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Verma Module of Quantum Group $GL(3)_q$, and the q-boson Realization and Cyclic Representations

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The structure and Verma module of the matrix element algebra $A(3)_q$ of the quantum group $GL(3)_q$ are studied using a similar method for studying the structure and Verma module of semisimple Lie algebras. The q-boson realization of $A(3)_q$ is constructed from its Verma representation and the cyclic representation of $A(3)_q$ is obtained in terms of the q-boson realization.

1. INTRODUCTION

Quantum groups [1], quantum algebras [2], and their representation theories play an important role in constructing solutions of the quantum Yang-Baxter equation arose from nonlinear integrable models [3]. Quantum groups can be specified using the quantum R-matrix that satisfy the quantum Yang-Baxter equation [4]. Florators [5], Weyers [6], and Chakrabarti *et al.* [7] studied the representations of matrix element algebra $A(n)_q$ of quantum group $GL(n)_q$ in terms of their Heisenberg-Weyl relation realizations, in particular, the non-generic representations where q is root of unity. A classification of irreducible representations of $A(2)_q$ were presented in [8]

In this paper, we propose a new procedure for studying the structure and Verma module of $A(n)_q$ of $GL(n)_q$. The method used here is similar to that for studying the structure and Verma module of semisimple Lie algebras. The concepts of Cartan subalgebra, raising and lowering matrix

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elements and their pairs are defined. Based on those concepts we studied the q-boson realization and cyclic representation of $A(3)_q$. The method used here can be generalized to the case $A(n)_q$.

Throughout this paper we denote by $\mathbf{Z}^+$ the set of all non-negative integers, by $\mathbb{C}$ the complex number field and $\mathbb{C}^\times \equiv \mathbb{C}\backslash\{0\}$.

2. STRUCTURE AND VERMA MODULE OF $A(3)_q$

Quantum group $GL(3)_q$ is a set of matrices $M = (m_{ij})$, $1 \le i, j \le 3$, whose matrix elements are non-commutative and satisfy the following relations

$$\begin{aligned} m_{ij}m_{ik} &= q^{-1}m_{ik}m_{ij} \quad j<k, \\ m_{ij}m_{kj} &= q^{-1}m_{kj}m_{ij} \quad i<k, \\ m_{ij}m_{kl} &= m_{kl}m_{ij} \quad i<k \text{ and } j>l, \\ m_{ij}m_{kl} &= m_{kl}m_{ij} + (q^{-1}-q)m_{il}m_{kj}, \quad i<k \text{ and } j<l. \end{aligned} \tag{2.1}$$

We also require that the quantum determinant $D_q(M)$ be not vanishing:

$$\begin{aligned} D_q(M) = {} & m_{11}(m_{22}m_{33} - q^{-1}m_{23}m_{32}) - q^{-1}m_{12}(m_{21}m_{33} - q^{-1}m_{23}m_{31}) \\ & + q^{-2}m_{13}(m_{21}m_{32} - q^{-1}m_{22}m_{31}) \end{aligned} \tag{2.2}$$

We note that $D_q(M)$ commutes with all the matrix elements m_{ij}.

The quantum matrix element algebra $A(3)_q$ is defined as an associative algebra generated by $\{m_{ij} \mid 1 \le i, j \le 3\}$ satisfying relations (2.1) and (2.2). Purpose of this paper is to study the structure and representations of $A(3)_q$.

