ij},\ \omega=\mathrm{e}^{\frac{2\pi\sqrt{-1}}{n}},\ (i,j\in Z_n)$$

$$\sigma_\alpha(z)\equiv\theta\begin{bmatrix}\frac{1}{2}+\frac{\alpha_1}{n}\\ \frac{1}{2}+\frac{\alpha_2}{n}\end{bmatrix}(z,\tau),\tag{2}$$

$$\theta\begin{bmatrix}a\\ b\end{bmatrix}(z,\tau)\equiv\sum_{m\in Z}\mathrm{e}^{\pi\sqrt{-1}(m+a)^2\tau+2\pi\sqrt{-1}(m+a)(z+b)}.$$

R 矩阵满足 Yang-Baxter 方程(YBE)[9,10]

$$R_{12}(z_1-z_2)R_{13}(z_1-z_3)R_{23}(z_2-z_3)=R_{23}(z_2-z_3)R_{13}(z_1-z_3)R_{12}(z_1-z_2).\tag{3}$$

这里 $R_{12}(z)$，$R_{13}(z)$ 和 $R_{23}(z)$ 作用在 $C^n\otimes C^n\otimes C^n$ 空间上，$R_{12}(z)=R(z)\otimes I$, $R_{23}(z)=I\otimes R(z)$ 等等. R 矩阵同时也满足幺正性和交叉幺正性

$$R_{12}(z_1-z_2)R_{21}(z_2-z_1)=\rho(z_1-z_2)\cdot id,\tag{4}$$

$$R_{21}^{t_2}(z_2-z_1-nw)R_{12}^{t_2}(z_1-z_2)=\tilde{\rho}(z_1-z_2)\cdot id.\tag{5}$$

其中

$$\rho(z)=\frac{\sigma_0(z+w)\sigma_0(-z+w)}{\sigma_0^2(w)},\tag{6}$$

$$\tilde{\rho}(z)=\frac{\sigma_0(z)\ \sigma_0(-z-nw)}{\sigma_0^2(w)},\tag{7}$$

$w=n\eta$, t_i 表示第 i 空间的转置. 如有一算子矩阵 L 满足 Yang-Baxter 关系(YBR)

$$R_{12}(z_1-z_2)L_1(z_1)L_2(z_2)=L_2(z_2)L_1(z_1)R_{12}(z_1-z_2),\tag{8}$$

$(L_1(z)=L(z)\otimes I, L_2(z)=I\otimes L(z))$，对于周期边界条件来说，有 $[\mathrm{tr}L(z_1),\mathrm{tr}L(z_2)]=0$; 对于非周期边界条件，Sklyanin[11] 提出了反射方程(RE)[12]

$$R_{12}(z_1-z_2)K_1(z_1)R_{21}(z_1+z_2)K_2(z_2)=K_2(z_2)R_{12}(z_1+z_2)K_1(z_1)R_{21}(z_1-z_2).\tag{9}$$

K 为一个 $n\times n$ 的数值矩阵. 为了构造可积模型，还必须考虑其对偶的反射方程(DRE). Z_n Belavin 模型的对偶反射方程为[13]

$$R_{12}(z_2-z_1)\tilde{K}_1(z_1)R_{21}(-z_1-z_2-nw)\tilde{K}_2(z_2)=$$
$$\tilde{K}_2(z_2)R_{12}(-z_1-z_2-nw)\tilde{K}_1(z_1)R_{21}(z_2-z_1). \tag{10}$$

若定义 transfer 矩阵为 $t(z)=\mathrm{tr}[\tilde{K}(z)L(z)K(z)L^{-1}(-z)]$，利用 R 矩阵的幺正性(4)和交叉幺正性(5)，可以证明 $[t(z_1),t(z_2)]=0$. 于是这就提供了一种可积模型.

$\tilde{K}(z)$ 和 $K(z)$ 之间还存在一种同态关系. 令

$$\tilde{K}(z)=K\left(-z-\frac{nw}{2}\right), \tag{11}$$

则由一个反射方程 RE(9)的解 $K(z)$，就可得到对偶反射方程 DRE(10)的一个解 $\tilde{K}(z)$. 侯伯宇教授等人[14]找到了 RE(9)的一个解

$$K(z)=K_0(z)K_0(0),$$

$$K_0(z)=\sum_{\gamma\in Z_n^2}U_{2\gamma}(z)\,\omega^{2\gamma_1\gamma_2}I_{2\gamma}, \tag{12}$$

$$U_{2\gamma}(z)=\frac{\sigma_{2\gamma}(z+c)}{\sigma_{2\gamma}(c)}.$$

c 为任意常数.

