非阿贝尔规范场中两种变换式的等价性

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摘　要

本文从要求拉氏函数密度不变性出发导出了非阿贝尔规范场的一般变换式，论证了当其中的群参数 $\theta(x)$ 等于反对易的鬼场 $C(x)$ 与反对易的无穷小数 ξ 的乘积时即得 B. R. S. 变换，此时 ξ 与时空坐标及内部空间分量指标无关；$\theta(x)$ 等于对上指标反对易的无穷小张量 ξ_i^b 与鬼场 $C(x)$ 乘积时得“推广”的变换式．证明了两者是等价的．

非阿贝尔规范场中的 B.R.S. 变换[3]，有它的推广变换式[4-6]．下面将论证这些变换式都可由我们得到的一般变换式推出并且是等价的．

非阿贝尔规范场的有效拉氏函数密度为[1]:

$$\mathscr{L}_{\text{eff}} = \mathscr{L} + \mathscr{L}_f + \mathscr{L}_g, \tag{1}$$

$$\mathscr{L} = -\frac{1}{4}(\partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + gf^{abc}A_\mu^b(x)A_\nu^c(x)) + \psi^+\gamma_4(\gamma_\mu D_\mu + m)\psi, \tag{2}$$

$$\mathscr{L}_f = -\frac{1}{2\alpha}(\partial_\mu A_\mu^a(x))^2, \tag{3}$$

$$\mathscr{L}_g = -(\partial_\mu C^{a+}(x))D_\mu^{ab}C^b(x) = C^{a+}(x)\partial_\mu(D_\mu^{ab}C^b(x)), \tag{4}$$

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - gf^{abc}A_\mu^c(x). \tag{5}$$

其中 $A_\mu^a(x)$ 为规范场，a、b、c 为内部空间分量指标，$C^a(x)$ 为鬼场、反对易，ψ 为费米场，g 为耦合常数，f^{abc} 是规范群的结构常数，α 为规范参数，x 为时空点坐标．

在规范变换

$$\delta\psi = -i\frac{\lambda^a}{2}\theta^a\psi,\quad \delta\bar{\psi} = i\bar{\psi}\frac{\lambda^a}{2}\theta^a,\quad \delta A_\mu^a(x) = -\frac{1}{g}D_\mu^{ab}\theta^b(x), \tag{7a}$$

下，$\mathscr{L}$ 具有不变性[2]，而 $\mathscr{L}_{\text{eff}}$ 不具有不变性，式中 $\theta(x)$ 为规范群的群参量，是无穷小量，λ^a 为盖尔曼矩阵．要 $\mathscr{L}_{\text{eff}}$ 也具有不变性，可令

$$\delta(\mathscr{L}_g + \mathscr{L}_f) = \delta\left\{-\frac{1}{2\alpha}(\partial_\mu A_\mu^a(x))^2 + C^{a+}(x)\partial_\mu(D_\mu^{ab}C^b(x))\right\}$$

$$= -\frac{1}{\alpha}(\partial_\mu A_\mu^a(x))(\partial_\nu\delta A_\mu^a(x))$$

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$$+\delta C^{a+}(x)\partial_\mu(D_\mu^{ab}C^b(x))+C^{a+}(x)\partial_\mu\delta(D_\mu^{ab}C^b(x))$$
$$=0. \tag{8}$$

要使上式成立，可取

$$\delta C^{a+}(x)=\frac{1}{\alpha}\frac{(\partial_\mu A_\mu^a(x))\partial_\nu(\delta A_\nu^a(x))}{\partial_\mu(D_\mu^{ab}C^b(x))}, \tag{7b}$$

和

$$\delta(D_\mu^{ab}C^b(x))=(\delta D_\mu^{ab})C^b(x)+D_\mu^{ab}\delta C^b(x)=0. \tag{9}$$

