A Two-Parameter Deformed Two-Dimensional Interacting Boson Model with $U_{qp}(U_3) \supset U_{qp}(U_2) \supset U_{qp}(SO_2)$ Symmetry

Zhou Huanqiang, Guan Xiwen¹, and He Jingsong²

(Department of Physics, Chongqing University, Chongqing, China)
¹(Qingdao Workers’ University, Qingdao, China)
²(Department of Physics, Yuzhou University, Chongqing, China)

A two-parameter $qp$-deformed two-dimensional interacting boson model (IBM) with the quantum symmetry $U_{qp}(U_3) \supset U_{qp}(U_2) \supset U_{qp}(SO_2)$ is constructed. It is found that the energy spectra and the transition matrix elements depend very sensitively on the second parameter of deformation.

Key words: quantum algebra, two-parameter deformation, interacting boson model.

1. INTRODUCTION

In the last decade, there has been considerable interest in the study of the theory of quantum algebras [1-3]. Physically, the theory has been successfully applied to a variety of physical problems, such as the rotational spectra in nuclei [4-6], the two-dimensional interacting boson model (IBM) [7], and the $U(3)$ shell model [8]. However, as emphasized in Ref. 9, most of the applications of quantum algebras to physics have been restricted to the those of one-parameter quantum algebras. This is due to the fact (cf., Drinfeld’s theorem [10]) that a two-parameter deformation of a semi-simple Lie algebra turns out to be essentially equivalent to a one-parameter deformation. Therefore, in order to

Received on December 20, 1994.
The commutation relation for $U_{\mathfrak{g}}(U_3)$.

<table>
<thead>
<tr>
<th>$A_{11}$</th>
<th>$A_{12}$</th>
<th>$A_{13}$</th>
<th>$A_{21}$</th>
<th>$A_{22}$</th>
<th>$A_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$-A_{12}$</td>
<td>$A_{12}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$A_{13}$ (q)</td>
</tr>
<tr>
<td>0</td>
<td>$-A_{23}$</td>
<td>$A_{23}$</td>
<td>0</td>
<td>$-q^{-1}A_{13}$ $A_{23}$</td>
<td>0</td>
</tr>
<tr>
<td>$-A_{13}$</td>
<td>0</td>
<td>$A_{13}$</td>
<td>0</td>
<td>$0 (p^{-1})$</td>
<td>0 (p)</td>
</tr>
<tr>
<td>$A_{21}$</td>
<td>$-A_{21}$</td>
<td>0</td>
<td>$-A_{21}$</td>
<td>0</td>
<td>$-A_{13} (p^{-1})$</td>
</tr>
<tr>
<td>0</td>
<td>$A_{32}$</td>
<td>$-A_{32}$</td>
<td>0</td>
<td>$-A_{13} (p^{-1})$</td>
<td>0</td>
</tr>
<tr>
<td>$A_{31}$</td>
<td>0</td>
<td>$A_{31}$</td>
<td>$q^{1/2}$ $A_{11}$ $A_{31}$</td>
<td>$-A_{13} (p^{-1})$</td>
<td></td>
</tr>
</tbody>
</table>

The commutation relation are given in the form $[A, B]_z = C$. $A$ is listed in the first column and $B$ in the first row. $C$ is given in the parentheses following $A$.

get nontrivial two-parameter deformed quantum algebras, one has to deform non-semisimple Lie algebras. Recently, Kibler [11] has presented a nontrivial two-parameter quantum algebra $U_{\mathfrak{g}}(U_3)$, which reduces to $U_q(U_3)$ in some limit, and then successfully constructed a $U_{\mathfrak{g}}(U_3)$ model for rotational bands of nuclei [9].

The aim of this article is to present a nontrivial two-parameter quantum algebra $U_{\mathfrak{g}}(U_3)$ which reduces to $U_q(U_3)$ in some sense, and to construct a two-parameter deformation of the two-dimensional toy interacting boson model (IBM) [12-15] with the symmetry $U_{\mathfrak{g}}(U_3) \supset U_{\mathfrak{g}}(U_2) \supset U_{\mathfrak{g}}(SO_2)$.