3 因式化的 L 矩阵

Jimbo 等人[15] 给出了 Z_n Belavin 模型与 $A_{n-1}^{(1)}$ 面模型的 intertwiner，它是一个列矢量 $\phi_{a,a+\hat{\mu}}(z)$，其分量为

$$\phi_{a,a+\hat{\mu}}^{(j)}(z)=\theta^{(j)}(z-nw\bar{a}_\mu,n\tau), \tag{13}$$

$$\theta^{(j)}(z-nw\bar{a}_\mu,n\tau)=\theta\begin{bmatrix}\frac{1}{2}-\frac{j}{n}\\ \frac{1}{2}\end{bmatrix}(z-nw\bar{a}_\mu,n\tau), \tag{14}$$

$a=(a_0,a_1,\cdots,a_{n-1})\in Z^n$，$\hat{\mu}=(0,0,\cdots,1\cdots,0)$ (第 μ 个位置为 1). $\bar{a}_\mu=a_\mu-\frac{1}{n}\sum_\nu a_\nu+\delta_\mu$，$\delta_\mu$ 为一些一般的复数. 利用 intertwiner，面–顶角对应可以写为

$$R_{12}(z_1-z_2)_{ij}^{i'j'}\,\phi_{a-\hat{\mu}-\hat{\nu},a-\hat{\mu}}^{(i')}(z_1)\phi_{a-\hat{\mu},a}^{(j')}(z_2)=$$
$$\sum_\kappa W\begin{bmatrix}a-\hat{\mu}-\hat{\nu} & a-\hat{\mu}\\ a-\hat{\kappa} & a\end{bmatrix}(z_1-z_2)\phi_{a-\hat{\mu}-\hat{\nu},a-\hat{\kappa}}^{(j)}(z_2)\phi_{a-\hat{\kappa},a}^{(i)}(z_1). \tag{15}$$

$W\begin{bmatrix} a-\hat{\mu}-\hat{\nu} & a-\hat{\mu} \\ a-\hat{\kappa} & a \end{bmatrix}(z)$ 为 $A^{(1)}_{n-1}$ 面模型的 Boltzmann 权．关于 μ, ν, κ 取值的以下三种情况，Boltzmann 权为

$$W\begin{bmatrix} a-2\hat{\mu} & a-\hat{\mu} \\ a-\hat{\mu} & a \end{bmatrix}(z) = \frac{\sigma_0(z+w)}{\sigma_0(w)},$$

$$\left.\begin{aligned} W\begin{bmatrix} a-\hat{\mu}-\hat{\nu} & a-\hat{\mu} \\ a-\hat{\mu} & a \end{bmatrix}(z) &= \frac{\sigma_0(z+a_{\mu\nu}w)}{\sigma_0(a_{\mu\nu}w)} \\ W\begin{bmatrix} a-\hat{\mu} & a-\hat{\nu} \\ a-\hat{\mu} & a \end{bmatrix}(z) &= \frac{\sigma_0(z)\sigma_0((1-a_{\mu\nu})w)}{\sigma_0(w)\sigma_0(-a_{\mu\nu}w)} \end{aligned}\right\} \mu \neq \nu, \tag{16}$$

其余情况，Boltzmann 权皆为零．式中 $a_{\mu\nu} = \bar{a}_\mu - \bar{a}_\nu$．还可以构造满足下列关系的行矢量 $\tilde{\phi}$ 和 $\bar{\phi}$ [16—18]

$$\tilde{\phi}^{(k)}_{a-\hat{\mu},a}(z)\,\phi^{(k)}_{a-\hat{\nu},a}(z) = \delta_{\mu\nu}, \tag{17}$$

$$\bar{\phi}^{(k)}_{a,a+\hat{\mu}}(z)\,\phi^{(k)}_{a,a+\hat{\nu}}(z) = \delta_{\mu\nu}. \tag{18}$$

上述关系又可写成

$$\sum_\mu \phi_{a-\hat{\mu},a}(z)\,\tilde{\phi}_{a-\hat{\mu},a}(z) = I, \tag{19}$$

$$\sum_\mu \phi_{a,a+\hat{\mu}}(z)\,\bar{\phi}_{a,a+\hat{\mu}}(z) = I. \tag{20}$$