而 (9) 式中的

$$\begin{aligned}(\delta D_\mu^{ab})C^b(x)&=[\delta(\delta^{ab}\partial_\mu-gf^{abc}A_\mu^c(x))]C^b(x)\\&=-gf^{abc}(\delta A_\mu^c(x))C^b(x)\\&=-gf^{abc}\left(-\frac{1}{g}D_\mu^{cd}\theta^d(x)\right)C^b(x)\\&=f^{abc}(D_\mu^{cd}\theta^d(x))C^b(x)\\&=f^{abc}[(\delta^{cd}\partial_\mu-gf^{cde}A_\mu^e(x))\theta^d(x)]C^b(x)\\&=f^{abc}(\partial_\mu\theta^c(x))C^b(x)-gf^{abc}f^{cde}A_\mu^e(x)\theta^d(x)C^b(x).\end{aligned} \tag{10}$$

对求和指标作代换：$b\to d$，$d\to b$，则 (10) 式右端第二项化为：

$$\begin{aligned}f^{abc}f^{cde}A_\mu^e(x)\theta^d(x)C^b(x)&=f^{adc}f^{cbe}A_\mu^e(x)\theta^b(x)C^d(x)\\&=-f^{adc}f^{ceb}A_\mu^e(x)\theta^b(x)C^d(x),\end{aligned}$$

所以

$$\begin{aligned}f^{abc}f^{cde}A_\mu^e(x)\theta^d(x)C^b(x)&=\frac{1}{2}f^{abc}f^{cde}A_\mu^e(x)\theta^d(x)C^b(x)\\&\quad-\frac{1}{2}f^{adc}f^{ceb}A_\mu^e(x)\theta^b(x)C^d(x)\\&=\frac{1}{2}(f^{abc}f^{cde}+f^{adc}f^{ceb})A_\mu^e(x)\theta^d(x)C^b(x)\\&\quad-\frac{1}{2}f^{adc}f^{ceb}A_\mu^e(x)(\theta^bC^d+\theta^dC^b)\\&=-\frac{1}{2}f^{aec}f^{cbd}A_\mu^e(x)\theta^d(x)C^b(x)\\&\quad-\frac{1}{2}f^{adc}f^{ceb}A_\mu^e(x)(\theta^bC^d+\theta^dC^b),\end{aligned} \tag{11}$$

已利用了关系式：

$$f^{abc}f^{cde}+f^{adc}f^{ceb}+f^{aec}f^{cbd}=0;$$

而 (10) 式右端的第一项为：

$$\begin{aligned}f^{abc}(\partial_\mu\theta^c(x))C^b(x)&=\frac{1}{2}f^{abc}\partial_\mu(\theta^c(x)C^b(x))+\frac{1}{2}f^{abc}(\partial_\mu\theta^c(x))C^b(x)\\&\quad-\frac{1}{2}f^{abc}\theta^c(x)\partial_\mu C^b(x).\end{aligned} \tag{12}$$

但是 (12) 式等号右端第一项减去 (11) 式等号右端第一项乘以 g 得：

$$
\begin{aligned}
&\frac{1}{2} f^{abc}\partial_\mu(\theta^c(x)C^b(x)) + \frac{g}{2} f^{aec}f^{cbd}A^e_\mu(x)\theta^d(x)C^b(x) \\
&\quad = \frac{1}{2}\delta^{ac}\partial_\mu(f^{cbd}\theta^d(x)C^b(x)) + \frac{g}{2} f^{aec}f^{cbd}A^e_\mu(x)\theta^d(x)C^b(x) \\
&\quad = -(\delta^{ac}\partial_\mu - g f^{ace}A^e_\mu(x))\left(\frac{1}{2} f^{cbd}\theta^b(x)C^d(x)\right) \\
&\quad = -(D^{ac}_\mu)\left(\frac{1}{2} f^{cbd}\theta^b(x)C^d(x)\right) \\
&\quad = -(D^{ac}_\mu)\delta C^c(x) \\
&\quad = -(D^{ab}_\mu)\delta C^b(x), \qquad (13)
\end{aligned}
$$

