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关于非亚贝尔规范群的对偶荷（单磁荷）问题（II）

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摘 要

本文把我们在前一工作[1]中所发展的使规范势与球面上的联络相对应的方法推广于讨论有 $O(5)$ 对称的 $SU(2)$ 磁单极势，得到了有双弦奇异性及单弦奇异性的磁单极势的表达式．最后通过一个坐标-规范联合变换，得出了无奇异弦的，有 $O(5)$ 对称的 $SU(2)$ 磁单极规范势．

（一）有奇异弦的 $O(5)$ 对称的 $SU(2)$ 磁单极势

1. 我们在前一工作中[1]，应用规范场的积分表述[2]，使规范势与球面上的联络对应，导出了 $U(1)$ 规范场的无奇异弦的磁单极势．$U(1)$ 群作为非亚贝尔规范群 $O(3)$ 的一个子群，所得的磁单极势具有 $O(3)$ 对称．在我们的讨论中无须引入 Higgs 标量，磁单极势是定域规范对称的自然结果．

本文把文[1]的方法推广用于讨论 $SU(2)$ 磁单极势：考虑在空间每一点上联系于一个 $SU(2)$ 变换，它相当于 $O(3)$ 转动，$O(3)$的转动可看成在一四维球面 S^4 上的转动，此四维球面应存在于五维欧氏空间 E^5 中．所寻求的 $SU(2)$ 磁单极势是对于 $SU(2)$ 及 $O(5)$ 同时具有对称性．从文[1]的观点看，这就相应于在 E^5 中的 S^4 球面上求出其联络，并使之与 $O(4)$ 定域规范势相对应；由于 $O(4) \approx O(3) \times O(3)$ [或 $SU(2) \times SU(2)$] 因此可导出两组独立的 $SU(2)$ 磁单极势．

2. 采用球坐标 $(r, \theta_1, \cdots \theta_4)$，半径为 r 的球面上的线元为（本节中希腊文附标取值 1, 2, 3, 4）

$$dS^2 = g_{\mu\nu} d\theta^\mu d\theta^\nu,$$
$$0 \leqslant \theta_1, \theta_2, \theta_3 \leqslant \pi, \quad 0 \leqslant \theta_4 \leqslant 2\pi, \tag{1}$$

式中 $g_{\mu\nu}$ 是球上的度规：

$$g_{\mu\nu} = \begin{pmatrix} r^2 & \cdot & \cdot & \cdot \\ \cdot & r^2\sin^2\theta_1 & \cdot & \cdot \\ \cdot & \cdot & r^2\sin^2\theta_1\sin^2\theta_2 & \cdot \\ \cdot & \cdot & \cdot & r^2\sin^2\theta_1\sin^2\theta_2\sin^2\theta_3 \end{pmatrix}. \tag{2}$$

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球面上的自然联络$\left\{ {\rho \atop \mu\nu} \right\}$可按下式由度规计算得出:

$$\left\{ {\rho \atop \mu\nu} \right\} = \frac{1}{2} g^{\rho\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial\theta^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial\theta^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial\theta^{\sigma}} \right), \tag{3}$$

计算的结果是:

$$\left\{ {\mu \atop 1\nu} \right\} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{\cos\theta_1}{\sin\theta_1} & \cdot & \cdot \\ \cdot & \cdot & \frac{\cos\theta_1}{\sin\theta_1} & \cdot \\ \cdot & \cdot & \cdot & \frac{\cos\theta_1}{\sin\theta_1} \end{pmatrix}, \tag{4a}$$

$$\left\{ {\mu \atop 2\nu} \right\} = \begin{pmatrix} \cdot & -\sin\theta_1\cos\theta_1 & \cdot & \cdot \\ \frac{\cos\theta_1}{\sin\theta_1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{\cos\theta_2}{\sin\theta_2} & \cdot \\ \cdot & \cdot & \cdot & \frac{\cos\theta_2}{\sin\theta_2} \end{pmatrix}, \tag{4b}$$