We note that the set of all antidiagonal matrix elements $\{m_{i4-i} \mid i = 1,2,3\}$ is maximal set of mutually commutative matrix elements, and generates a maximal commutative subalgebra $H(3)_q$. Following the terminology of semisimple Lie algebra, we call it the Cartan subalgebra, and m_{i4-i} the Cartan elements. There then exists a common eigenvector v_0 of m_{i4-i} on the algebraic closed field $\mathbb{C}$ such that

$$m_{i4-i}v_0 = \lambda_i v_0, \quad \lambda_i \in \mathbb{C}, \quad 1 \leqslant i \leqslant 3. \tag{2.3}$$

To define the Verma module, we must first define the raising and lowering generators and a maximal vector. What are the raising generators? Actually, because $D_q(M)$ commutes with all the matrix elements, the requirement that $D_q(M)$ is a nonzero constant becomes

$$D_q(M)v_0 = \Gamma v_0, \quad \Gamma \in \mathbb{C}. \tag{2.4}$$

If we require that

$$m_{ij}v_0 = 0 \quad (j > 4-i), \tag{2.5}$$

then Eq. (2.4) is satisfied and $\Gamma = -q^{-3}\lambda_1\lambda_2\lambda_3$. Therefore, we define the matrix elements m_{ij} $(j > 4 - i)$, and m_{ij} $(j < 4 - i)$ as raising and lowering generators, respectively. In the sense of Eqs. (2.3) and (2.5), v_0 is the maximal vector.

Noting that

$$[m_{ij}, m_{4-j,4-i}] = (q^{-1}-q)m_{i4-i}m_{4-jj} \in H(3)_q, \quad (j < 4-i),$$

we say that $\{m_{ij}, m_{4-j,4-i}\}$ $(j < 4 - i)$ is a pair of a raising generator and a lowering operator. This concept is similar to the pair $\{x_\alpha, y_\alpha\}$ of raising and lowering generators corresponding to a positive root α of semisimple Lie algebras. It is obvious that for $A(3)_q$ there are three pairs

$$\{m_{11}m_{33}\}\ \{m_{12}m_{32}\}\ \{m_{21}m_{23}\}$$

After analyzing the structure of $A(3)_q$, we turn to the Verma module of $V(\lambda_i) \equiv V(\lambda_1, \lambda_2, \lambda_3) = A(3)_q \cdot v_0$ of $A(3)_q$. It is obvious that $V(\lambda_i)$ is spanned by

$$\{\tilde{X}(m,n,r) \equiv m_{12}^m m_{21}^n m_{11}^r v_0 \mid m,n,r \in Z^+\} \tag{2.6}$$

However, m_{22} is not diagonal on elements (2.6)

$$m_{22}\tilde{X}(m,n,r) = q^{m+n}\lambda_2\tilde{X}(m,n,r) - (1-q^{2r})q^{m+n-2r-1}\tilde{X}(m+1,n+1,r).$$

We hope to choose a basis for $A(3)_q$, such that m_{22} is diagonal on the bases, as is the situation in Lie algebras. For this purpose, we define a set of new vectors

$$\{X(m,n,r) \equiv m_{12}^m m_{21}^n \Delta^r \mid \Delta = m_{11}m_{22} - q^{-1}m_{12}m_{21}, m,n,r \in Z^+\}, \tag{2.7}$$

We want to prove that if $q^p \neq 1$, elements (2.7) form a set of bases for $V(\lambda_i)$. In fact, by making use of the following recursion relation

$$\tilde{X}(m,n,r) = \lambda_2^{-1}\Delta\tilde{X}(m,n,r-1) - \lambda_2^{-1}q^{-2r+1}\tilde{X}(m+1,n+1,r-1),$$

$\tilde{X}(m,n,r)$ can be written as finite sum of $\tilde{X}(m,n,r)$, i.e., bases $X(m,n,r)$ are complete. We can also prove that, if $q^p \neq 1$, $X(m,n,r)$ are linearly independent because they are the eigenvectors of different eigenvalues of the operator $(m_{13} + m_{22} + m_{31})$:

$$(m_{13} + m_{22} + m_{31})\ X(m,n,r) = (\lambda_1 q^{m+r} + \lambda_2 q^{m+n} + \lambda_3 q^{n+r})\ X(m,n,r)$$

