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Factorial Moments of Continuous Order and Multifractal Analysis in 400 GeV/c pp Collisions

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The method of factorial moments F_q of continuous order suggested by Hwa has been tested. It was found that using this method to analyze the experimental data will not produce satisfactory results. Some improvements were made for Hwa's method, making it suitable for multifractal analysis of experimental data. The analytic results for the experimental pseudorapidity distributions of charged particles produced in 400 GeV/c pp collisions indicated that the method of the factorial moments of continuous order is feasible. There is possibly multifractal behavior in the process of multiplicity production in pp collisions at 400 GeV/c.

Key words: factorial moments of continuous order, negative binomial distribution, maximum likelihood function method, multifractal behavior.

1. INTRODUCTION

In order to investigate the intermittent phenomenon of multiparticle productions in high energy collisions, one may calculate the moments of the multiplicity distribution and study their dependence on the size of phase space. The multiplicity distribution originates from the dynamic and statistical fluctuations. In high energy hadron collisions, the multiplicity is usually small, especially when the phase space becomes small, and the statistical fluctuation will be a dominant factor compared to the dynamic one. Bialas and Peschanski suggested investigating the intermittent phenomenon using the scaled factorial moment F_q [1] which was defined as

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$$F_q = \frac{\langle n(n-1)\cdots(n-q+1)\rangle}{\langle n\rangle^q} \quad , \tag{1}$$

which gives a nonbiased estimation of the moments of dynamic fluctuation. It greatly improved the study of intermittency. However, the defect of the factorial moment is that the order of moment should be an integer greater than or equal to 2. Consequently, one cannot obtain two important fractal dimensions D_0 and D_1 and the dip signal of the pseudorapidity distributions. It was later suggested to use the G_q moments and various modified ones, which have arbitrary q values and overcome the defects of F_q. However, the G_q moments have an inherent defect, i.e., they did not eliminate the influence of statistical background. Subtraction of the statistical component has to be done by hand [2]. Because of this, it is necessary to make the order of F_q moments continuously varying in order to have both merits of F_q and G_q and to overcome both defects. Hwa recently proposed a new method [3] to obtain F_q moment of continuous order. However, it is unsatisfactory for use in experimental data analysis. We have made some modifications for this method and made it feasible for multifractal analysis of experimental data.

2. HWA'S METHOD OF FACTORIAL MOMENT OF CONTINUOUS ORDER

The multiplicity distribution P_n can be represented as

$$P_n = S \times D, \tag{2}$$

where S represents the statistical fluctuation which was described by Poisson distribution, and D represents the dynamic component. Further, it can be written as

$$P_n = \int_0^\infty \mathrm{d}t \frac{t^n}{n!} \mathrm{e}^{-t} D(t) . \tag{3}$$

The scaled factorial moment F_q can be written as

$$F_q = f_q / f_1^q, \tag{4}$$

where

$$f_q = \sum_{n=q}^{\infty} \frac{n!}{(n-q)!} P_n. \tag{5}$$

Because q is a positive integer, substituting Eq. (3) into Eq. (5) and performing the summation over n, one can obtain

$$f_q = \int_0^\infty \mathrm{d}t\, t^q D(t). \tag{6}$$

It is the q-th moment of the dynamic fluctuation $D(t)$. However, for continuous order q, Eq. (5) cannot be applied to obtain Eq. (6) of the q-th moment f_q of the dynamic fluctuation. So, other methods are needed. It is demanded in Ref. [3] that Eq. (6) is suitable for all q. In order to obtain the expression of $D(t)$, the multiplicity distribution P_n is expanded using negative binomial distribution (NBD), i.e.,

$$P_n = \sum_{j=0}^{N} a_j P_n^{\mathrm{NB}}(k_j, x_j), \tag{7}$$

where $n = 0, 1, \cdots, N$ (N is the maximum multiplicity), and

$$P_n^{\mathrm{NB}}(k_j, x_j) = \frac{\Gamma(n+k_j)}{\Gamma(n+1)\,\Gamma(k_j)}\left(\frac{k_j}{k_j+x_j}\right)^{k_j}\left(\frac{x_j}{k_j+x_j}\right)^{n}. \tag{8}$$