利用 $\bar{\phi}$, $\tilde{\phi}$ 和 ϕ 的关系，面–顶角对应可写成以下几种形式

$$\tilde{\phi}^{(i)}_{a-\hat{\mu},a}(z_1)\,R_{12}(z_1-z_2)^{i'j'}_{ij}\,\phi^{(j')}_{a-\hat{\nu},a}(z_2) =$$

$$\sum_\kappa W\begin{bmatrix} a-\hat{\mu}-\hat{\kappa} & a-\hat{\nu} \\ a-\hat{\mu} & a \end{bmatrix}(z_1-z_2)\,\phi^{(j)}_{a-\hat{\mu}-\hat{\kappa},a-\hat{\mu}}(z_2)\,\tilde{\phi}^{(i')}_{a-\hat{\mu}-\hat{\kappa},a-\hat{\nu}}(z_1), \tag{21}$$

$$\bar{\phi}^{(j)}_{a,a+\hat{\mu}}(z_2)\,R_{12}(z_1-z_2)^{i'j'}_{ij}\,\phi^{(i')}_{a,a+\hat{\nu}}(z_1) =$$

$$\sum_\kappa W\begin{bmatrix} a & a+\hat{\nu} \\ a+\hat{\mu} & a+\hat{\mu}+\hat{\kappa} \end{bmatrix}(z_1-z_2)\,\phi^{(i)}_{a+\hat{\mu},a+\hat{\mu}+\hat{\kappa}}(z_1)\,\bar{\phi}^{(j')}_{a+\hat{\nu},a+\hat{\mu}+\hat{\kappa}}(z_2), \tag{22}$$

$$\tilde{\phi}^{(i)}_{a-\hat{\mu},a}(z_1)\,\tilde{\phi}^{(j)}_{a-\hat{\mu}-\hat{\nu},a-\hat{\mu}}(z_2)\,R_{12}(z_1-z_2)^{i'j'}_{ij} =$$

$$\sum_\kappa W\begin{bmatrix} a-\hat{\mu}-\hat{\nu} & a-\hat{\kappa} \\ a-\hat{\mu} & a \end{bmatrix}(z_1-z_2)\,\tilde{\phi}^{(j')}_{a-\hat{\kappa},a}(z_2)\,\tilde{\phi}^{(i')}_{a-\hat{\mu}-\hat{\nu},a-\hat{\kappa}}(z_1), \tag{23}$$

$$\bar{\phi}^{(i)}_{a+\hat{\mu},a+\hat{\mu}+\hat{\nu}}(z_1)\,\bar{\phi}^{(j)}_{a,a+\hat{\mu}}(z_2)\,R_{12}(z_1-z_2)^{i'j'}_{ij} =$$

$$\sum_{\kappa} W\begin{bmatrix} a & a+\hat{\kappa} \\ a+\hat{\mu} & a+\hat{\mu}+\hat{v} \end{bmatrix}(z_1-z_2)\,\bar{\phi}^{(j')}_{a+\hat{\kappa},a+\hat{\mu}+\hat{v}}(z_2)\bar{\phi}^{(i')}_{a,a+\hat{\kappa}}(z_1). \tag{24}$$

利用这些关系,可以构造因式化的 L 矩阵[8].

令

$$L(z)^{j}_{i}=\sum_{\mu}\Gamma_{\mu}\,\phi^{(i)}_{a-\hat{\mu},a}(z)\,\tilde{\phi}^{(j)}_{a-\hat{\mu},a}(z), \tag{25}$$

其中Γ_{μ}为差分算子,定义为

$$\Gamma_{\mu}f(a)=f(a+\hat{\mu})\,\Gamma_{\mu}. \tag{26}$$

利用面–顶角关系(15)和(23)式,可证明

$$R(z_1-z_2)^{i'j'}_{ij}\,L(z_1)^{i''}_{i'}\,L(z_2)^{j''}_{j'}=L(z_2)^{j'}_{j}\,L(z_1)^{i'}_{i}\,R(z_1-z_2)^{i''j''}_{i'j'}. \tag{27}$$

即 $L(z)$满足 Yang-Baxter 关系. 另外,可以证明

$$L^{-1}(z)^{j}_{i}=\sum_{\mu}\Gamma_{-\mu}\phi^{(i)}_{a,a+\hat{\mu}}(z)\,\bar{\phi}^{(j)}_{a,a+\hat{\mu}}(z). \tag{28}$$

因此 transfer 矩阵为

$$t(z)=\mathrm{tr}[\tilde{K}(z)L(z)K(z)L^{-1}(-z)]=\sum_{\mu v}\Gamma_{-\mu}\Gamma_{v}\,G_{\mu v}(z). \tag{29}$$

其中

$$G_{\mu v}(z)=F^{(1)}_{\mu v}(z)F^{(2)}_{\mu v}(z), \tag{30}$$

$$F^{(1)}_{\mu v}(z)=\tilde{\phi}^{(i)}_{a+\hat{\mu}-\hat{v},a+\hat{\mu}}(z)K(z)^{j}_{i}\,\phi^{(j)}_{a,a+\hat{\mu}}(-z), \tag{31}$$

$$F^{(2)}_{\mu v}(z)=\bar{\phi}^{(k)}_{a,a+\hat{\mu}}(-z)\tilde{K}(z)^{l}_{k}\,\phi^{(l)}_{a+\hat{\mu}-\hat{v},a+\hat{\mu}}(z). \tag{32}$$