2. QUANTUM ALGEBRA $U_{\mathfrak{g}}(U_3)$ AND SUBALGEBRA CHAIN $U_{\mathfrak{g}}(U_3) \supset U_{\mathfrak{g}}(U_2) \supset U_{\mathfrak{g}}(SO_2)$

The two-parameter deformation $U_{\mathfrak{g}}(U_3)$ of the Lie algebra $U_3$ is spanned by nine operators $A_{ij}$ ($i, j = 1, 2, 3$) which satisfy the commutation relations given in Table 1, where the following convention has been adopted:

$$[A, B]_z = AB - zBA,$$  \hspace{1cm} (1)

where $\alpha = p^{\pm 1}, q^{\pm 1}$. 


As in the case of one-parameter deformation, quantum algebra $U_{q^p}(U_3)$ may be realized in terms of the two-parameter boson operators [16,17]. The latter satisfy the following defining relations:

$$a_\alpha^\dagger a_\beta^\dagger - pa_\beta^\dagger a_\alpha^\dagger = q^{\alpha\beta}, \quad a_\alpha^\dagger a_\beta = qa_\beta^\dagger a_\alpha = p^{\beta\alpha},$$

$$x = 1, 2, 3.$$  \hspace{1cm} (2)

$$[a_\alpha^\dagger, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0,$$

$$x, \beta = 1, 2, 3.$$ \hspace{1cm} (3)

$$[a_\alpha^\dagger, a_\beta^\dagger] = [a_\alpha^\dagger, a_\beta] = 0,$$

$$x \neq \beta, \quad x, \beta = 1, 2, 3.$$ \hspace{1cm} (4)

with

$$a_\alpha = [N_{\phi}]_p, \quad a_\alpha^\dagger = [N_{\phi} + 1]_p,$$

$$x = 1, 2, 3.$$ \hspace{1cm} (5)

where

$$[x]_p = \frac{q^x - p^x}{q - p},$$ \hspace{1cm} (6)

with $x$ being an operator or a number. Then, the $q^p$-deformed boson realization may be explicitly written as

$$A_{11} = N_1, \quad A_{22} = N_2, \quad A_{33} = N_3,$$

$$A_{12} = a_1^\dagger a_2, \quad A_{21} = a_2^\dagger a_1, \quad A_{33} = a_3^\dagger a_3,$$

$$A_{32} = a_3^\dagger a_2, \quad A_{13} = A_{12}A_{33} - qA_{32}A_{12} = a_1^\dagger a_2 p^{\alpha_3},$$

$$A_{31} = A_{32}A_{13} - pA_{13}A_{32} = a_3^\dagger a_1 q^{\alpha_2}.$$ \hspace{1cm} (7)

Omitting the generators involving $a_\alpha$, one may obtain the $U_{q^p}(U_3)$ subalgebra with the generators that just are the $q^p$-deformed spherical angular momentum operators:

$$J_+ = a_2^\dagger a_1^\dagger, \quad J_- = a_1^\dagger a_2, \quad J_3 = \frac{1}{2} (N_1 - N_2),$$

$$J_\phi = \frac{1}{2} (N_1 + N_2).$$ \hspace{1cm} (8)

They satisfy the following commutation relations:

$$[J_3, J_\pm] = \pm J_\pm,$$

$$[J_+, J_-] = (q^p)^{\lambda_3 - \lambda_2} [2J_3]_p,$$

$$[J_\phi, J_\lambda] = 0, \quad \lambda = +, - , 3.$$ \hspace{1cm} (9)

Further, $J_3$ itself forms a $U_{q^p}(SO_2)$ subalgebra. Therefore, the relevant chain of subalgebras is

$$U_{q^p}(U_3) \supset U_{q^p}(U_2) \supset U_{q^p}(SO_2).$$ \hspace{1cm} (10)

For later use, let us briefly discuss the representations of the quantum algebra $U_{q^p}(U_2)$. First, we introduce four sets of independent $q^p$-deformed boson operators $a_\alpha^\dagger, a_\beta^\dagger, b_\alpha, b_\beta (i = 1, 2)$, with the
vacuum $|0\rangle$ annihilated by $a_1$, $a_2$ and $b_1$, $b_2$. It is easy to see that

$$
J_0 = N_1 + N_2 + \overline{N}_1 + \overline{N}_2 ,
$$

$$
J_+ = a_1^* a_2 + b_1^* b_2 ,
$$

$$
J_- = a_2^* a_1 + b_2^* b_1 ,
$$

$$
J_z = \frac{1}{2} (N_1 - N_2) + \frac{1}{2} (\overline{N}_1 - \overline{N}_2) .
$$