已取

$$
\delta C^c(x) = \frac{1}{2} f^{cbd}\theta^b(x)C^d(x). \qquad (7c)
$$

将(11)、(12)、(13)式代入(10)式得:

$$
\begin{aligned}
(\delta D^{ab}_\mu)C^b(x) = &-D^{ab}_\mu\delta C^b(x) \\
&+ \frac{g}{2} f^{adc}f^{ceb}A^e_\mu(x)(\theta^b(x)C^d(x) + \theta^d(x)C^b(x)) \\
&+ \frac{1}{2} f^{abc}(\partial_\mu\theta^c(x))C^b(x) - \frac{1}{2} f^{abc}\theta^c(x)\partial_\mu C^b(x),
\end{aligned}
$$

即

$$
\begin{aligned}
\delta(D^{ab}_\mu C^b(x)) = &\frac{g}{2} f^{adc}f^{ceb}A^e_\mu(x)(\theta^b(x)C^d(x) + \theta^d(x)C^b(x)) \\
&+ \frac{1}{2} f^{abc}[(\partial_\mu\theta^c(x))C^b(x) - \theta^c(x)\partial_\mu C^b(x)]. \qquad (14)
\end{aligned}
$$

(14)式中为

$$
\theta^b(x)C^d(x) = -\theta^d(x)C^b(x), \qquad (15)
$$

和

$$
f^{abc}(\partial_\mu\theta^c(x))C^b(x) = f^{abc}\theta^c(x)\partial_\mu C^b(x), \qquad (7d)
$$

则得

$$
\delta(D^{ab}_\mu C^b(x)) = 0.
$$

(7d) 式即

$$
\begin{aligned}
&\frac{1}{2} f^{abc}(\partial_\mu\theta^c(x))C^b(x) - \frac{1}{2} f^{abc}\theta^c(x)\partial_\mu C^b(x) \\
&\quad = \partial_\mu\left(\frac{1}{2} f^{abc}\theta^c(x)C^b(x)\right) - f^{abc}\theta^c(x)\partial_\mu C^b(x) \\
&\quad = -\partial_\mu\delta C^a - f^{abc}\theta^c(x)\partial_\mu C^b(x) = 0
\end{aligned}
$$

即

$$
\delta(\partial_\mu C^a(x)) = -f^{abc}\theta^c(x)\partial_\mu C^b(x) = f^{abc}\theta^b(x)\partial_\mu C^c(x). \qquad (7d)'
$$

这样在变换 (7a)、(7b)、(7c)、(7d) 下,即在变换:

$$
\begin{cases}
\delta\psi = -i\dfrac{\lambda^a}{2}\theta^a(x)\psi,\ \delta\bar{\psi} = i\bar{\psi}\dfrac{\lambda^a}{2}\theta^a(x),\ \delta A_\mu^a(x) = -\dfrac{1}{g}D_\mu^{ab}\theta^b(x), & (7a)\\[2ex]
\delta C^{a+}(x) = \dfrac{1}{\alpha}\dfrac{(\partial_\mu A_\mu^a(x))\partial_\nu(\delta A_\nu^a(x))}{\partial_\mu(D_\mu^{ab}C^b(x))}, & (7b)\\[2ex]
\delta C^a(x) = \dfrac{1}{2}f^{abc}\theta^b(x)C^c(x), & (7c)\\[2ex]
\delta(\partial_\mu C^a(x)) = f^{abc}\theta^b(x)\partial_\mu C^c(x) = f^{abc}(\partial_\mu\theta^b(x))C^c(x), & (7d)
\end{cases}
$$