The assertion is proved. The representation on $V(\lambda_i)$ is obtained as

$$\begin{aligned}
m_{22}X(m,n,r) &= q^{m+n}\lambda_2 X(m,n,r),\\
m_{13}X(m,n,r) &= q^{m+r}\lambda_1 X(m,n,r),\\
m_{31}X(m,n,r) &= q^{n+r}\lambda_3 X(m,n,r),\\
m_{12}X(m,n,r) &= X(m+1,n,r),\\
m_{21}X(m,n,r) &= X(m,n+1,r),\\
m_{11}X(m,n,r) &= q^{-(m+n)}\lambda_2^{-1}X(m,n,r+1) + q^{-(m+n+1)}\lambda_2^{-1}X(m+1,n+1,r),\\
m_{23}X(m,n,r) &= -q^{n+r-1}\lambda_1\lambda_2(1-q^{2m})X(m-1,n,r),\\
m_{32}X(m,n,r) &= -q^{m+r-1}\lambda_2\lambda_3(1-q^{2n})X(m,n-1,r),\\
m_{33}X(m,n,r) &= q^{-3}\lambda_1\lambda_2\lambda_3(1-q^{2r})X(m,n,r-1)\\
&\quad + q^{2r-2}\lambda_1\lambda_2\lambda_3(1-q^{2m})(1-q^{2n})X(m-1,n-1,r).
\end{aligned} \tag{2.8}$$

We now prove that Eq. (2.8) defines an infinite dimensional irreducible representation if $q^p \neq 1$. Suppose that $\tilde{V}$ is a nonzero invariant subspace of $V(\lambda_i)$. There then exists a nonzero vector v in V

$$0 \neq v = \sum_{m,n,r} C_{mnr}X(m,n,r) \in \tilde{V},\quad 0 \neq C_{mnr} \in C.$$

Let $\tilde{V}$ be the largest one among m, then

$$m_{23}^{\tilde{m}}v = \sum_{n\,r}(-q^{(n+r-1)\tilde{m}})(\lambda_1\lambda_2)^{\tilde{m}}(1-q^{2\tilde{m}})\cdots(1-q^2)C_{\tilde{m}nr}X(0,n,r)\in\tilde{V}.$$

In the same way, applying $m_{32}^{\tilde{n}}$ and $R^{\tilde{r}}$ on $m_{23}^{\tilde{m}}v$, where $\tilde{n}$ and $\tilde{r}$ are the largest ones among n and r, respectively, and

$$R = m_{33} - q m_{23} m_{32} m_{22}^{-1},$$

$$RX(m,n,r) = q^{-3}\lambda_1\lambda_2\lambda_3(1-q^{2r})X(m,n,r-1),$$

and noting that all the coefficients are not vanishing in the case $q^p \not\equiv 1$, we conclude that $X(0,0,0) \in \tilde{V}$. Applying m_{12}^m, m_{21}^n, $(m_{11} - q^{-1}m_{12}m_{21}m_{22}^{-1})^r$ on $X(0,0,0)$, and noting

$$(m_{11} - q^{-1}m_{12}m_{21}m_{22}^{-1})X(m,n,r) = q^{-(m+n)}\lambda_2^{-1}X(m,n,r+1),$$

we have $X(m,n,r) \in \tilde{V}$ $(m,n,r \in \mathbf{Z}^+)$. Therefore $\tilde{V} \equiv V(\lambda_i)$. This means that Eqs. (2.8) is an infinite dimensional irreducible representation if $q^p \not\equiv 1$.