Using Eq. (3) and the following relation

$$P_n^{\mathrm{NB}}(k_j, x_j) = \int_0^\infty \mathrm{d}t \frac{t^n}{n!} \mathrm{e}^{-t} D^{\mathrm{NB}}(t, j), \tag{9}$$

where

$$D^{\mathrm{NB}}(t, j) = \left(\frac{k_j}{x_j}\right)^{k_j} \frac{t^{k_j-1}}{\Gamma(k_j)} \mathrm{e}^{-(k_j t / x_j)}, \tag{10}$$

one gets

$$D(t) = \sum_{j=0}^{N} a_j D^{\mathrm{NB}}(t, j). \tag{11}$$

Substituting (11) to Eq. (6) and integrating over t, one finds

$$f(q) = \sum_{j=0}^{N} a_j f^{\mathrm{NB}}(q, j), \tag{12}$$

where

$$f^{\mathrm{NB}}(q, j) = \left(\frac{x_j}{k_j}\right)^{q} \frac{\Gamma(q+k_j)}{\Gamma(k_j)}. \tag{13}$$

Then one obtains the relation similar to Eq. (4)

$$F(q) = f(q) \,/\, f(1)^q. \tag{14}$$

To determine the value of a_j, one should assign $N + 1$ pairs of x_j, k_j, then calculate $P_n^{\mathrm{NB}}(k_j, x_j)$ according to Eq. (8), and get the values of a_j by solving the $N + 1$ linear algebraic Eq. (7). Therefore, the factorial moments of continuous order $F(q)$ can be obtained by Eqs. (12) and (14).

3. TEST OF HWA'S METHOD

To have a test of Hwa's method, we apply Hwa's method to the multiplicity distribution with Poissonian distribution

$$P_n = \mathrm{e}^{-\langle n \rangle} \langle n \rangle^n \,/\, n!, \tag{15}$$

P_n is calculated analytically according to the above equation with $\langle n \rangle = 1$, $n = 0, 1, \cdots, N$; $N = 10$. We used the above method to expand P_n and calculated $N + 1$ expansion coefficients a_j, then calculated the factorial moment of continuous order F_q. In the calculation, the program package called MATHEMATICA is adopted to ensure numerical accuracy. The results are shown in Fig. 1 with a solid line. It can be seen that $F(q)$ has a constant value of 1. In an experiment, the access to an accurate value of P_n would require measurement of an infinite number of events. However, in a real experiment,

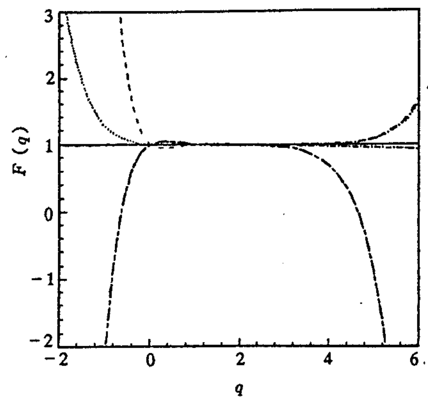

Fig. 1

The results obtained by Hwa's method and by the maximum likelihood method for Poisson distribution ($\langle n \rangle = 1$).

Solid line: the results of the analytic P_n; long-short line, dotted line, dashed line: the results for 10^5, 10^6, 10^7 MC events; dot-dashed line: the calculated results using the maximum likelihood method for 10^5 MC events sample.

one can only measure a finite number of events. So each P_n is measured with a statistical error. In order to see the effect of the statistical fluctuation, we performed a Monte Carlo simulation according to the Poisson distribution. Three samples are generated. Then we counted up the P_n and calculated $F(q)$ according to the Hwa's method. The results are shown in Fig. 1. It can be seen from Fig. 1 that the $F(q)$'s deviated from 1 obviously and the deviations cannot be reduced only by increasing the event number. It is because the expansion (7) demands penetrating all the experimental points P_n precisely. So large statistical noises are included in the expansion coefficients a_j, i.e., a set of very large a_j values with alternating signs may appear and cause great instability for the results of $F(q)$. The a_j's of the 10^5 event sample are listed in Table 1.