4 $t(z)$的计算

把 $K(z)$(12)代入(31),得

$$F^{(1)}_{\mu v}(z)=\tilde{\phi}^{(i)}_{a+\hat{\mu}-\hat{v},a+\hat{\mu}}(z)\sum_{\gamma\in Z_n^2}U_{2\gamma}(z)\,\omega^{2\gamma_1\gamma_2}\times[g^{2\gamma_2}h^{2\gamma_1}K_0(0)]^{j}_{i}\,\phi^{(j)}_{a,a+\hat{\mu}}(-z). \tag{33}$$

可以证明(参见附录)

$$g^{\beta-1}h^{\alpha}K_0(0)\,\phi_{a,a+\hat{\mu}}(z)=$$

$$(-1)^{\beta-1}\mathrm{e}^{2\pi\sqrt{-1}\frac{\alpha}{n}\left(\frac{\alpha}{2}\tau+z-nw\bar{a}_{\mu}+\frac{1}{2}\right)}\,\phi_{a,a+\hat{\mu}}(-z+2nw\bar{a}_{\mu}-\alpha\tau-\beta). \tag{34}$$

利用上述结果

$$F^{(1)}_{\mu v}(z)=\sum_{\gamma\in Z_n^2}U_{2\gamma}(z)\,\mathrm{e}^{2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}}\mathrm{e}^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}\left(\frac{2\gamma_1}{2}\tau-z-nw\bar{a}_{\mu}+\frac{1}{2}\right)}\times$$

$$\tilde{\phi}^{(i)}_{a+\hat{\mu}-\hat{v},a+\hat{\mu}}(z)\,\phi^{(i)}_{a,a+\hat{\mu}}(z+2nw\bar{a}_\mu-2\gamma_1\tau-2\gamma_2-1). \tag{35}$$

由(19)式可以知道$\tilde{\phi}_{a+\hat{\mu}-\hat{v},a+\hat{\mu}}(z)$能从矩阵 $\tilde{M}$的逆矩阵获得．其中

$$\tilde{M}_{i\lambda}(z)=\phi^{(i)}_{a+\hat{\mu}-\hat{\lambda},a+\hat{\mu}}(z)=\theta^{(i)}(z-nw(\bar{a}_\lambda+\delta_{\mu\lambda}-1)), \tag{36}$$

同时定义矩阵 $\tilde{M}'$ 为把矩阵 $\tilde{M}$第 v 列元素用列矢量$\phi_{a,a+\hat{\mu}}(z+2nw\bar{a}_\mu-2\gamma_1\tau-(2\gamma_2+1))$相应的元素替换而得到的矩阵，可得

$$\tilde{\phi}^{(i)}_{a+\hat{\mu}-\hat{v},a+\hat{\mu}}(z)\,\phi^{(i)}_{a,a+\hat{\mu}}(z+2nw\bar{a}_\mu-2\gamma_1\tau-(2\gamma_2+1))=\frac{\det\tilde{M}'}{\det\tilde{M}}. \tag{37}$$

如有一矩阵 $A_{ij}=\theta^{(i)}(nz_j)$[17]，则

$$\det A=C(\tau)\sigma_0\left(\sum_i z_i-\frac{n-1}{2}\right)\prod_{i<k}\sigma_0(z_i-z_k). \tag{38}$$

利用上式

$$\frac{\det\tilde{M}'}{\det\tilde{M}}=$$

$$\frac{\sigma_0\left(-z+w\delta+w(1-n)+\frac{n-1}{2}-w(\bar{a}_v+\delta_{\mu v}+\bar{a}_\mu-1)+\frac{1}{n}(2\gamma_1\tau+2\gamma_2+1)\right)}{\sigma_0\left(-z+w\delta+w(1-n)+\frac{n-1}{2}\right)}\times$$

$$\prod_{j\neq v}\frac{\sigma_0(-w(\bar{a}_j+\delta_{\mu j}+\bar{a}_\mu-1)+\frac{1}{n}(2\gamma_1\tau+2\gamma_2+1))}{\sigma_0(-w(\bar{a}_j+\delta_{\mu j}-\bar{a}_v-\delta_{\mu v}))}. \tag{39}$$

$\left(\delta=\sum_i\delta_i\right)$．运用公式

$$\sigma_0\left(z+\frac{1}{n}(\alpha\tau+\beta)\right)=\mathrm{e}^{-2\pi\sqrt{-1}\frac{\alpha}{n}\left(\frac{\alpha}{2n}\tau+z+\frac{1}{2}+\frac{\beta}{n}\right)}\sigma_{(\alpha,\beta)}(z,\tau), \tag{40}$$