We then discuss the case where q is a root of unity, i.e., the case $q^p = 1$. In this case, $V(\lambda_i)$ is no longer an irreducible module. In fact, noting that in $V(\lambda_i)$ elements m_{12}^p, m_{21}^p and Δ^p commute with any element, we conclude that $\{m_{12}\mu_1, m_{21}\text{-}\mu_2, \Delta^p\text{-}\mu_3 \mid \mu_i \in \mathbf{C}\}$ generates a normal submodule $I(\mu_i)$ of $V(\lambda_i)$. The bases for the quotient module $W(\lambda_i, \mu_i) \equiv V(\lambda_i)/I(\mu_i)$ can be obviously chosen as

$$\{Y(m,n,r) \equiv X(m,n,r)\mathrm{Mod}I(\mu_i) \mid 1 \leqslant m,n,r \leqslant p-1\},$$
$$\dim W(\lambda_i,\mu_i) = p^3. \tag{2.9}$$

Equation (2.8) induces on $W(\lambda_i, \mu_i)$, a p^3-dimensional representation

$$\begin{aligned}
&m_{22}Y(m,n,r) = q^{m+n}\lambda_2 Y(m,n,r),\\
&m_{13}Y(m,n,r) = q^{m+r}\lambda_1 Y(m,n,r),\\
&m_{31}Y(m,n,r) = q^{n+r}\lambda_3 Y(m,n,r),\\
&m_{12}Y(m,n,r) = Y(m+1,n,r), \quad (m \neq p-1)\\
&m_{12}Y(p-1,n,r) = \mu_1 Y(0,n,r),\\
&m_{21}Y(m,n,r) = Y(m,n+1,r), \quad (n \neq p-1)\\
&m_{21}Y(m,p-1,r) = \mu_2 Y(m,0,r),\\
&m_{11}Y(m,n,r) = q^{-(m+n)}\lambda_2^{-1}Y(m,n,r+1) + q^{-(m+n+1)}\lambda_2^{-1}Y(m+1,n+1,r),\\
&\qquad\qquad (m,n,r \neq p-1)\\
&m_{11}Y(p-1,n,r) = q^{-n+1}\lambda_2^{-1}Y(p-1,n,r+1) + q^{-n}\lambda_2^{-1}\mu_1 Y(0,n+1,r),\\
&m_{11}Y(m,p-1,r) = q^{-m+1}\lambda_2^{-1}Y(m,p-1,r+1) + q^{-m}\lambda_2^{-1}\mu_2 Y(m+1,0,r),\\
&m_{11}Y(m,n,p-1) = q^{-(m+n)}\lambda_2^{-1}\mu_3 Y(m,n,0)\\
&\qquad\qquad + q^{-(m+n-1)}\lambda_2^{-1}Y(m+1,n+1,p-1).
\end{aligned} \tag{2.10}$$

Using the same method and noting $q^t \neq 1$ $(1 \leq t \leq p-1)$, we can prove that Eq. (2.10) defines a p-dimensional irreducible representation.

It is easy to verify that in representation (2.10) we have

$$m_{12}^p = \mu_1, \quad m_{21}^p = \mu_2, \quad \Delta^p = \mu_3, \quad m_{32}^p = m_{23}^p = m_{33}^p = 0,$$

Therefore, (2.10) is not a pure cyclic representation. To obtain pure cyclic representation, we first construct its q-boson realization.

3. q-BOSON REALIZATION OF $A(3)_q$

To construct the q-boson realization of $A(3)_q$, we define that the q-Fock space $\mathscr{F}_q(3)$ of three q-bosons:

$$\mathscr{F}_q(3):\{|m,n,r\rangle \equiv (b_1^+)^m(b_2^+)^n(b_3^+)^r|0\rangle | b_i|0\rangle = 0,\ q^{N_i}|0\rangle = |0\rangle \quad i = 1,2,3;\ m,n,r \in \mathbf{Z}^+\}. \tag{3.1}$$

Then the mapping φ: $V(\lambda_i) \rightarrow \mathscr{F}_q(3)$ defined by

$$\varphi: X(m,n,r) \longmapsto |m,n,r\rangle \tag{3.2}$$

is isomorphism of a linear space. Define

$$\Gamma \equiv \varphi\rho\varphi^{-1}, \tag{3.3}$$

where ρ is the representation (2.8) of $A(3)_q$. Then Γ is a representation of $A(3)_q$ on $\mathscr{F}_q(3)$. It is easy to prove that

$$\Gamma(x)|m,n,r\rangle = \sum_{m',n',r'} \rho(x)_{m,n,r}^{m',n',r'}|m',n',r'\rangle,\quad \forall x \in A(3)_q. \tag{3.4}$$

