High Energy Physics and Nuclear Physics

Vol. 13, No. 2

Vacuum State of 2 + 1 Dimensional SU(2) Lattice Gauge Theory With Fermions and Spontaneous Chiral Symmetry Breaking

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The vacuum structure of 2 + 1 dimensional $SU(2)$ LGT with fermions is studied by incorporating the exact ground state of pure gauge theory and the variational fermion vacuum state. We have calculated the fermion condensation $<\bar{\psi}\psi>$, and obtained an improved scaling behavior.

1. INTRODUCTION

Chiral symmetry breaking is one of the most important features of QCD. The fermion condensation $<\bar{\psi}\psi>$, which represents this character, has been extensively computed by means of Monte Carlo simulations and analytic approaches[1]. The scaling behavior of $<\bar{\psi}\psi>$ was observed in Monte Carlo calculations, but most of the analytic investigations were restricted to the strong coupling regime or large N limit. In Ref.[2], we presented a different approach of treating lattice fermions. It consists of unitary transformations and variational approximations. The calculation of $<\bar{\psi}\psi>$ was extended to the crossover region, but the scaling behavior was not obtained, because only the tree diagrams of gauge fields were considered.

Considerable progress has been made in the analytic investigation of pure lattice gauge theory. In Ref.[3], we found a Hamiltonian of the pure gauge theory with exact ground state, and applied the

Supported in part by the Foundation of Zhongshan University Advanced Center and by the Science Fund of the Chinese Education Committee
Received on December 29, 1987

variational method to obtain rigorous upper bounds of the mass gap in the 2+1 dimensional U(1), SU(2) and SU(3) LGT[4,5]. Good scaling behaviors of the mass gap and the string tension were obtained in a wide weak coupling region.

In this paper, we will investigate the vacuum structure and calculate $<\bar{\psi}\psi>$ by combining the exact ground state of the 2+1 dimensional $SU(2)$ pure gauge theory and the variational fermion state. Our result shows that the Wilson loops of gauge fields play a very important role in the weak coupling region, and the scaling behavior is greatly improved.

2. VACUUM STATE AND HAMILTONIAN WITH FERMIONS

We take the Hamiltonian of 2 + 1 dimensional LGT with naive fermions as

$$H = H_f + H_g \tag{1}$$

$$H_f = \frac{1}{2a}\sum_{x,k}\bar{\psi}(x)\sigma_k U(x,k)\psi(x+k) + m\sum_x \bar{\psi}(x)\psi(x) \tag{2}$$

$$H_g = \frac{g^2}{2a}\sum_{x,j}E_j^2(x) - \frac{1}{ag^2}\sum_p \mathrm{tr}(U_p + U_p^+) + \Delta H \tag{3}$$

where $\Delta H$is a function of plaquette variables U_p and U_p^+, and $\Delta H \rightarrow 0$ in the continuum limit $a \rightarrow 0$. The dimensionless coupling constant g is related to charge e by $g^2 = e^2a$. In Ref.[5], we showed that H_g could be rewritten as

$$H_g = \frac{g^2}{2a}\sum_{x,j} e^{-s_g}E_j^a(x)e^{2s_g}E_j^a(x)e^{-s_g} \tag{4}$$

$$s_g = \frac{1}{2g^4C_N}\sum_p \mathrm{tr}(U_p + U_p^+) \tag{5}$$

which has the exact ground state $\exp(s_g)\,|\,0\!>$ C_N is the Casimir invariant of the SU(N) gauge group. For SU(2) group, $C_N = 3/4$. The fermion field $\psi(x)$ is represented by

$$\psi(x) = \begin{pmatrix} \xi(x) \\ \eta^+(x) \end{pmatrix} \tag{6}$$

and the bare vacuum is defined by

$$\xi(x)|0\rangle = \eta(x)|0\rangle = E_j^2(x)|0\rangle = 0 \tag{7}$$

Following Ref.[2], we can show that under the unitary transformation $\exp(-i\Theta s_f)H\cdot\exp(i\Theta s_f)$, the naive Hamiltonian becomes

$$e^{-i\theta s_f}H_m e^{i\theta s_f} = e^{-i\theta s_f} m \sum_{x} \bar{\Psi}(x)\psi(x) e^{i\theta s_f}$$

$$= m \sum_{n=0}^{\infty} \frac{(-1)^n (2\theta)^n}{n!\, 4^{n/2}} \sum_{x,\pm k_i} \bar{\Psi}(x)\sigma_{k_1}\cdots\sigma_{k_n} U(x, k_1, \cdots, k_n)\psi(x + k_1 + \cdots + k_n)$$

$$e^{-i\theta s_f}H_k e^{i\theta s_f} = e^{-i\theta s_f} \frac{1}{2a} \sum_{x,\pm k_1} \bar{\Psi}(x)\sigma_k U(x,k)\psi(x+k) e^{i\theta s_f}$$