最终得到

$$F^{(1)}_{\mu v}(z)=\sum_{\gamma\in Z_n^2}U_{2\gamma}(z)\,\mathrm{e}^{-2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}}\,\mathrm{e}^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}}\times$$

$$\frac{\sigma(2\gamma_1,2\gamma_2+1)\left(-z+w\delta+w(1-n)+\frac{n-1}{2}-w(\bar{a}_v+\delta_{\mu v}+\bar{a}_\mu-1)\right)}{\sigma_0\left(-z+w\delta+w(1-n)+\frac{n-1}{2}\right)}\times$$

$$\prod_{j\neq v}\frac{\sigma(2\gamma_1,2\gamma_2+1)(-w(\bar{a}_j+\delta_{\mu j}+\bar{a}_\mu-1))}{\sigma_0(-w(\bar{a}_j+\delta_{\mu j}-\bar{a}_v-\delta_{\mu v}))}. \tag{41}$$

重复上述过程并利用 $\tilde{K}(z)$ 与 $K(z)$ 之间的同态关系(11),得到

$$F_{\mu\nu}^{(2)}(z)=\sum_{\gamma\in Z_n^2}U_{2\gamma}\left(-z-\frac{nw}{2}\right)\mathrm{e}^{-2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}}\mathrm{e}^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}}\times$$

$$\frac{\sigma(2\gamma_1,2\gamma_2+1)\left(z+w\delta+\frac{n-1}{2}-w(\bar{a}_\mu+\delta_{\mu\nu}+\bar{a}_\nu-1)\right)}{\sigma_0\left(z+w\delta+\frac{n-1}{2}\right)}\times$$

$$\prod_{j\neq\mu}\frac{\sigma(2\gamma_1,2\gamma_2+1)(-w(\bar{a}_j+\delta_{\mu\nu}+\bar{a}_\nu-1))}{\sigma_0(-w(\bar{a}_j-\bar{a}_\mu))}.\tag{42}$$

5 $t(z)$的三角和标度极限

取$\tau\to\sqrt{-1}\,\infty$,就可得到 $t(z)$的三角极限. 当$\frac{2\gamma_1}{n}\neq m$($m$ 为整数)时

$$\theta\begin{bmatrix}\frac{1}{2}+\frac{2\gamma_1}{n}\\ \frac{1}{2}+\frac{2\gamma_2+1}{n}\end{bmatrix}(z,\tau)\to f(\tau)\mathrm{e}^{2\pi\sqrt{-1}\left(\frac{2\gamma_1}{n}+\frac{1}{2}+m\right)\left(z+\frac{1}{2}+\frac{2\gamma_2+1}{n}\right)},\tag{43}$$

则

$$U_{2\gamma}(z)\to\mathrm{e}^{2\pi\sqrt{-1}\left(\frac{2\gamma_1}{n}+\frac{1}{2}+m\right)z},$$

$$F_{\mu\nu}^{(1)}(z)\to\sum_{\gamma_1}\frac{\mathrm{e}^{2\pi\sqrt{-1}\left(\frac{2\gamma_1}{n}+\frac{1}{2}+m\right)z}\,\mathrm{e}^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}}}{\sin\pi\left(-z+w\delta+w(1-n)+\frac{n-1}{2}\right)}\times$$

$$\prod_{j\neq\nu}\frac{1}{\sin\pi(\bar{a}_j+\delta_{\mu j}-\bar{a}_\nu-\delta_{\mu\nu})}\times$$

$$\sum_{\gamma_2}\mathrm{e}^{-2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}}f^n(\tau)\mathrm{e}^{2\pi\sqrt{-1}\left(\frac{2\gamma_1}{n}+\frac{1}{2}+m\right)\left(-z+n-\frac{1}{2}-nw\bar{a}_\mu+2\gamma_2\right)}\to$$

$$\sum_{\gamma_2}\mathrm{e}^{2\pi\sqrt{-1}\frac{2\gamma_1\gamma_2}{n}}=\sum_{\gamma_2}\left(\mathrm{e}^{2\pi\sqrt{-1}\frac{2\gamma_1}{n}}\right)^{\gamma_2}=0.\tag{44}$$