They provide a realization of $U_q(U_2)$. Now, introducing a basis

$$
|\Lambda n_1 n_2 \rangle = N_{\Lambda n_1 n_2} \left( \frac{a_1^* b_2^* \overline{N}_2 + 1}{[N_2 + 1]_{qp}} - a_2^* b_1^* \frac{N_2 + 1}{[N_2 + 1]_{qp}} \right)^\Lambda (a_1^*)^{n_1} (a_2^*)^{n_2} |0\rangle ,
$$

where $N_{\Lambda n_1 n_2}$ is a normalization constant, we have

$$
J_0 |\Lambda n_1 n_2 \rangle = (2\Lambda + n_1 + n_2) |\Lambda n_1 n_2 \rangle ,
$$

$$
J_+ |\Lambda n_1 n_2 \rangle = \sqrt{[n_1 + 1]_{qp}[n_2]_{qp}} |\Lambda n_1 - 1 n_2 + 1 \rangle ,
$$

$$
J_- |\Lambda n_1 n_2 \rangle = \sqrt{[n_1]_{qp}[n_2 + 1]_{qp}} |\Lambda n_1 + 1 n_2 + 1 \rangle ,
$$

$$
J_z |\Lambda n_1 n_2 \rangle = \frac{n_1 - n_2}{2} |\Lambda n_1 n_2 \rangle .
$$

Thus, for each value of $\Lambda$, one may get a $2j + 1$-dimensional representation of $U_q(U_2)$. Letting $j = 1/2(n_1 + n_2)$ and $m = 1/2(n_1 - n_2)$, one has

$$
|\Lambda j m \rangle = N_{\Lambda j m} \left( \frac{a_1^* b_2^* \overline{N}_2 + 1}{[N_2 + 1]_{qp}} - a_2^* b_1^* \frac{N_2 + 1}{[N_2 + 1]_{qp}} \right)^\Lambda (a_1^*)^{2j} (a_2^*)^{2m} |0\rangle ,
$$

$$
J_0 |\Lambda j m \rangle = (2\Lambda + 2j) |\Lambda j m \rangle ,
$$

$$
J_+ |\Lambda j m \rangle = \sqrt{[j + m + 1]_{qp}[j - m]_{qp}} |\Lambda j m + 1 \rangle ,
$$

$$
J_- |\Lambda j m \rangle = \sqrt{[j]_{qp}[j - m + 1]_{qp}} |\Lambda j m - 1 \rangle ,
$$

$$
J_z |\Lambda j m \rangle = m |\Lambda j m \rangle .
$$

From now on, we will restrict our discussion to the case $\Lambda = 0$. For each quantum $qp$-deformed oscillator, its basis is [8]

$$
|n_1\rangle = \left( \frac{a_1^*}{{[n_1]_{qp}}} \right)^{n_1} |0\rangle ,
$$
Volume 20, Number 1

Here $[n]_{qp} = [n]_{qp}[n - 1]_{qp} \cdots [2]_{qp}[1]_{qp}$, with

$$N | n \rangle = n | n \rangle,$$
$$a_n^* | n \rangle = \sqrt{[n + 1]_{qp}} | n + 1 \rangle,$$

$$a^*_n | n \rangle = \sqrt{[n]_{qp}} | n - 1 \rangle.$$  \hspace{1cm} (16)

Then, for the quantum algebra $U_{qp}(U_3)$, the full bases may be chosen as

$$| N, n_1, M \rangle = \frac{(a^*_n)_{N-n_1} (a^*_n)_{\frac{n_1}{2} + \frac{M}{4}} (a^*_n)_{\frac{n_1}{2} - \frac{M}{4}}}{[N-n_1]_{qp} \left[ \frac{n_1}{2} + \frac{M}{4} \right]_{qp} \left[ \frac{n_1}{2} - \frac{M}{4} \right]_{qp}} | 0 \rangle.$$ \hspace{1cm} (17)

Here $N = N_1 + N_2 + N_3$ is the total number of bosons, $n_2 = N_1 + N_2$ is the number of bosons with angular momentum 2, and $M$ is the eigenvalue of $L = 4J_3$. $n_1$ takes values from 0 to $N$, while for a given value of $n_1$, $M$ takes the values $\pm 2n_1, \pm 2(n_1 - 1), \ldots, \pm 2$ or 0, depending on whether $n_1$ is odd or even [6,13].