4. IMPROVEMENT OF HWA'S METHOD

Because the negative binomial distribution can fit the experimental results well, we can choose a smaller number of a_j ($j = 0, 1, \cdots, J, J \leq N$) to fit the experimental data

$$\tilde{P}_n = \sum_{j=0}^{J} a_j P_n^{\mathrm{NB}}(k_j, x_j), \tag{16}$$

where

$$x_j = x(1 + \Delta_j);\ k_j = k(1 + \Delta_j);$$

$$x = \langle n \rangle = \sum_{n=0}^{N} nP_n;$$

$$k = (F_2 - 1)^{-1},\ F_2 = \langle n(n-1) \rangle / x^2;$$

$$\Delta_j = \Delta\left(-\frac{1}{2} + \frac{j}{J}\right),\ j = 0, 1, \cdots, J. \tag{17}$$

Table 1
The calculated results of a 10^5 events sample for the Poisson distribution with $\langle n \rangle = 1$.

n	N_n	N_n(MC)	$\tilde{N}_n$	j	a_j	$\tilde{a}_j$
0	36787.9	36571	36659.0	0	$-6.0310967075 \times 10^5$	−0.0436
1	36787.9	37130	36887.5	1	5.1107712192×10^6	1.0867
2	18394.0	18274	18456.9	2	$-1.8943023020 \times 10^7$	−0.0431
3	6131.3	6120	6125.1	3	4.0111196201×10^7	
4	1532.8	1558	1516.5	4	$-5.3070399844 \times 10^7$	
5	306.6	293	298.7	5	4.4927260786×10^7	
6	51.1	41	48.7	6	$-2.3765120027 \times 10^7$	
7	7.3	12	6.8	7	7.1816970312×10^6	
8	0.9	1	0.8	8	$-9.4927165712 \times 10^5$	

Notes: N_n: the event number distribution calculated according to Eq. (14); N_n(MC): the event number distribution for the Monte Carlo sample; $\tilde{N}n$: the fit values of the MC sample using the maximum likelihood method; a_j: the coefficient obtained by using Hwa's method to expand N_n(MC); $\tilde{a}_j$: the coefficient obtained by using the maximum likelihood method to fit N_n(MC).

Δ can be set to 0.5 [3] and the Δ_j's range from $-\Delta/2$ to $\Delta/2$ in equal steps. The fit was performed for $N + 1$ points P_n and using the maximum likelihood method. The a_j's are chosen so that the following likelihood function L reached its maximum,

$$L = \frac{N_{\mathrm{ev}}!}{\Pi_{n=0}^{N} N_n!} \Pi_{n=0}^{N} (N_{\mathrm{ev}} \tilde{P}_n)^{N_n} , \tag{18}$$

where N_{eu} is the total number of events, and N_n is the number of events with multiplicity n.

From Eq. (16), one obtains

$$\sum_{j=0}^{J} a_j = 1 . \tag{19}$$

which should be fulfilled in the whole process of finding the proper values of a_j. At first, we set all a_j's to $1/J$. Then, we change each a_j in turn by step $h = 1/2J$ to acquire maximum L value. When the best values of a_j are obtained at step h, we reduce the step h and find the better values of a_j at the new step. The change direction could be positive or negative. If a change is made for one a_j, an extra factor $1/(1 + h)$ or $1/(1 - h)$ should be multiplied to each a_j in order to fulfill Eq. (19). The last step used in our calculation is 0.0001. Usually, a suitable number of a_j is chosen according to the experimental multiplicity distribution when the fit was performed. The dot-dashed line in Fig. 1 is the result of $J = 2$ for the Monte Carlo sample which includes 10^5 events. The fit values (N_n) of the event number distribution are listed in Table 1. It can be seen from Fig. 1 that the maximum likelihood method can well repress the effect of the measurement error of P_n. We also calculated factorial moment of continuous order for the distribution [3]