$F_{\mu\nu}^{(2)}(z)$也同样为零. 因此只考虑$\frac{2\gamma_1}{n}=m$时的情形. 在这种情况下,对 $U_{2\gamma}(z)$中的参数 c 做变量替换 $c=s\tau+c'\left(s\ll\frac{1}{n}\right)$,则有

$$\theta\begin{bmatrix}\frac{1}{2}+\frac{2\gamma_1}{n}\\ \frac{1}{2}+\frac{2\gamma_2}{n}\end{bmatrix}(z+c,\tau)=\mathrm{e}^{-2\pi\sqrt{-1}\,s\left(\frac{2s}{n}\tau+z+c'+\frac{1}{2}+\frac{2\gamma_2}{n}\right)}\theta\begin{bmatrix}\frac{1}{2}+\frac{2\gamma_1}{n}+s\\ \frac{1}{2}+\frac{2\gamma_2}{n}\end{bmatrix}(z+c',\tau), \tag{45}$$

$$U_{2\gamma}(z)=\mathrm{e}^{-2\pi\sqrt{-1}\,sz}\frac{\theta\begin{bmatrix}\frac{1}{2}+\frac{2\gamma_1}{n}+s\\ \frac{1}{2}+\frac{2\gamma_2}{n}\end{bmatrix}(z+c',\tau)}{\theta\begin{bmatrix}\frac{1}{2}+\frac{2\gamma_1}{n}+s\\ \frac{1}{2}+\frac{2\gamma_2}{n}\end{bmatrix}(c',\tau)}$$

$$\xrightarrow[c'\text{不变}]{\tau\to\sqrt{-1}\,\infty}\mathrm{e}^{-2\pi\sqrt{-1}\,sz}\mathrm{e}^{2\pi\sqrt{-1}\left(\frac{1}{2}+\frac{2\gamma_1}{n}+s+m\right)z}. \tag{46}$$

可以看出，$U_{2\gamma}(z)$ 与 γ_2 无关且不含任何动力学变量，因此在这样地取三角极限时，可以当作常数来处理．结论(44)仍然成立．在 $\frac{2\gamma_1}{n}=m$ 时

$$F_{\mu\nu}^{(1)}(z)\to\sum_{\gamma_2}\frac{\sin\pi\left(-z+w\delta+w(1-n)+\frac{n-1}{2}-w(\bar a_\nu+\delta_{\mu\nu}+\bar a_\mu-1)+\frac{2\gamma_2+1}{n}\right)}{\sin\pi\left(-z+w\delta+w(1-n)+\frac{n-1}{2}\right)}\times$$

$$\prod_{j\neq\nu}\frac{\sin\pi\left(-w(\bar a_j+\delta_{\mu j}+\bar a_\mu-1)+\frac{2\gamma_2+1}{n}\right)}{\sin\pi(-w(\bar a_j+\delta_{\mu j}-\bar a_\nu-\delta_{\mu\nu}))}, \tag{47}$$

$$F_{\mu\nu}^{(2)}(z)\to\sum_{\gamma_2}\frac{\sin\pi\left(z+w\delta+\frac{n-1}{2}-w(\bar a_\nu+\delta_{\mu\nu}+\bar a_\mu-1)+\frac{2\gamma_2+1}{n}\right)}{\sin\pi\left(z+w\delta+\frac{n-1}{2}\right)}\times$$

$$\prod_{j\neq\mu}\frac{\sin\pi\left(-w(\bar a_j+\delta_{\mu\nu}+\bar a_\mu-1)+\frac{2\gamma_2+1}{n}\right)}{\sin\pi(-w(\bar a_j-\bar a_\mu))}. \tag{48}$$

令 $z\to-\sqrt{-1}\,\infty$，移去 $F_{\mu\nu}^{(i)}(z)$ 中的谱参数 z，并把分母上含 γ_2 的正弦函数用指数函数来表示，发现，$F_{\mu\nu}^{(i)}(z)$ 的所有求和项均具有 $\sum_{\gamma_2}\mathrm{e}^{\frac{2\pi\sqrt{-1}}{n}m\gamma_2}$ $(m=1\overbrace{\pm1\pm\cdots\pm1}^{n-1}=n,n-2,$

$\cdots, -n+2)$的形式．只有 $m=n$ 或 0 时，该求和项才不为零．当 n 为偶数时，$F_{\mu\nu}^{(i)}(z)$的结果仍相当复杂，为简单起见，只考虑 n 为奇数时的情况．这时

$$F_{\mu\nu}^{(1)}(z) \xrightarrow{z\to -\sqrt{-1}\,\infty} \mathrm{e}^{\pi\sqrt{-1}\sum_i\left(-w(\bar{a}_i+\delta_{\mu i}+\bar{a}_\mu-1)+\frac{1}{n}\right)} \times$$

$$\prod_{j\neq\nu} \frac{1}{\sin\pi(-w(\bar{a}_j+\delta_{\mu j}-\bar{a}_\nu-\delta_{\mu\nu}))}, \tag{49}$$