$$= \frac{1}{2a} \sum_{n=0}^{\infty} \frac{(-1)^n (2\theta)^n}{n!\, 4^{n/2}} \sum_{x,\pm k_i} \bar{\Psi}(x)\sigma_{k_1}\cdots\sigma_{k_{n+1}} U(x, k_1, \cdots, k_{n+1})\psi(x + k_1 + \cdots + k_{n+1})$$

$$e^{-i\theta s_f} \sum_{y,j} \frac{g^2}{2a} E_j^2(y) e^{i\theta s_f} = \frac{g^2}{2a} \sum_{y,j} E_j^{a\prime}(y) \cdot E_j^{a\prime}(y) \tag{8}$$

where

$$s_f = \frac{i}{\sqrt{4}} \sum_{x,\pm k} \psi^+(x)\sigma_k U(x, k)\psi(x+k)$$

$$E_j^{a\prime}(y) = E_j^a(y) + \sum_{n=1}^{\infty} \frac{\theta^n}{n!\, 4^{n/2}} \sum_{x,\pm k_i} \psi^+(x)\sigma_{k_1}\cdots\sigma_{k_n}[U(x,k_1,\cdots,k_n), E_j^a(y)]\psi(x + k_1 + \cdots + k_n)\eta_n(y)$$

$$\eta_n(y) = (-1)^{n+i+1}\binom{n-1}{i}\delta_{y,x+k_1+\cdots+k_i}\delta_{j,k_{i+1}}, \quad i = 0,1,\cdots,n-1 \tag{9}$$

Now, let the physical vacuum be

$$|\Omega\rangle = e^{i\theta s_f} e^{s_g}|0\rangle \tag{10}$$

Where Θ is determined by the condition of the lowest vacuum energy $\partial E_\Omega/\partial\Theta = 0$, and E_Ω is the mean vacuum energy given by

$$E_\Omega = \langle\Omega|H|\Omega\rangle/\langle\Omega|\Omega\rangle \tag{11}$$

3. FERMION CONDENSATION

Combining the method developed in Refs [2-6], we have the following expression for the mean vacuum energy

$$E_{\Omega} = m \sum_{n=0}^{\infty} \frac{(2\theta)^{2n}}{(2n)!4^n} \sum_{x,\pm k_i} W_{2n}(x, k_i) - \frac{1}{2a} \sum_{n=0}^{\infty} \frac{(2\theta)^{2n+1}}{(2n+1)!4^{n+1/2}} \sum_{x,\pm k_i} W_{2n+2}(x, k_i) + \frac{g^2}{2a} C_N \sum_{n=0}^{\infty} \frac{(2\theta)^{2n+2}}{(2n+2)!4^{n+1}} \sum_{y,j,x,\pm k_i} V_{2n+2}(y, j, x, k_i) \tag{12}$$

Where $\sum_{x,\pm k_i} W_{2n}(x,k_i)$ is the sum of all the closed graphs with $2n$ links

$$\sum_{x,\pm k_i} W_{2n}(x, k_i) = \sum_{x,\pm k_i} \times \frac{\langle 0| e^{S_g}\bar{\psi}(x)\sigma_{k_1}\cdots\sigma_{k_2} U(x, k_1, \cdots, k_{2n})\psi(x + k_1 + \cdots + k_{2n}) e^{S_g}|0\rangle}{\langle 0| e^{2S_g}|0\rangle}$$

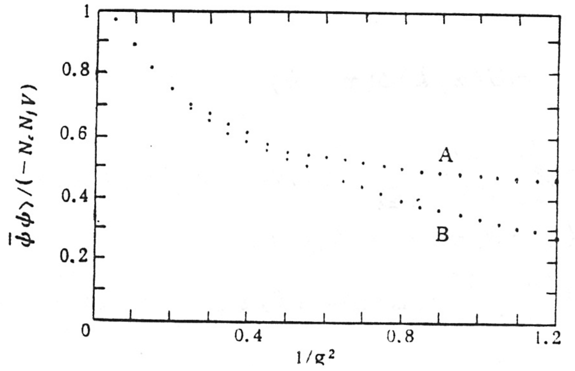


FIG.1 $<\bar{\psi}\psi>$ as a function of $1/g^2$.
A: Only tree graphs considered.
B: Loop graphs are included.