By making use of representation on $\mathscr{F}_q(3)$ of q-Heisenberg-Weyl algebra

$$\begin{aligned}
&q^{N_1}|m,n,r\rangle = q^m|m,n,r\rangle, q^{N_2}|m,n,r\rangle = q^n|m,n,r\rangle,\\
&q^{N_3}|m,n,r\rangle = q^r|m,n,r\rangle, b_1^+|m,n,r\rangle = |m+1,n,r\rangle,\\
&b_2^+|m,n,r\rangle = |m,n+1,r\rangle, b_3^+|m,n,r\rangle = |m,n,r+1\rangle,\\
&b_1|m,n,r\rangle = [m]|m-1,n,r\rangle = -\frac{1}{q-q^{-1}}q^{-m}(1-q^{2m})|m-1,n,r\rangle,\\
&b_2|m,n,r\rangle = -\frac{1}{q-q^{-1}}q^{-n}(1-q^{2n})|m,n-1,r\rangle,\\
&b_3|m,n,r\rangle = -\frac{1}{q-q^{-1}}q^{-r}(1-q^{2r})|m,n,r-1\rangle,
\end{aligned} \tag{3.5}$$

we rewrite $\Gamma(x)$ in the form of q-boson operators

$$\begin{aligned}
&m_{22} = \lambda_2 q^{N_1}q^{N_2},\ m_{13} = \lambda_1 q^{N_1}q^{N_3},\ m_{31} = \lambda_3 q^{N_2}q^{N_3},\\
&m_{12} = b_1^+,\ m_{21} = b_2^+,\ m_{11} = \lambda_2^{-1}q^{-N_1}q^{-N_2}(b_3^+ + qb_1^+b_2^+),\\
&m_{23} = (q-q^{-1})\lambda_1\lambda_2 q^{N_1}q^{N_2}q^{N_3}b_1,\ m_{32} = (q-q^{-1})\lambda_2\lambda_3 q^{N_1}q^{N_2}q^{N_3}b_2,\\
&m_{33} = -\lambda_1\lambda_2\lambda_3 q^{-2}(q-q^{-1})q^N b_3 + (q-q^{-1})^2\lambda_1\lambda_2\lambda_3 q^{N_1}q^{N_2}q^{2N_3}b_1b_2.
\end{aligned} \tag{3.6}$$

which is the desired q-boson realization of $A(3)_q$. The method used above is a generalization of that for constructing the q-boson realizations of quantum universal enveloping algebras [9].

It is worth noting that the q-boson realization (3.6) is valid in both the case $q^p \neq 1$ and the case $q^p = 1$. This fact can be directly verified using the basic defining relations of q-Heisenberg-Weyl algebra.

4. CYCLIC REPRESENTATION OF $A(3)_q$

In this section we suppose that q is the p-th root of unity, i.e., $q^p = 1$.

Our aim is to construct cyclic representation of $A(3)_q$ in terms of its q-boson realization. In [9], we presented the cyclic representation of the q-Heisenberg-Weyl algebra. For the case with three q-bosons, letting $V_p(3)$ be a linear space spanned by $\{v(m,n,r) \mid 1 \le m,n,r \le p-1\}$, the cyclic representation of the q-Heisenberg-Weyl algebra is defined as ($\xi_i \in \mathbb{C}^\times$, $\zeta_i^p = 1$):