$$\left\{ {\mu \atop 3\nu} \right\} = \begin{pmatrix} \cdot & \cdot & -\sin\theta_1\cos\theta_1\sin^2\theta_2 & \cdot \\ \cdot & \cdot & -\sin\theta_2\cos\theta_2 & \cdot \\ \frac{\cos\theta_1}{\sin\theta_1} & \frac{\cos\theta_2}{\sin\theta_2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{\cos\theta_3}{\sin\theta_3} \end{pmatrix}, \tag{4c}$$

$$\left\{ {\mu \atop 4\nu} \right\} = \begin{pmatrix} \cdot & \cdot & \cdot & -\sin\theta_1\cos\theta_1\sin^2\theta_2\sin^2\theta_3 \\ \cdot & \cdot & \cdot & -\sin\theta_2\cos\theta_2\sin^2\theta_3 \\ \cdot & \cdot & \cdot & -\sin\theta_3\cos\theta_3 \\ \frac{\cos\theta_1}{\sin\theta_1} & \frac{\cos\theta_2}{\sin\theta_2} & \frac{\cos\theta_3}{\sin\theta_3} & \cdot \end{pmatrix}. \tag{4d}$$

我们知道,相应于球坐标的自然基虽然正交,但不是归一的. 和规范势相对应的是自然联络在正交归一标架 (tetrad) 上的投影 $\Gamma^A_{\mu B}$ (在本节中拉丁文大写附标是标架指标,取值 1, 2, 3, 4):

$$\Gamma^A_{\mu B} = e^A_\rho \left\{ {\rho \atop \mu\sigma} \right\} e^\sigma_B + e^A_\rho \frac{\partial}{\partial\theta^\mu} e^\rho_B, \tag{5}$$

其中正交归一标架向量 e^A_μ 及其逆 e^μ_B 的定义是

$$g_{\mu\nu} = e^A_\mu \delta_{AB} e^B_\nu, \tag{6}$$

$$e^A_\mu e^\mu_B = \delta^A_B. \tag{7}$$

由式 (6) 及 (2), 我们有

$$e^A_\mu = \begin{pmatrix} r & \cdot & \cdot & \cdot \\ \cdot & r\sin\theta_1 & \cdot & \cdot \\ \cdot & \cdot & r\sin\theta_1\sin\theta_2 & \cdot \\ \cdot & \cdot & \cdot & r\sin\theta_1\sin\theta_2\sin\theta_3 \end{pmatrix}, \tag{8a}$$

$$e_B^\nu = \begin{pmatrix} \frac{1}{r} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{r\sin\theta_1} & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{r\sin\theta_1\sin\theta_2} & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{r\sin\theta_1\sin\theta_2\sin\theta_3} \end{pmatrix}, \tag{8b}$$

把式 (4) 和 (8) 代入式 (5)，便可算得

$$\Gamma_1 = 0, \tag{9a}$$

$$\Gamma_2 = \begin{pmatrix} \cdot & -\cos\theta_1 & \cdot & \cdot \\ \cos\theta_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{9b}$$

$$\Gamma_3 = \begin{pmatrix} \cdot & \cdot & -\cos\theta_1\sin\theta_2 & \cdot \\ \cdot & \cdot & -\cos\theta_2 & \cdot \\ \cos\theta_1\sin\theta_2 & \cos\theta_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{9c}$$

$$\Gamma_4 = \begin{pmatrix} \cdot & \cdot & \cdot & -\cos\theta_1\sin\theta_2\sin\theta_3 \\ \cdot & \cdot & \cdot & -\cos\theta_2\sin\theta_3 \\ \cdot & \cdot & \cdot & -\sin\theta_3 \\ \cos\theta_1\sin\theta_2\sin\theta_3 & \cos\theta_2\sin\theta_3 & \sin\theta_3 & \cdot \end{pmatrix}. \tag{9d}$$

式 (9) 中矩阵 Γ_μ 的元素就是 $\Gamma^A_{\mu B}$.