$$F_{\mu\nu}^{(2)}(z) \xrightarrow{z\to -\sqrt{-1}\,\infty} \mathrm{e}^{-\pi\sqrt{-1}\sum_i\left(-w(\bar{a}_i+\delta_{\mu\nu}+\bar{a}_\nu-1)+\frac{1}{n}\right)} \times$$

$$\prod_{j\neq\mu} \frac{1}{\sin\pi(-w(\bar{a}_j-\bar{a}_\mu))}. \tag{50}$$

又 $\bar{a}_j-\bar{a}_k=a_j-a_k+\delta_j-\delta_k$, $H=\sum_{\mu\nu}\Gamma_{-\mu}\Gamma_\nu G_{\mu\nu}(z)$，则

$$H=\sum_{\mu\nu}\Gamma_\nu\left[\mathrm{e}^{\pi\sqrt{-1}\,(nw(a_\nu+\delta_\nu))}\prod_{j\neq\nu}\frac{1}{\sin\pi(-w(a_j-a_\nu+\delta_j-\delta_\nu))}\right]\times$$

$$\Gamma_{-\mu}\left[\mathrm{e}^{-\pi\sqrt{-1}\,(nw(a_\mu+\delta_\mu))}\prod_{k\neq\mu}\frac{1}{\sin\pi(-w(a_k-a_\mu+\delta_k-\delta_\mu))}\right]. \tag{51}$$

在量子力学中，$\hat{p}_\mu=\dfrac{\hbar}{\sqrt{-1}}\dfrac{\partial}{\partial x_\mu}$

$$\mathrm{e}^{\frac{\hbar}{\sqrt{-1}}\frac{\partial}{\partial x_\mu}}f(x)=f\left(x+\frac{\hbar}{\sqrt{-1}}\hat{\mu}\right)\mathrm{e}^{\frac{\hbar}{\sqrt{-1}}\frac{\partial}{\partial x_\mu}}. \tag{52}$$

同$\Gamma_\mu f(a)=f(a+\hat{\mu})\Gamma_\mu$ 比较，令$\Gamma_\mu\to\mathrm{e}^{\frac{\hbar}{\sqrt{-1}}\frac{\partial}{\partial x_\mu}}$, $w\to\hbar$, $-\sqrt{-1}\,w(a_\mu+\delta_\mu)\to x_\mu$，则有

$$H=\sum_{\mu\nu}\mathrm{e}^{\hat{p}_\nu}\left[\mathrm{e}^{-n\pi x_\nu}\prod_{j\neq\nu}\frac{1}{\sinh\pi(x_j-x_\nu)}\right]\times$$

$$\bar{\mathrm{e}}^{\hat{p}_\mu}\left[\mathrm{e}^{n\pi x_\mu}\prod_{k\neq\mu}\frac{1}{\sinh\pi(x_k-x_\mu)}\right]. \tag{53}$$

令 $\hat{p}_\lambda\to p_\lambda$，则得到经典可积的哈密顿量．

可积体系要求力学量集合 $H=f_1,f_2,\cdots,f_n$满足

$$\{f_i,f_j\}=\sum_{lm}\left(\frac{\partial f_i}{\partial p_l}\frac{\partial f_j}{\partial q_m}-\frac{\partial f_i}{\partial q_l}\frac{\partial f_j}{\partial p_m}\right)=0, \tag{54}$$

对于一个可积体系，做变量替换，例如令 $q'_i=\chi q_i+\rho_i$，(χ, ρ_i 均为参数)，ρ_i不变，会发现

在新坐标系下(54)仍然成立，即仍为可积体系．再适当的取极限仍可能得到新的可积体系．例如令(53)中的 $x'_i = \chi x_i$，则

$$H' = \sum_{\mu\nu} e^{p_\nu}\left[e^{-n\pi\frac{1}{\chi}x'_\nu}\prod_{j\neq\nu}\frac{1}{\sinh\frac{\pi}{\chi}(x'_j - x'_\nu)}\right]\times$$

$$e^{-p_\mu}\left[e^{n\pi\frac{1}{\chi}x'_\mu}\prod_{k\neq\mu}\frac{1}{\sinh\frac{\pi}{\chi}(x'_k - x'_\mu)}\right]; \tag{55}$$

令 $\chi\to\infty$，则

$$H' = \sum_{\mu\nu} e^{p_\nu - p_\mu}\prod_{j\neq\nu}\frac{1}{x'_j - x'_\nu}\prod_{k\neq\mu}\frac{1}{x'_k - x'_\mu}. \tag{56}$$