及条件(15)式:

$$\theta^b(x)C^d(x) = -\theta^d(x)C^b(x) \tag{15}$$

下,

$$\delta\mathscr{L}_{\rm eff} = 0,$$

即拉格朗日函数密度在上述条件及变换下保持不变。

下面来考察(15)式。

一、因为鬼场 $C(x)$ 是相互独立的，所以若 $\theta^b(x)$ 是与 $C(x)$ 无关的系数，则要(15)式成立,必须 θ^b 和 θ^d 全为零。但群参数 $\theta(x)$ 不为零,否则(7)式全为零，不合要求。因而 $\theta(x)$ 必须与 $C(x)$ 有关，(15)式才有可能成立。

二、由(15)式可见，对 $\theta(x)$ 和 $C(x)$ 上的上指标 b、d 说是反对易的，而且当 $b=d$ 时有 $\theta^b(x)C^b(x)=0$，而 $C^b(x)$ 反对易，可见 $\theta^b(x)$ 也必含 $C^b(x)$。又由于 $\theta^b(x)$ 为对易的无穷小群参量，所以必是一带上指标 b 的反对易量与另一反对易量的乘积,其中之一为无穷小量。因而可设 $\theta(x)$ 与 $C(x)$ 成正比，$C(x)$ 是反对易的有限量，所以比例系数 ξ 必是反对易的无穷小量。当 ξ 与时空坐标及内部空间分量指标 a、b、c、…无关时有:

$$\theta^b(x) = \xi C^b(x),\ \theta^c(x) = \xi C^c(x),\ \theta^d(x) = \xi C^d(x),\ \cdots, \tag{16}$$

并有 $\xi c^b = -c^b\xi, \xi\psi = -\psi\xi, \xi\bar{\psi} = -\bar{\psi}\xi, \xi^2 = 0$。(16)式右端尚可乘一与 a、b、c、…无关的常数。当 ξ 与时空坐标无关与内部空间分量指标有关时,可设

$$\theta^b = \xi_i^b C^i(x),\ \theta^c(x) = \xi_i^c C^i(x),\ \theta^d(x) = \xi_i^d C^i(x). \tag{17}$$

先考虑(16)式。将它代入(15)式得

$$\xi C^b(x)C^d(x) = -\xi C^d(x)C^b(x)$$

是一恒等式,显然满足。

将(16)式代入(7c)式再对时空坐标微商并注意 ξ 与时空坐标无关得:

$$
\begin{aligned}
\partial_\mu\delta C^a(x) &= \partial_\mu\left(\frac{\xi}{2}f^{abc}C^b(x)C^c(x)\right)\\
&= \frac{\xi}{2}f^{abc}[(\partial_\mu C^b(x))C^c(x) - (\partial_\mu C^c(x))C^b(x)],
\end{aligned}
\tag{18}
$$

$C^c(x)$ 对时空坐标求微商前,与 $C^b(x)$ 先交换位置，即 $C^b(x)C^c(x) = -C^c(x)C^b(x)$。将(18)式右端第二项的求和指标换为: $b\to c$, $c\to b$，并注意 $f^{acb} = -f^{abc}$ 即得

$$
\begin{aligned}
\partial_\mu\delta C^a(x) &= \frac{\xi}{2}f^{abc}[(\partial_\mu C^b(x))C^c(x) + (\partial_\mu C^b(x))C^c(x)]\\
&= \xi f^{abc}(\partial_\mu C^b(x))C^c(x).
\end{aligned}
$$

与(7d)式以 $\theta^b(x)=\xi C^b(x)$ 代入后的一致。故(7d)式可由(7c)式求微商得到。于是(16)式代入到(7a)、(7b)、(7c)式，便由上述方法求得通常所称的 B.R.S. 变换[3]:

$$\begin{cases}\delta\psi=-i\dfrac{\lambda^a}{2}\xi C^a(x)\psi,\quad \delta\bar{\psi}=i\bar{\psi}\dfrac{\lambda^a}{2}\xi C^a(x),\quad \delta A^a_\mu(x)=-\dfrac{\xi}{g}D^{ab}_\mu C^b(x),\\ \delta C^{a+}(x)=-\dfrac{\xi}{\alpha g}(\partial_\mu A^a_\mu(x))\\ \delta C^a(x)=\dfrac{1}{2}f^{abc}\xi C^b(x)C^c(x)\end{cases}\tag{19}$$