引入 $O(4)$ 群的生成元 $X_{\mu\nu}$，其元素 $(X_{\mu\nu})^A_B$ 为：

$$(X_{\mu\nu})^A_B = -\delta_{\mu A}\delta_{\nu B} + \delta_{\mu B}\delta_{\nu A}, \tag{10}$$

式 (9a)—(9d) 可写成为

$$\Gamma_1 = 0, \tag{11a}$$

$$\Gamma_2 = \cos\theta_1 X_{12}, \tag{11b}$$

$$\Gamma_3 = \cos\theta_1\sin\theta_2 X_{13} + \cos\theta_2 X_{23}, \tag{11c}$$

$$\Gamma_4 = \cos\theta_1\sin\theta_2\sin\theta_3 X_{14} + \cos\theta_2\sin\theta_3 X_{24} + \cos\theta_3 X_{34}, \tag{11d}$$

由于 $O(4) \approx O(3) \times O(3)$，$X_{\mu\nu}$ 可用两组对易的 $O(3)$ 生成元 $X_i^{(+)}$ 及 $X_i^{(-)}$ $(i = 1, 2, 3)$ 表示：

$$X_{23} = X_1^{(+)} + X_1^{(-)}, \quad X_{31} = X_2^{(+)} + X_2^{(-)},$$
$$X_{12} = X_3^{(+)} + X_3^{(-)}, \quad X_{14} = X_1^{(+)} - X_1^{(-)},$$
$$X_{24} = X_2^{(+)} - X_2^{(-)}, \quad X_{34} = X_3^{(+)} - X_3^{(-)}, \tag{12}$$

于是矩阵 Γ_μ 可用 $X^{(+)}$ 及 $X^{(-)}$ 表示．由文 [1]，规范势与联络的对应关系为

$$\Gamma^{(\pm)A}_{\mu B} = (gW_\mu^{(\pm)i}X_i^{(\pm)})^A_B \tag{13}$$

其中 g 是规范场的自耦常数，$W_\mu^{(\pm)i}$ 是规范势在正交归一标架上的分量，i 是 $SU(2)$ 指标．由此可得两组独立的 $SU(2)$ 规范势 $W_\mu^{(\pm)i}$.把这些分量变换回球面上自然基的分量 $W_A^{(\pm)i}$，

$W_A^{(\pm)i}$ 的关系为

$$W_A^{(\pm)i} = e_A^\mu W_\mu^{(\pm)i}, \tag{14}$$

也即

$$W_{\theta_1}^{(\pm)i} = \frac{1}{r} W_1^{(\pm)i}, \tag{14a}$$

$$W_{\theta_2}^{(\pm)i} = \frac{1}{r\sin\theta_1} W_2^{(\pm)i}, \tag{14b}$$

$$W_{\theta_3}^{(\pm)i} = \frac{1}{r\sin\theta_1\sin\theta_2} W_3^{(\pm)i}, \tag{14c}$$

$$W_{\theta_4}^{(\pm)i} = \frac{1}{r\sin\theta_1\sin\theta_2\sin\theta_3} W_4^{(\pm)i}, \tag{14d}$$

便可得到如下两组 $SU(2)$ 磁单极势：

$$\text{(I)}\quad W_{\theta_1}^{(+)i} = 0,\qquad\qquad \text{(II)}\quad W_{\theta_1}^{(-)i} = 0,$$

$$\begin{cases} W_{\theta_2}^{(+)1} = 0, \\ W_{\theta_3}^{(+)1} = \dfrac{\cos\theta_2}{gr\sin\theta_1\sin\theta_2}, \\ W_{\theta_4}^{(+)1} = \dfrac{\cos\theta_2}{gr\sin\theta_1}, \end{cases}\qquad \begin{cases} W_{\theta_2}^{(-)1} = 0, \\ W_{\theta_3}^{(-)1} = \dfrac{\cos\theta_2}{gr\sin\theta_1\sin\theta_2}, \\ W_{\theta_4}^{(-)1} = -\dfrac{\cos\theta_1}{gr\sin\theta_1}, \end{cases}$$