For the quantum algebra $U_{qp}(U_3)$, the second-order Casimir operator is

$$C_2(U_{qp}(U_3)) = \frac{1}{2} (J_+ J_+ + J_- J_-) + \frac{1}{2} \left[2]_{qp} \right)^{a_0 - a_1} \left[ J_3 \right]_{qp}^2.$$ \hspace{1cm} (18)

Acting on the above basis, one gets the eigenvalue

$$C_2(U_{qp}(U_3)) | N, n_1, M \rangle = \left[ \frac{n_1}{2} \right]_{qp} \left[ \frac{n_1}{2} + 1 \right]_{qp} | Nn_1 M \rangle.$$ \hspace{1cm} (19)

3. A TWO-PARAMETER DEFORMED TWO-DIMENSIONAL INTERACTING BOSON MODEL

A $qp$-deformed version of a two-dimensionsal toy IBM with the symmetry $U_{qp}(U_3) \supset U_{qp}(U_2) \supset U_{qp}(SO_2)$ contains the limit $p = q^{-1}$ which leads to the IBM with the symmetry $U_3(3) \supset SU_3(2) \supset SO_4(2)$ [6]. The model Hamiltonian is

$$H = E_0 + AC_2(U_{qp}(U_3)) + BC_2(U_{qp}(SO_2)),$$ \hspace{1cm} (20)

where $E_0$, $A$, and $B$ are constants. From the above discussion, we obtain the energy spectrum

$$E = E_0 + A \left[ \frac{n_1}{2} \right]_{qp} \left[ \frac{n_1}{2} + 1 \right]_{qp} + BM^2.$$ \hspace{1cm} (21)

At this point, one may introduce the quadrupole transition operators in the model [15]:

$$Q_+ = a^*_n a_n + a^*_n a_n,$$
$$Q_- = a^*_n a_n + a^*_n a_n.$$ \hspace{1cm} (22)
From Eqs. (16), (17), and (22), we have

\[
\langle N, n_d + 1, M \pm 2 | Q_\tau | N, n_d, M \rangle = \sqrt{[N-n_d]_{ep} \left\{ \frac{n_d}{2} \pm \frac{M}{4} + 1 \right\}_{ep}},
\] (23-1)

\[
\langle N, n_d - 1, M \pm 2 | Q_\tau | N, n_d, M \rangle = \sqrt{[N-n_d+1]_{ep} \left\{ \frac{n_d}{2} \pm \frac{M}{4} \right\}_{ep}}.
\] (23-2)

From this one can conclude that the selection rules are \( \Delta M = \pm 2, \Delta n_d = \pm 1 \), and that both intraband and interband transitions are possible.

We should examine the problem of the dependence of the energy spectrum and the transition matrix elements in the model on the deformation parameters. For simplicity, we consider a system with twenty bosons \((N = 20)\). Consider the following two cases:

(i) if \( q = S^{-1}e^\tau, p = S^{-1}e^{-\tau} \), where \( \tau \) and \( S \) are real parameters, the \( qp \)-number may be written as

\[
[x]_{ep} = S^{-\tau} \frac{\sin \tau x}{\sin \tau};
\] (24)

(ii) if \( q = S^{-1}e^\tau, p = S^{-1}e^{-\tau} \), where \( \tau \) and \( S \) are real parameters, the corresponding \( qp \)-number is

\[
[x]_{ep} = S^{-\tau} \frac{\sin \tau x}{\sin \tau}.
\] (25)

In order to see the dependence of the energy spectrum on the \( qp \)-deformation, we shall restrict ourselves to a Hamiltonian (21) with \( E_0 = 0, A = 1, \) and \( B = 0 \). As pointed out in Refs. 7 and 15, the ground-state band contains states characterized by \( M = 2n_d \). In Table 2, we report the numerical