$$P_n^{(2)} = (n + 1)^{0.5} e^{-n} / Z \tag{20}$$

where Z is a normalization factor. Unlike the Poissonian distribution, there exist not only statistical fluctuations, but also dynamic fluctuations in $P_n^{(2)}$ distribution. At first, we generate a sample of 10^5 Monte Carlo events according the $P_n^{(2)}$ distribution. Then we count up the multiplicity distribution P_n

Table 2

The calculated results of a 10^5 Monte Carlo event sample ($P_n^{(2)}$ distribution).

n	$N_n^{(2)}$	N_n(MC)	$\tilde{N}_n$	q	$F(q)$
0	52016.2	52231	52206.9	−1.0	7.7765×10^{18}
1	27061.9	26955	27047.2	−0.8	2.1717×10^{15}
2	12193.0	12137	12103.6	−0.6	1.5422×10^{13}
3	5179.5	5230	5124.6	−0.4	3.1475×10^{11}
4	2130.3	2043	2107.2	−0.2	9.1117×10^{9}
5	858.5	850	850.9	0.0	0.9978
6	341.1	343	339.3	0.2	-8.5623×10^{7}
7	134.2	132	134.1	1.0	1.0000
8	52.3	51	52.7	1.2	5.1967×10^{4}
9	20.3	14	20.6	2.0	0.7846
10	7.8	10	8.0	3.0	−18.959
11	3.0	2	3.1	4.0	−477.21
12	1.2	1	1.2	5.0	-9.8628×10^{3}
13	0.4	1	0.5	6.0	-1.9438×10^{5}

Notes: $N_n^{(2)}$: the event number distribution calculated according to Eq. (20); N_n(MC): the event number distribution for the Monte Carlo sample; $\tilde{N}n$: the fit values of the Monte Carlo sample using the maximum likelihood method.

and calculate a_j and $F(q)$ by the maximum likelihood method. $F(q)$ are shown in Fig. 2 with solid circles. We have also applied Hwa's method to expand this Monte Carlo sample P_n and calculated the a_j and $F(q)$. The results are listed in Table 2. It can be seen from Table 2 that the $F(q)$ is totally covered by statistical noise. The calculated results obtained by using Hwa's method to analytical P_n are also shown in Fig. 2 in order to compare with the results of the maximum likelihood method for the Monte Carlo sample. It can be seen that they are coincident with each other in the range $q > 0$, while in the range $q < 0$, there are deviations and the deviation is much greater for large N. The above results show that the original Haw's method has distinct defect.

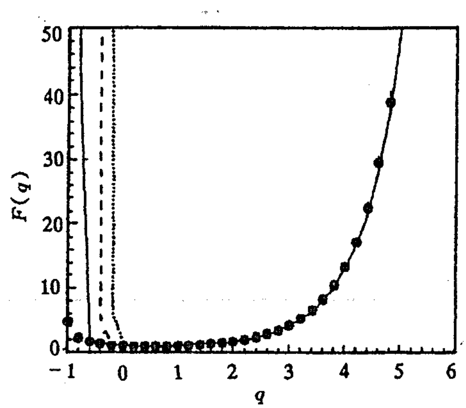

Fig. 2

The calculated results of $F(q)$ for the $P_n^{(2)}$ distribution.

Black circle: the calculated results using the maximum likelihood method for the MC events sample. Solid line, dashed line, dotted line: the calculated using Hwa's method for analytic $P_n^{(2)}$ with $N = 10, 15,$ and 20, respectively.