$$\begin{cases} W_{\theta_2}^{(+)2} = 0, \\ W_{\theta_3}^{(+)2} = -\dfrac{\cos\theta_1}{gr\sin\theta_1}, \\ W_{\theta_4}^{(+)2} = \dfrac{\cos\theta_2}{gr\sin\theta_1\sin\theta_2}, \end{cases}\qquad \begin{cases} W_{\theta_2}^{(-)2} = 0, \\ W_{\theta_3}^{(-)2} = -\dfrac{\cos\theta_1}{gr\sin\theta_1}, \\ W_{\theta_4}^{(-)2} = -\dfrac{\cos\theta_2}{gr\sin\theta_1\sin\theta_2}, \end{cases} \tag{15}$$

$$\begin{cases} W_{\theta_2}^{(+)3} = \dfrac{\cos\theta_1}{gr\sin\theta_1}, \\ W_{\theta_3}^{(+)3} = 0, \\ W_{\theta_4}^{(+)3} = \dfrac{\cos\theta_3}{gr\sin\theta_1\sin\theta_2\sin\theta_3}, \end{cases}\qquad \begin{cases} W_{\theta_2}^{(-)3} = \dfrac{\cos\theta_1}{gr\sin\theta_1}, \\ W_{\theta_3}^{(-)3} = 0, \\ W_{\theta_4}^{(-)3} = -\dfrac{\cos\theta_3}{gr\sin\theta_1\sin\theta_2\sin\theta_3}, \end{cases}$$

这两组磁单极势在 θ_1, θ_2 及 θ_3 为 0 及 π 时有奇异弦．这是两组双弦奇异解．

3. 可以证明，选取适当的规范变换，能把这两组双弦奇异解变换成单弦奇异解．例如，取规范变换 S 为：

$$S = e^{\theta_4 X_{23}} e^{\theta_3 X_{12}} e^{\theta_2 X_{23}} \tag{16}$$

$$\Gamma'_\mu = S\Gamma_\mu S^{-1} + S\frac{\partial}{\partial\theta_\mu}S^{-1}, \tag{17}$$

由此可得

$$\Gamma'_{\theta_1} = 0, \tag{18a}$$

$$\begin{aligned}\Gamma'_{\theta_2} = &(1-\cos\theta_1)(\cos\theta_3 X_1^{(+)} + \sin\theta_3\cos\theta_4 X_2^{(+)} + \sin\theta_4 X_3^{(+)}) \\ &+ (1+\cos\theta_1)(\cos\theta_3 X_1^{(-)} + \sin\theta_3\cos\theta_4 X_2^{(-)} + \sin\theta_4 X_3^{(-)}),\end{aligned} \tag{18b}$$

$$
\begin{aligned}
\Gamma'_{\theta_3} = &(1-\cos\theta_1)\sin\theta_2[\cos\theta_2\sin\theta_3 X_1^{(+)} + (\cos\theta_2\cos\theta_3\cos\theta_4 \\
&- \sin\theta_2\sin\theta_4)X_2^{(+)} + (\cos\theta_2\cos\theta_3\sin\theta_4 + \sin\theta_2\cos\theta_4)X_3^{(+)}] \\
&+ (1+\cos\theta_1)\sin\theta_2[\text{同前项以 } X_i^{(-)} \text{ 代 } X_i^{(+)}],
\end{aligned} \tag{18c}
$$

$$
\begin{aligned}
\Gamma'_{\theta_4} = &(1-\cos\theta_1)\sin\theta_2\sin\theta_3[\cos\theta_2 X_1^{(+)} - (\sin\theta_2\cos\theta_3\cos\theta_4 - \cos\theta_2\sin\theta_4)X_2^{(+)} \\
&- (\sin\theta_2\cos\theta_3\sin\theta_4 - \cos\theta_2\cos\theta_4)X_3^{(+)}] \\
&+ (1+\cos\theta_1)\sin\theta_2\sin\theta_3[\text{同前项以 } X_i^{(-)} \text{ 代 } X_i^{(+)}],
\end{aligned} \tag{18d}
$$