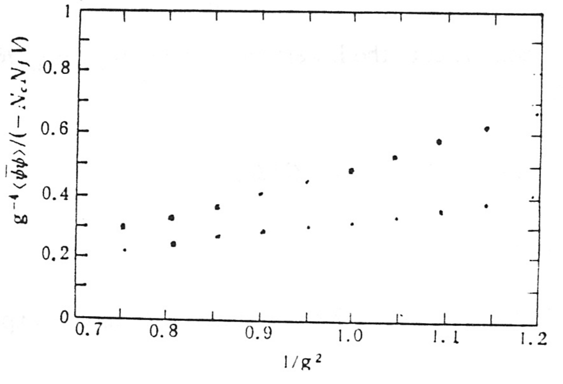
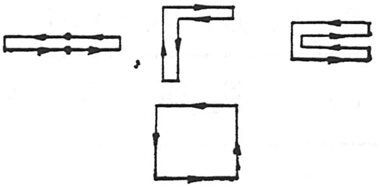
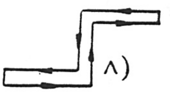
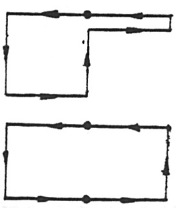


FIG.2 $<\bar{\psi}\psi>/g^4$ as a function of $1/g^2$ (the meanings of A and B are the same as those of Fig.1).

TABLE 1.
Number of Diagrams Contributing to $\sum_{x,\pm k_i} W_{2n}(x,k_i)/(-N_cN_fV)$

n	lattice graphs	number	$\sum_{x,\pm k_i} W_{2n}(x,k_i)/(-N_cN_fV)$
1		4	-4
2		28 8	$28-8R$
	tree graphs (e.g.  ∧) 	232 144 24	$-232+144R-24R^2$
⋮	⋮	⋮	⋮

$$= \left(-\frac{1}{2}N_fV\right)\sum_{\pm k_i}\mathrm{Tr}\,\sigma_{k_1}\cdots\sigma_{k_{2n}}\frac{\int [dU]e^{2sg}\mathrm{Tr}\,U_1\cdots U_{2n}}{\int [dU]e^{2sg}} \tag{13}$$

We used the computer to count the number of graphs which satisfy $k_1+\ldots+k_{2n}=0$. The results are shown in Tab.1, Where $R = I_2(16/3g^4)/I_1(16/3g^4)$, and I_i is the i-th order modified Bessel function. $\sum_{y,j,x,\pm k_i} V_{2n+2}(y,j,x,k_i)$ can be computed in a similar way, and the results are given in Tab.2.

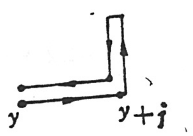
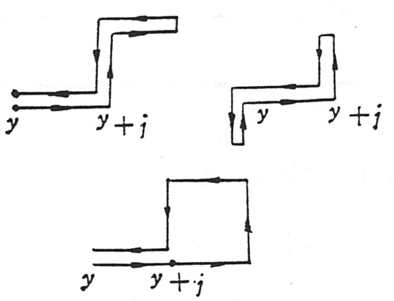
The values of Θ as a function of the coupling constant g are determined by solving the equations $\partial E_\Omega/\partial\Theta = 0$. We obtained the relation between $<\bar\psi\psi>$ and g by substituting $\Theta(g^2)$ into the following formula

$$\langle\Omega|\bar\psi\psi|\Omega\rangle/(-N_cN_fV\langle\Omega|\Omega\rangle) = 1-\frac{(2\theta)^2}{2!}+\frac{(2\theta)^4\times(28-8R)}{4!\times 4^2} + \frac{(2\theta)^6\times(-232+144R-24R^2)}{6!\times 4^3}+\cdots \tag{14}$$

In a wide range of g, this series converges so rapidly that the high order contributions can be neglected.

Since 2 + 1 dimensional LGT is super-renormalizable, the charge e does not vary with the cutoff. If we use the dimensionless coupling constant g, then $g^2 \propto a$. Introducing ψ_L and ψ_C as the fermion fields on the lattice and in the continuum, respectively, we have the scaling behavior $\langle\bar{\psi}\psi\rangle_L = a^2\langle\bar{\psi}\psi\rangle_C \propto a^2 \propto g^4$.

TABLE 2.
Number of Diagrams Contributing to $\sum_{y,j,x,\pm k_i} V_{2n+2}(y,j,x,k_i)/(N_cN_fV)$

n	lattice graphs	number	$\sum_{y,j,x,\pm k_i} V_{2n+2}(y,j,x,k_i)/(N_cN_fV)$
0		2	2
1		2×8	−16
2		2×152 2×16	304 − 32R
⋮	⋮	⋮	⋮

In the case of $m = 0$, we present $\langle\bar{\psi}\psi\rangle$ as a function of $1/g^2$ in Fig.1, and $\langle\bar{\psi}\psi\rangle/g^4$ versus $1/g^2$ in Fig.2. From these figures, we can see that the results are extended to the weak coupling region more effectively than those given in Ref.[2]. However, there is still a small deviation from the scaling behavior. This is probably due to the only one-link terms we have taken from s_f. But our results confirm that the Wilson loops are of fundamental importance in improving the scaling behavior.

ACKNOWLEDGMENT

We would like to thank G. F. Li, Z. B. Li and W. H. Zheng for the useful discussions we had with them.

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