<table>
<thead>
<tr>
<th>( n_d )</th>
<th>( S = 0.95 )</th>
<th>( S = 1.05 )</th>
<th>( S = 0.95 )</th>
<th>( S = 1.05 )</th>
<th>( S = 0.95 )</th>
<th>( S = 1.05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>2</td>
<td>2.11</td>
<td>2.00</td>
<td>1.90</td>
<td>2.11</td>
<td>2.00</td>
<td>1.90</td>
</tr>
<tr>
<td>3</td>
<td>4.17</td>
<td>3.76</td>
<td>3.41</td>
<td>4.17</td>
<td>3.76</td>
<td>3.44</td>
</tr>
<tr>
<td>4</td>
<td>7.03</td>
<td>6.03</td>
<td>5.21</td>
<td>7.13</td>
<td>6.11</td>
<td>5.28</td>
</tr>
<tr>
<td>5</td>
<td>10.82</td>
<td>8.81</td>
<td>7.25</td>
<td>11.04</td>
<td>8.99</td>
<td>7.39</td>
</tr>
<tr>
<td>9</td>
<td>38.16</td>
<td>29.25</td>
<td>17.09</td>
<td>40.41</td>
<td>26.81</td>
<td>18.15</td>
</tr>
<tr>
<td>10</td>
<td>48.79</td>
<td>39.75</td>
<td>19.82</td>
<td>52.47</td>
<td>33.07</td>
<td>21.32</td>
</tr>
</tbody>
</table>

High Energy Physics and Nuclear Physics
Table 3
The first ten transition matrix elements among the levels of the ground state band (in which $M = 2n_d$) of a system with twenty bosons.

<table>
<thead>
<tr>
<th>$n_d$</th>
<th>$\tau = 0.05$</th>
<th>$\tau = 0.1$</th>
<th>$\tau = 0.05$</th>
<th>$\tau = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S = 0.95$</td>
<td>$S = 1$</td>
<td>$S = 1.05$</td>
<td>$S = 0.95$</td>
</tr>
<tr>
<td>0</td>
<td>7.29</td>
<td>4.85</td>
<td>3.05</td>
<td>9.80</td>
</tr>
<tr>
<td>1</td>
<td>10.79</td>
<td>6.63</td>
<td>4.17</td>
<td>13.19</td>
</tr>
<tr>
<td>2</td>
<td>12.79</td>
<td>7.86</td>
<td>4.94</td>
<td>15.38</td>
</tr>
<tr>
<td>3</td>
<td>14.28</td>
<td>8.77</td>
<td>5.52</td>
<td>16.94</td>
</tr>
<tr>
<td>4</td>
<td>15.42</td>
<td>9.47</td>
<td>5.96</td>
<td>18.09</td>
</tr>
<tr>
<td>5</td>
<td>16.28</td>
<td>10.00</td>
<td>6.29</td>
<td>18.92</td>
</tr>
<tr>
<td>6</td>
<td>16.95</td>
<td>10.41</td>
<td>6.55</td>
<td>19.53</td>
</tr>
<tr>
<td>7</td>
<td>17.42</td>
<td>10.70</td>
<td>6.73</td>
<td>19.96</td>
</tr>
<tr>
<td>8</td>
<td>17.71</td>
<td>10.88</td>
<td>6.84</td>
<td>20.23</td>
</tr>
<tr>
<td>9</td>
<td>17.86</td>
<td>10.97</td>
<td>6.90</td>
<td>20.37</td>
</tr>
</tbody>
</table>

Results for the lowest 10 members of the ground-state band for the $q$-deformed case as well as for the $qp$-deformed case. Also, the numerical results for the transition matrix elements $\langle N, n_d, M + 2 | Q_+ | N, n_d, M \rangle$ are reported in Table 3, where Eq. (23) becomes Eq. (26) for the ground-state band ($M = n_d$):

\[
\langle N, n_d + 1, 2n_d + 2 | Q_+ | N, n_d, 2n_d \rangle = \sqrt{[N - n_d + 1][n_d + 1]} \]

(26)

From Tables 2 and 3, one can conclude that the energy spectra and the transition matrix elements depend very sensitively on the second parameter of deformation.

4. CONCLUSION

In conclusion, we have presented the $qp$-deformed version of a two-dimensional toy IBM with the symmetry $U_q(U_2) \supset U_q(U_2) \supset U_q(SO_3)$. The results show that the energy spectra and the transition matrix elements depend very sensitively on the second parameter of deformation. This indicates that the $qp$-deformed algebraic models applying to a realistic nuclei will be more flexible than the $q$-deformed ones.

REFERENCES