应用对应关系 (13) 及式 (14)，我们得到变换后的规范势在正交归一标架上的分量为

$$
W'^{(\pm)i}_{\theta_1} = 0,
$$

$$
\begin{cases}
W'^{(\pm)1}_{\theta_2} = \dfrac{1\mp\cos\theta_1}{gr\sin\theta_1}\cos\theta_3, \\
W'^{(\pm)1}_{\theta_3} = \dfrac{1\mp\cos\theta_1}{gr\sin\theta_1}\cos\theta_2\sin\theta_3, \\
W'^{(\pm)1}_{\theta_4} = \dfrac{1\mp\cos\theta_1}{gr\sin\theta_1}\cos\theta_2,
\end{cases}
$$

$$
\begin{cases}
W'^{(\pm)2}_{\theta_2} = \dfrac{1\mp\cos\theta_1}{gr\sin\theta_1}\sin\theta_3\cos\theta_4, \\
W'^{(\pm)2}_{\theta_3} = \dfrac{1\mp\cos\theta_1}{gr\sin\theta_1}(\cos\theta_2\cos\theta_3\cos\theta_4 - \sin\theta_2\sin\theta_4), \\
W'^{(\pm)2}_{\theta_4} = \dfrac{1\mp\cos\theta_1}{gr\sin\theta_1}(-\sin\theta_2\cos\theta_3\cos\theta_4 + \cos\theta_2\sin\theta_4),
\end{cases} \tag{19}
$$

$$
\begin{cases}
W'^{(\pm)3}_{\theta_2} = \dfrac{1\mp\cos\theta_1}{gr\sin\theta_1}\sin\theta_4, \\
W'^{(\pm)3}_{\theta_3} = \dfrac{1\mp\cos\theta_1}{gr\sin\theta_1}(\cos\theta_2\cos\theta_3\sin\theta_4 + \sin\theta_2\cos\theta_4), \\
W'^{(\pm)3}_{\theta_4} = \dfrac{1\mp\cos\theta_1}{gr\sin\theta_1}(-\sin\theta_2\cos\theta_3\sin\theta_4 + \cos\theta_2\cos\theta_4).
\end{cases}
$$

不难看出，由于因子 $(1\pm\cos\theta_1)$，磁单极势 $W'^{(\pm)i}_{\theta_\mu}$ 对于 θ_1 已化为单弦奇异解. $W'^{(+)i}_{\theta_\mu}$ 在 E^5 空间中沿第 5 轴的上半球解析，$W'^{(-)i}_{\theta_\mu}$ 在下半球解析.

4. 在本节中我们导出杨振宁的单弦奇异解[3].

代替球坐标 $(r, \theta_1, \cdots, \theta_4)$，我们采用投影坐标 $(r, \xi_1, \xi_2, \xi_3, \theta\equiv\theta_1)$，
其中 ξ_i 由下式定义

$$
\begin{cases}
\xi_1 = \rho\sin\theta_3\cos\theta_4, \\
\xi_2 = \rho\sin\theta_3\sin\theta_4, \\
\xi_3 = \rho\cos\theta_3, \\
\rho = \tan\dfrac{\theta_2}{2},
\end{cases} \tag{20}
$$

亦即为图 1 所示的单位半径的 S^3 球面上的点 P 的测地投影点 Q 的坐标. 采用这些坐标，度规(2)变为

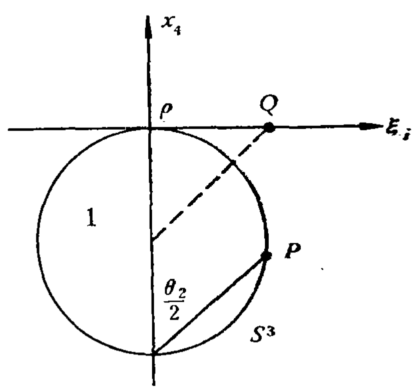

图 1

$$g_{\mu\nu} = \begin{pmatrix} \dfrac{4r^2\sin^2\theta}{(1+\rho^2)^2} & \cdot & \cdot & \cdot \\ \cdot & \dfrac{4r^2\sin^2\theta}{(1+\rho^2)^2} & \cdot & \cdot \\ \cdot & \cdot & \dfrac{4r^2\sin^2\theta}{(1+\rho^2)^2} & \cdot \\ \cdot & \cdot & \cdot & r^2 \end{pmatrix}, \tag{21}$$