显然 ξ 是与内部空间分量指标无关的。反之，若对不同的内部空间分量指标有不同的 ξ，则无穷小群参量 $\theta(x)$ 可写成:

$$\theta^b(x)=\xi_{(b)}C^b(x),\quad \theta^c(x)=\xi_{(c)}C^c(x),\quad \theta^d(x)=\xi_{(d)}C^d(x),\ \cdots,\tag{20}$$

$\xi_{(b)}$、$\xi_{(c)}$、$\xi_{(d)}$ 各是与 $C^b(x)$、$C^c(x)$、$C^d(x)$ 相应的与内部空间分量指标有关的系数，对下标 (b)、(c)、(d) 不求和。(20)式代入(15)式得

$$\xi_{(b)}C^b(x)C^d(x)=-\xi_{(d)}C^d(x)C^b(x),\tag{21}$$

由于鬼场的反对易性，(21)式右端可改写为

$$-\xi_{(d)}C^d(x)C^b(x)=\xi_{(d)}C^b(x)C^d(x)\tag{22}$$

由(22)、(21)式得:

$$\xi_{(b)}C^b(x)C^d(x)=\xi_{(d)}C^b(x)C^d(x),$$

如

$$\xi_{(b)}\neq\xi_{(d)}$$

则(21)式不成立即(15)式不成立，便得不到 B.R.S. 变换。只有

$$\xi_{(b)}=\xi_{(d)}=\xi$$

时(21)式才成立，即(15)式才成立。即 B.R.S. 变换的系数 ξ 对不同的内部空间分量指标 a、b、c、$\cdots$是共同的量。而且可证只有一个。实际上若存在另一与内部空间指标无关的量 ξ'，则由(16)式得

$$\theta^{b\prime}(x)=\xi' C^b(x),$$

代入 B.R.S. 变换中，例如:

$$\delta A^a_\mu(x)=-\frac{1}{g}D^{ab}_\mu\xi' C^b(x),$$

另一方面

$$\delta A^a_\mu(x)=-\frac{1}{g}D^{ab}_\mu\xi C^b(x)$$

所以必

$$\xi'=\xi.$$

若取

$$\theta^b(x)=\xi_1\xi_2\xi C^b(x),\quad \theta^c(x)=\xi_1\xi_2\xi C^c(x),\quad \theta^d(x)=\xi_1\xi_2\xi C^d(x)\tag{16$'$}$$

时，(15)式成为

$$\xi_1\xi_2\xi C^b(x)C^d(x)=-\xi_1\xi_2\xi C^d(x)C^b(x),$$

所以也成立。式中 ξ_1 和 ξ_2 为反对易的数，与内部空间分量指标 a、b、c、$\cdots$无关。因而

乘积 $(\xi_1\xi_2)$ 为对易的常数。将 (16′) 式代入 (7a)—(7d) 式得推广式[4]

$$\delta\psi = -i\frac{\lambda^a}{2}\xi_1\xi_2 C^a(x)\psi,\quad \delta\bar{\psi} = i\bar{\psi}\frac{\lambda^a}{2}\xi_1\xi_2\xi C^a(x),$$

$$\delta A_\mu = -\frac{1}{g}D_\mu^{ab}\xi_1\xi_2\xi C^b(x),$$

$$\delta C^{a+}(x) = -\frac{\xi_1\xi_2\xi}{\alpha g}(\partial_\mu A_\mu^a(x)),$$

$$\delta C^a(x) = \frac{1}{2}f^{abc}\xi_1\xi_2\xi C^b(x)C^c(x),$$

(16′) 式代入 (7c) 式再求微商便得以 (16′) 式代入 (7d) 的式。显然，当

$$\xi_1\xi_2 = 1$$

时，即为 B.R.S. 变换。说明 B.R.S. 变换可以乘一对易的常数。

三、现在来讨论 (17) 式：

$$\theta^b(x) = \xi_i^b C^i(x), \tag{17}$$

$\theta^b(x)$ 是等于对上指标反对易的无穷小张量 ξ_i^b 与鬼场 $C^i(x)$ 的缩并，ξ_i^b 与时空坐标无关。将 (17) 式代入 (7a)—(7d) 及 (15) 式即得在变换：