5. EXPERIMENTAL RESULT OF THE FACTORIAL MOMENT OF CONTINUOUS ORDER $F(q)$

Using the LEBC films offered by the CERN NA27 Collaboration, we measured the pseudorapidity distribution of charged particles produced in 400 GeV/c pp collisions. 3730 non-single-diffractive events were measured. The details about the measurement are described elsewhere [4]. The fractal behavior of multiplicity production was investigated by using the method of factorial moment of continuous order as mentioned above. In order to eliminate the influence of uneven pseudorapidity distribution, the following normalized pseudorapidity $x(\eta)$ [5] is used in the analysis

$$x(\eta) = \int_{\eta_{min}}^{\eta} \rho(\eta')\,d\eta' \Big/ \int_{\eta_{min}}^{\eta_{max}} \rho(\eta')\,d\eta'. \tag{21}$$

where $[\eta_{min}, \eta_{max}]$ is chosen to be $[-2, 2]$ and x is uniformly distributed in $[0, 1]$. x space is divided into M bins with equal size δ. Counting up the experimental multiplicity distribution $P_{n,m}$ in bin m,

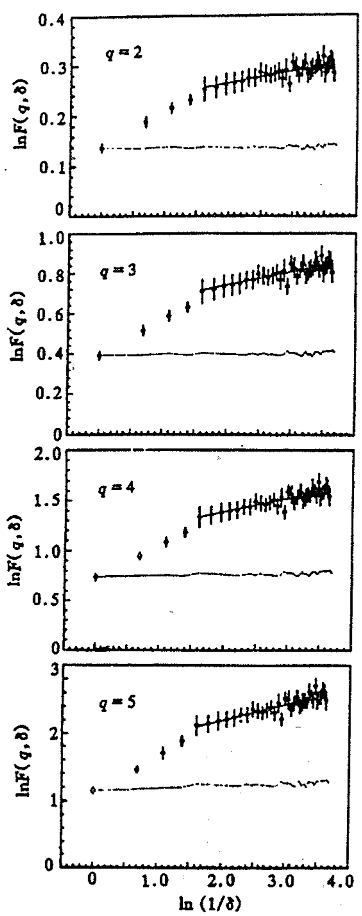

Fig. 3

The experimental results of the $F(q)$ obtained by the maximum likelihood method.

Black points: the experimental results; solid line: the results of fitting the $F(q, \delta)$ with Eq. (23); dotted line: the calculated results for MC events.

Table 3
A comparison of $D(q)$ obtained using different methods.

q	$D(q)(F(q))$	$D(q)(F(q$ order$))$	$D^{d}_{qyn}(G_q$ order$)$
0	1		
1	0.9868 ± 0.0004		
2	0.9764 ± 0.0060	0.973 ± 0.003	0.963 ± 0.004
3	0.9654 ± 0.0076	0.968 ± 0.005	0.959 ± 0.005
4	0.9535 ± 0.0087	0.968 ± 0.009	0.963 ± 0.007
5	0.9400 ± 0.0094	0.942 ± 0.015	0.944 ± 0.009

calculation $P^{NB}_{n,m}(x_{j,m}, k_{j,m})$ by (8), determining $a_{j,m}$ $(j = 0, \cdots, J)$ by the maximum likelihood method according to Eqs. (16) and (18) ($J = 4$ for $M \leq 4$ and $J = 2$ for $M > 4$), then $f_m(q)$ is calculated for bin m according to Eq. (12). At last, making an average over each bin

$$F(q, \delta) = \frac{1}{M} \sum_{m=1}^{M} \frac{f_m(q)}{[f_m(1)]^q}, \tag{22}$$

the factorial moment $F(q, \delta)$ of continuous order q at bin size δ is obtained. The results are shown in Fig. 3 with solid points. In order to see the statistical contribution to $F(q, \delta)$ we made a sample of Monte Carlo events. Comparing to the experimental data, the sample of Monte Carlo events has the same multiplicity distribution in the x space but no correlation. For event i with n_i particles, we distribute these particles randomly through x space with uniform distribution. A total of $100 N_{ev}$ events have been simulated. The calculated results are also shown in Fig. 3 with dotted lines. It can be seen that if there is only statistical fluctuation, the $F(q, \delta)$ remain constant approximately when δ is decreased and hence the intermittent exponents $\phi(q)$ equal zero. This indicated that the statistical fluctuations are filtered out.