其中第四维的坐标为 θ. 此时 S^4 球面上的联络可由度规(20)算得为:

$$\left\{ {\mu \atop 1\nu} \right\} = \begin{pmatrix} \dfrac{-2\xi_1}{1+\rho^2} & \dfrac{-2\xi_2}{1+\rho^2} & \dfrac{-2\xi_3}{1+\rho^2} & \dfrac{\cos\theta}{\sin\theta} \\ \dfrac{2\xi_2}{1+\rho^2} & \dfrac{-2\xi_1}{1+\rho^2} & \cdot & \cdot \\ \dfrac{2\xi_3}{1+\rho^2} & \cdot & \dfrac{-2\xi_1}{1+\rho^2} & \cdot \\ \dfrac{-4\sin\theta\cos\theta}{(1+\rho^2)^2} & \cdot & \cdot & \cdot \end{pmatrix}, \tag{22a}$$

$$\left\{ {\mu \atop 2\nu} \right\} = \begin{pmatrix} \dfrac{-2\xi_2}{1+\rho^2} & \dfrac{2\xi_1}{1+\rho^2} & \cdot & \cdot \\ \dfrac{-2\xi_1}{1+\rho^2} & \dfrac{-2\xi_2}{1+\rho^2} & \dfrac{-2\xi_3}{1+\rho^2} & \dfrac{\cos\theta}{\sin\theta} \\ \cdot & \dfrac{2\xi_3}{1+\rho^2} & \dfrac{-2\xi_2}{1+\rho^2} & \cdot \\ \cdot & \dfrac{-4\sin\theta\cos\theta}{(1+\rho^2)^2} & \cdot & \cdot \end{pmatrix}, \tag{22b}$$

$$\left\{ {\mu \atop 3\nu} \right\} = \begin{pmatrix} \dfrac{-2\xi_3}{1+\rho^2} & \cdot & \dfrac{2\xi_1}{1+\rho^2} & \cdot \\ \cdot & \dfrac{-2\xi_3}{1+\rho^2} & \dfrac{2\xi_2}{1+\rho^2} & \cdot \\ \dfrac{-2\xi_1}{1+\rho^2} & \dfrac{-2\xi_2}{1+\rho^2} & \dfrac{-2\xi_3}{1+\rho^2} & \dfrac{\cos\theta}{\sin\theta} \\ \cdot & \cdot & \dfrac{-4\sin\theta\cos\theta}{(1+\rho^2)^2} & \cdot \end{pmatrix}, \tag{22c}$$

$$\left\{ {\mu \atop 4\nu} \right\} = \begin{pmatrix} \dfrac{\cos\theta}{\sin\theta} & \cdot & \cdot & \cdot \\ \cdot & \dfrac{\cos\theta}{\sin\theta} & \cdot & \cdot \\ \cdot & \cdot & \dfrac{\cos\theta}{\sin\theta} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{22d}$$

如前把联络投影在正交归一标架上. 在此情况下,标架向量 e_μ^A 及其逆 e_B^ν 为

$$e_\mu^A = \begin{pmatrix} \frac{2r\sin\theta}{1+\rho^2} & \cdot & \cdot & \cdot \\ \cdot & \frac{2r\sin\theta}{1+\rho^2} & \cdot & \cdot \\ \cdot & \cdot & \frac{2r\sin\theta}{1+\rho^2} & \cdot \\ \cdot & \cdot & \cdot & r \end{pmatrix}, \tag{23}$$

$$e_B^\nu = \begin{pmatrix} \frac{1+\rho^2}{2r\sin\theta} & \cdot & \cdot & \cdot \\ \cdot & \frac{1+\rho^2}{2r\sin\theta} & \cdot & \cdot \\ \cdot & \cdot & \frac{1+\rho^2}{2r\sin\theta} & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{r} \end{pmatrix}. \tag{24}$$