这依然可能是一个可积的哈密顿量．

也可令 $x'_0 = x_0 + \rho_0, \rho_0\to\infty$，保持其它变量不变，使 $\sinh\pi(x_0 - x_i)$ 变成 $\frac{1}{2}e^{\pi(x_0 - x_i)}$ 来获得一个新的可积哈密顿量．也可改变更多的变量去获得更多的哈密顿量．例如同时改变 n 个变量并令 $\delta_0 \gg \delta_1 \gg \cdots \gg \delta_{n-1} \gg 1, x'_i = x_i + \delta_i$，则有 $\sin\pi(x_i - x_j)\to\frac{1}{2}e^{\pi(x_i - x_j + \delta_i - \delta_j)}$ $(i < j)$，这时(53)变为

$$H\to\sum_{\mu\nu} e^{p_\nu - p_\mu}\, e^{-n\pi(x_\mu - x_\nu)}\prod_{j<\nu} e^{-\pi(x_j - x_\nu)}\times$$

$$\prod_{j>\nu} e^{-\pi(x_\nu - x_j)}\prod_{k<\mu} e^{-\pi(x_k - x_\mu)}\prod_{k>\mu} e^{-\pi(x_\mu - x_k)}. \tag{57}$$

这仍然可能是一个可积的哈密顿量．

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附录

$$A=[g^{\beta}h^{\alpha}K_0(0)\phi_{a,\alpha+\hat{\mu}}(z)]^{(i)}=\omega^{\beta i}\delta_{ij}\bar{\delta}_{j+\alpha,k}\bar{\delta}_{k+l,0}\theta^{(l)}(z-nw\bar{a}_{\mu},n\tau)=$$

$$\omega^{\beta i}\theta\begin{bmatrix}\frac{1}{2}+\frac{i+\alpha}{n}\\ \frac{1}{2}\end{bmatrix}(z-nw\bar{a}_{\mu},n\tau)=\omega^{\beta i}\theta\begin{bmatrix}-\frac{1}{2}-\frac{i+\alpha}{n}\\ -\frac{1}{2}\end{bmatrix}(-z+nw\bar{a}_{\mu},n\tau)=$$

$$(-1)\omega^{\beta i}e^{2\pi\sqrt{-1}\frac{i+\alpha}{n}}\theta\begin{bmatrix}\frac{1}{2}-\frac{i+\alpha}{n}\\ \frac{1}{2}\end{bmatrix}(-z+nw\bar{a}_{\mu},n\tau).$$

$\bar{\delta}_{ij}$ 表示下指标对 n 取余．利用公式

$$\theta^{(i)}(z+\alpha\tau+\beta,n\tau)=e^{-2\pi\sqrt{-1}\frac{\alpha}{n}\left(\frac{\alpha}{2}\tau+z+\beta+\frac{1}{2}\right)}e^{2\pi\left(\frac{1}{2}-\frac{i-\alpha}{n}\right)\beta}\times$$

$$\theta\begin{bmatrix}\frac{1}{2}-\frac{i-\alpha}{n}\\ \frac{1}{2}\end{bmatrix}(z,n\tau),$$

有

$$A=(-1)^{(\beta-1)}e^{2\pi\sqrt{-1}\frac{\alpha}{n}\left(\frac{\alpha}{2}\tau+z-nw\bar{a}_{\mu}+\frac{1}{2}\right)}\omega^{i}\theta\begin{bmatrix}\frac{1}{2}-\frac{i}{n}\\ \frac{1}{2}\end{bmatrix}(-z+nw\bar{a}_{\mu}-\alpha\tau-\beta,n\tau)=$$

$$(-1)^{(\beta-1)}e^{2\pi\sqrt{-1}\frac{\alpha}{n}\left(\frac{\alpha}{2}\tau+z-nw\bar{a}_{\mu}+\frac{1}{2}\right)}\omega^{i}\phi^{(i)}_{a,\alpha+\hat{\mu}}(-z+2nw\bar{a}_{\mu}-\alpha\tau-\beta)=$$

$$(-1)^{(\beta-1)}e^{2\pi\sqrt{-1}\frac{\alpha}{n}\left(\frac{\alpha}{2}\tau+z-nw\bar{a}_{\mu}+\frac{1}{2}\right)}[g\phi_{a,\alpha+\hat{\mu}}(-z+2nw\bar{a}_{\mu}-\alpha\tau-\beta)]^{(i)}.$$

把等式右边 g 移到等式左边，即得(34)式．

A Kind of Classically Integrable Dynamics System Constructed by Factorized L Operator

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Abstract An integrable model with non-period boundary condition is constructed by use of factorized L operator. Taking trigonometric limit and scalar limit to the transfer matrix, we obtain the classical Hamiltonian of the n dimentional system (n is odd number). The result shows that this integrable system is similar to those found by Calogero et al.

Key words factorized L operator, non-period boundary condition, transfer matrix

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