When $\delta \to 0$, we can get the intermittent exponent $\phi(q)$ by fitting $F(q, \delta)$ with the following formula (for $M > 4$)

$$F(q, \delta) \approx \delta^{-\phi(q)} \tag{23}$$

Then we can calculate multifractal dimension and multifractal spectrum using the relations

$$\tau(q) = q-1-\phi(q), \quad D(q) = \tau(q) / (q-1);$$
$$\alpha = \mathrm{d}\tau(q) / \mathrm{d}q, \quad f(\alpha) = q\alpha - \tau(q). \tag{24}$$

The multifractal dimension $D(q)$ versus q obtained from experiment is shown in Fig. 4. It can be seen from Fig. 4 that the $D(q)$ is monotonously decreased with increasing q. This means that there is multifractal behavior in the process of multiplicity production in pp collisions. The results of the Monte Carlo samples with particles randomly distributed in pseudorapidity space are shown in Fig. 4 with a solid line. It can be seen that $D(q) = 1$ at each q for the Monte Carlo samples approximately. It means that the statistical fluctuation is filtered out by the method of maximum likelihood.

The calculated values of $D(q)$ at integer q are listed in Table 3 in comparison with that obtained by the ordinary scaled factorial moment and modified G moments. It can be seen that they are well consistent with each other. So the method of the continuous order of factorial moment is successful. The results of multifractal spectrum $f(\alpha)$ are shown in Fig. 5. It is a convex curve with a maximum at $q = 0$, $f(\alpha(0)) = D(0) = 1$. The straight line $f(\alpha) = \alpha$ is tangent to the $f(\alpha)$ curve at $q = 1$. The black point represents the results of the Monte Carlo sample, which essentially condense to a single

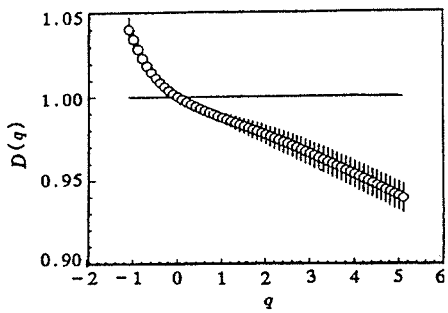

Fig. 4

Fractal dimension $D(q)$ versus q.

Open circle: the experimental value; solid line: the result of the Monte Carlo events.

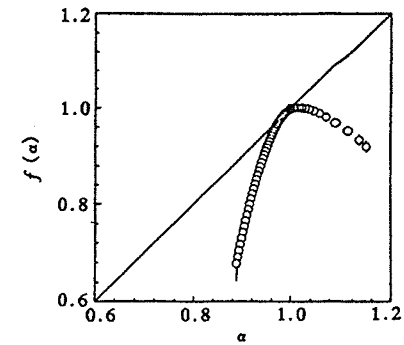

Fig. 5

Multifractal spectrum $f(\alpha)$ versus α.

Open circle: the experimental results; black circle: the results of the Monte Carlo events.

point, $a = f(\alpha) = 1$. The experimental multifractal spectrum is not a point showing that there is multifractal behavior in multiparticle production at 400 GeV/c pp collisions.

6. CONCLUSION

The method of the factorial moments F_q of continuous order suggested by Hwa has been tested. It is found that using this method to analyze the experimental data cannot obtain satisfactory results. Some improvements have been made for Hwa's method, which makes it suitable for the analysis of experimental data. The analytic results for the experimental pseudorapidity distributions of charged particles produced in 400 GeV/c pp collisions indicated that the method of the factorial moments of continuous order is feasible and correct. There is possibly multifractal behavior in the process of multiplicity production in pp collisions at 400 GeV/c.

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