$$\begin{cases}\delta\psi = -i\dfrac{\lambda^a}{2}\xi_i^a C^i(x)\psi,\quad \delta\bar{\psi} = i\bar{\psi}\dfrac{\lambda^a}{2}\xi_i^a C^i(x),\quad \delta A_\mu^a(x) = -\dfrac{1}{g}D_\mu^{ab}\xi_i^a C^i(x),\\ \delta C^{a+}(x) = -\dfrac{1}{\alpha g}\dfrac{(\partial_\mu A_\mu^a(x))\partial_\nu(D_\nu^{ab}\xi_i^b C^i(x))}{\partial_\mu(D_\mu^{ab}C^b(x))},\\ \delta C^a(x) = \dfrac{1}{2}f^{abc}\xi_i^b C^i(x)C^c(x),\\ \delta(\partial_\mu C^a(x)) = f^{abc}\xi_i^b C^i(x)\partial_\mu C^c(x) = f^{abc}(\partial_\mu\xi_i^b C^i(x))C^c(x),\end{cases} \tag{23}$$

及条件：

$$\xi_i^b C^i(x)C^d(x) = -\xi_i^d C^i(x)C^b(x), \tag{24}$$

下，

$$\delta\mathscr{L}_{\text{eff}} = 0.$$

(23) 式即为 B.R.S. 变换式的“推广式”[5,6]。

四、现在来论证变换式 (23)(24) 和 B.R.S. 变换是等价的。

将 (24) 式求和指标 $j\to i$，而且若 $i\neq b,\ i\neq d$，得

$$(\xi_i^b C^d(x) + \xi_i^d C^b(x))C^i(x) = 0.$$

由于 $C^i(x)$ 是任意的，所以

$$\xi_i^b C^d + \xi_i^d C^b = 0, \tag{25}$$

因 C^d 和 C^b 彼此独立，要 (25) 式成立，必须系数都为零，即

$$\xi_i^b = \xi_i^d = 0.\ (i\neq b,\ i\neq d) \tag{26}$$

当 $b\neq d$ 而 $i=b,\ j=d$ 时，由 (24) 式得：

$$\xi_b^b C^b C^d + \xi_d^d C^d C^b = 0,$$

$$(\xi_b^b - \xi_d^d)C^b C^d = 0$$

所以

$$\xi_b^b = \xi_a^a. \tag{27}$$

因而将(26)和(27)式代入(17)式得

$$\theta^b = \xi_i^b C^i = \xi_b^b C^b = \xi C^b$$

与(16)式：$\theta^b = \xi C^b$ 等价。

由此可见，由 $\theta^b = \xi_i^b C^i(x)$ 得到的变换(23)式与令 $\theta^b = \xi C^b$ 得到的 B.R.S. 变换式(19)是等价的。

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THE EQUIVALENCE OF TWO TRANSFORMATIONS OF THE INVARIANCE OF LAGRANGIAN FUNCTION IN NON-ABELIAN GAUGE FIELD

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ABSTRACT

In this paper, the general transformation of non-abelian gauge field is derived from the invariance reguirement of the effective lagrangian function. It is shown that when the group parameter $\theta(x)$ is the product of the anti-commuting ghost field and an infinitesimal number anti-commuting ξ which is independent of the component indices of intrinsic space, the B. R. S. transformation is obtained. When $\theta(x)$ is equal to the ghost field $C(x)$ times an infinitesimal tensor ξ_i^b which is commuting with respect to the upper index b, we get another transformation. Both transformations are proved to be equivalent.