$$\begin{aligned}
&q^{N_1}v(m,n,r) = q^{m+\zeta_1}v(m,n,r), \quad q^{N_2}v(m,n,r) = q^{n+\zeta_2}v(m,n,r),\\
&q^{N_3}v(m,n,r) = q^{r+\zeta_3}v(m,n,r), \quad b_1^+v(m,n,r) = v(m+1,n,r)(m \neq p-1),\\
&b_1^+v(p-1,n,r) = \xi_1 v(0,n,r), \quad b_2^+v(m,n,r) = v(m,n+1,r)(n \neq p-1),\\
&b_2^+v(m,p-1,r) = \xi_2 v(m,0,r), \quad b_3^+v(m,n,r) = v(m,n,r+1)(r \neq p-1),\\
&b_3^+v(m,n,p-1) = \xi_3 v(m,n,0), \quad b_1 v(m,n,r) = [m+\zeta_1]v(m-1,n,r)(m \neq 0),\\
&b_1 v(0,n,r) = \xi_1^{-1}[\zeta_1]v(p-1,n,r), \quad b_2 v(m,n,r) = [n+\zeta_2]v(m,n-1,r)(n \neq 0),\\
&b_2 v(m,0,r) = \xi_2^{-1}[\zeta_2]v(m,p-1,r), \quad b_3 v(m,n,r) = [r+\zeta_3]v(m,n,r-1)\\
&\qquad\qquad (r \neq 0),\\
&b_3 v(m,n,0) = \xi_3^{-1}[\zeta_3]v(m,n,p-1).
\end{aligned} \tag{4.1}$$

By making use of the q-boson realization (3.6) we immediately obtain the p^3-dimensional cyclic representation of $A(3)_q$

$$\begin{aligned}
&m_{22}v(m,n,r) = \lambda_2 q^{m+n+\zeta_1+\zeta_2}v(m,n,r),\\
&m_{13}v(m,n,r) = \lambda_1 q^{m+r+\zeta_1+\zeta_3}v(m,n,r),\\
&m_{31}v(m,n,r) = \lambda_3 q^{n+r+\zeta_2+\zeta_3}v(m,n,r),\\
&m_{12}v(m,n,r) = v(m+1,n,r)\ (m \neq p-1),\\
&m_{12}v(p-1,n,r) = \xi_1 v(0,n,r),\\
&m_{21}v(m,n,r) = v(m,n+1,r)\ (n \neq p-1),\\
&m_{21}v(m,p-1,r) = \xi_2 v(m,0,r),\\
&m_{11}v(m,n,r) = \lambda_2^{-1}q^{-(m+n+\zeta_1+\zeta_2)}v(m,n,r+1) + \lambda_2^{-1}q^{-(m+n+\zeta_1+\zeta_2+1)}v(m+1,n+1,r)\\
&\qquad\qquad (m,n,r \neq p-1),\\
&m_{11}v(p-1,n,r) = \lambda_2^{-1}q^{-(m+\zeta_1+\zeta_2-1)}v(p-1,n,r+1) + \lambda_2^{-1}q^{-(n+\zeta_1+\zeta_2)}\xi_1 v(0,n+1,r)\\
&\qquad\qquad (n,r \neq p-1),\\
&m_{11}v(m,p-1,r) = \lambda_2^{-1}q^{-(m+\zeta_1+\zeta_2-1)}v(m,p-1,r) + \lambda_2^{-1}q^{-(m+\zeta_1+\zeta_2)}\xi_2 v(m,0,r)\\
&\qquad\qquad (m,r \neq p-1),\\
&m_{11}v(m,n,p-1) = \lambda_2^{-1}q^{-(m+n+\zeta_1+\zeta_2)}\xi_3(m,n,0) + \lambda_2^{-1}q^{-(m+n+\zeta_1+\zeta_2+1)}v(m+1,n+1,\\
&\qquad p-1)\quad (m,n \neq p-1),\\
&m_{23}v(m,n,r) = (q-q^{-1})\lambda_1\lambda_2 q^{m+n+r+\zeta_1+\zeta_2+\zeta_3-1}[m+\zeta_1]v(m-1,n,r)\ (m \neq 0),\\
&m_{23}v(0,n,r) = (q-q^{-1})\lambda_1\lambda_2 q^{n+r+\zeta_1+\zeta_2+\zeta_3-1}\xi_1^{-1}[\zeta_1]v(p-1,n,r),\\
&m_{32}v(m,n,r) = (q-q^{-1})\lambda_2\lambda_3 q^{m+n+r+\zeta_1+\zeta_2+\zeta_3-1}[n+\zeta_2]v(m,n-1,r)\ (n \neq 0),\\
&m_{32}v(m,0,r) = (q-q^{-1})\lambda_2\lambda_3 q^{m+r+\zeta_1+\zeta_2+\zeta_3-1}\xi_2^{-1}[\zeta_2]v(m,p-1,r),\\
&m_{33}v(m,n,r) = -\lambda_1\lambda_2\lambda_3 q^{-3}(q-q^{-1})q^{r+\zeta_3}[r+\zeta_3]v(m,n,r-1)\\
&\qquad + (q-q^{-1})^2\lambda_1\lambda_2\lambda_3 q^{m+n+2r+\zeta_1+\zeta_2+2\zeta_3}[m+\zeta_1][n+\zeta_2]v(m-1,n-1,r),
\end{aligned}$$