联络在此正交归一标架上的投影 $\Gamma_{\mu B}^A(\xi)$ 为：

$$\Gamma_{\mu B}^A(\xi) = e_\rho^A \begin{Bmatrix} \rho \\ \mu\sigma \end{Bmatrix} e_B^\sigma + e_\rho^A \frac{\partial}{\partial \xi^\mu} e_B^\rho, \tag{25}$$

式中第四维坐标定义为 $\xi^4 \equiv \theta$. 由式 (22)—(24) 可得

$$\begin{aligned} &\Gamma_i(\xi) = -\frac{2\cos\theta}{1+\rho^2} X_{i4} + \frac{2\xi_j}{1+\rho^2} X_{ij}, \\ &\Gamma_4(\xi) = 0, \\ &ij = 1, 2, 3. \end{aligned} \tag{26}$$

引入规范变换 S：

$$S = \exp\left\{\frac{\theta_2}{\rho} \xi^i \left(X_i^{(+)} + X_i^{(-)}\right)\right\}, \tag{27}$$

$$\Gamma_\mu \to \Gamma'_\mu = S\Gamma_\mu S^{-1} + S\partial_\mu S^{-1}, \tag{28}$$

由式 (26) 及 (27)，有

$$\begin{aligned} \Gamma'_i = &-\frac{2(\cos\theta+1)}{(1+\rho^2)^2}\left[(1-\rho^2)X_i^{(+)} + 2\xi_i \mathbf{X}^{(+)}\cdot\boldsymbol{\xi}\right. \\ &\left.+ 2\epsilon_{ijl} X_j^{(+)}\xi_l\right] - \frac{2(1-\cos\theta)}{(1+\rho^2)^2}\left[(1-\rho^2)X_i^{(-)}\right. \\ &\left.+ 2\xi_i \mathbf{X}^{(-)}\cdot\boldsymbol{\xi} + 2\epsilon_{ijl} X_j^{(-)}\xi_l\right], \\ \Gamma'_4 = &\ 0. \end{aligned} \tag{29}$$

$X_i^{(+)}$ 和 $X_i^{(-)}$ 是两组相互独立的 $SU(2)$ 生成元，由式 (13) 及 (14)，它们各对应一组 $SU(2)$ 磁单极势：

$$\begin{aligned} &W_i^{(\pm)j} = \frac{-(1\pm\cos\theta)}{gr(1+\rho^2)\sin\theta}\left[(1-\rho^2)\delta_{ij} + 2\xi_i\xi_j + 2\epsilon_{ijl}\xi_l\right], \\ &W_4^{(\pm)j} = 0, \end{aligned} \tag{30}$$

这也就是杨振宁用投影坐标所给出的 $SU(2)$ 磁单极规范势的形式[3]，它们是**单弦奇异解**。

(二) 无奇异弦的 $O(5)$ 对称的 $SU(2)$ 磁单极势

5. 我们在前几节中，应用规范势与球面上的联络对应的方法[1]，导出了有奇异弦的 $O(5)$ 对称的 $SU(2)$ 磁单极势. 在这里,我们将导出无奇异弦的 $O(5)$ 对称的磁单极势.