$$
\begin{aligned}
&(m, n, r \neq 0), \\
m_{33}v(0,n,r) &= -\lambda_1\lambda_2\lambda_3 q^{-3}(q - q^{-1})q^{r+\zeta_3}[r + \zeta_3]v(0,n,r-1) \\
&\quad + (q - q^{-1})^2\lambda_1\lambda_2\lambda_3 q^{n+2r+\zeta_1+\zeta_2+2\zeta_3}\xi_1^{-1}[\zeta_1][n+\zeta_2]v(p-1,n,r), (n, r \neq 0), \\
m_{33}v(m,0,r) &= -\lambda_1\lambda_2\lambda_3 q^{-3}(q - q^{-1})q^{r+\zeta_3}[r + \zeta_3]v(m,0,r-1) \\
&\quad + (q - q^{-1})^2\lambda_1\lambda_2\lambda_3 q^{m+2r+\zeta_1+\zeta_2+2\zeta_3}[m+\zeta_1]\xi_2^{-1}[\zeta_2]v(m,p-1,r), (m, r \neq 0), \\
m_{33}v(m,n,0) &= -\lambda_1\lambda_2\lambda_3 q^{-3}(q - q^{-1})q^{\zeta_3}\xi_3^{-1}[\zeta_3]v(m,n,p-1) \\
&\quad + (q - q^{-1})^2\lambda_1\lambda_2\lambda_3 q^{m+n+\zeta_1+\zeta_2+2\zeta_3}[m+\zeta_1][n+\zeta_2]v(m-1,n-1,0), \\
&(m, n \neq 0)
\end{aligned}
\tag{4.2}
$$

It is easy to verify that Eq. (4.2) is a pure cyclic representation when ζ_i is generic, i.e., $\zeta_i^p \neq 1$, and the central elements m_{ij}^p are nonzero multiples of identity matrix.

It is worth noting that the representation (4.2) reduces to (2.10) if $\zeta_i^p = 1$. Therefore, representation (2.10) is only a special case of the general cyclic representation (4.2).

5. CONCLUSIONS

In this paper, we have studied the structure and Verma module of $A(3)_q$ using a similar method for studying the structure and Verma module of semisimple Lie algebras, constructed its q-boson realization from the Verma module, and obtained its cyclic representation at $q^p = 1$. The method used here is expected to generalize to the study of the structure and Verma module of the matrix element algebra $A(n)_q$ of the quantum group $GL(n)_q$, in particular, to the classification of finite dimensional irreducible representations of $A(n)_q$. There are the further work of authors.

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