我们用测地投影坐标 η_μ (以下,希腊文小写附标取值 1, 2, 3, 4) 来描述 E^5 空间中的 S^4 球面. η_μ 是半径为 1 的球面上的点 P 的测地投影 Q 的坐标:

$$\begin{aligned}\eta_1 &= \lambda \sin\theta_2 \sin\theta_3 \cos\theta_4,\\ \eta_2 &= \lambda \sin\theta_2 \sin\theta_3 \sin\theta_4,\\ \eta_3 &= \lambda \sin\theta_2 \cos\theta_3,\\ \eta_4 &= \lambda \cos\theta_2,\\ \lambda &= \tan\frac{\theta_1}{2},\end{aligned} \tag{31}$$

其中 $\theta_1, \theta_2, \theta_3, \theta_4$ 是极坐标所用的角，亦即在 E^5 中直坐标 x_a (以下，拉丁文小写附标取值 1, 2, 3, 4, 5) 可写为

$$\begin{aligned}x_1 &= r \sin\theta_1 \sin\theta_2 \sin\theta_3 \cos\theta_4,\\ x_2 &= r \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4,\\ x_3 &= r \sin\theta_1 \sin\theta_2 \cos\theta_3,\\ x_4 &= r \sin\theta_1 \cos\theta_2,\\ x_5 &= r \cos\theta_1.\end{aligned} \tag{32}$$

比较式 (31) 及式 (32), 便有

$$\begin{aligned}x_\mu &= \frac{2r\eta_\mu}{1+\lambda^2}, \qquad x_5 = r\frac{1-\lambda^2}{1+\lambda^2},\\ \lambda^2 &= \eta_\mu\eta^\mu, \qquad r^2 = x_a x^a.\end{aligned} \tag{33}$$

对于投影坐标 (31), 球面上度规 $g_{\mu\nu}(\eta)$ 为

$$g_{\mu\nu}(\eta) = \frac{4r^2}{(1+\lambda^2)^2}\delta_{\mu\nu}. \tag{34}$$

由此度规,便可算出球面上的自然联络 $\left\{ {\rho \atop \mu\nu} \right\}$ 为

$$\left\{ {\rho \atop \mu\nu} \right\} = \frac{-2}{1+\lambda^2}(\eta_\mu\delta^\rho_\nu + \eta_\nu\delta^\rho_\mu - \eta^\rho\delta_{\mu\nu}). \tag{35}$$

现在引入球面上的正交归一标架向量 $e^A_\mu(\eta)$ 及其逆 $e^\nu_B(\eta)$ (以下, 拉丁文大写附标是标架指标,取值 1, 2, 3, 4), 其定义为

$$\begin{aligned}g_{\mu\nu}(\eta) &= e^A_\mu(\eta)\delta_{AB}e^B_\nu(\eta),\\ e^A_\mu(\eta)e^\mu_B(\eta) &= \delta_{AB}.\end{aligned} \tag{36}$$

由式 (34) 及 (36), 有

$$e^A_\mu(\eta) = \frac{2r}{1+\lambda^2}\delta^A_\mu, \qquad e^\nu_B(\eta) = \frac{1+\lambda^2}{2r}\delta^\mu_B. \tag{37}$$

将自然联络投影在此标架上,得到联络在此标架上的投影 $\Gamma^A_{\mu B}(\eta)$ 为:

$$\Gamma^A_{\mu B}(\eta) = e^A_\rho(\eta)\left\{\begin{matrix}\rho\\ \mu\nu\end{matrix}\right\} e^\nu_B(\eta) + e^A_\rho(\eta)\frac{\partial}{\partial\eta^\mu} e^\rho_B(\eta), \tag{38}$$

把式(35)、(37)代入(38)，便有

$$\begin{aligned}\Gamma^A_{\mu B}(\eta) &= \frac{2}{1+\lambda^2}(\eta^A\delta_{\mu B} - \eta_B\delta^A_\mu)\\ &= \frac{2\eta^\nu}{1+\lambda^2}(X_{\mu\nu})^A_B,\end{aligned} \tag{39}$$

式中 $X_{\mu\nu}$ 是 $O(4)$ 群的生成元，其元素 $(X_{\mu\nu})^A_B$ 为

$$(X_{\mu\nu})^A_B = \delta_{\mu B}\delta^A_\nu - \delta^A_\mu\delta_{\nu B}. \tag{40}$$

6. 与式(9)相对应的规范势是与在(一)中导出的有奇异弦的单弦奇异势等价，这可由把式(39)中的投影坐标换回球面上的坐标看出来．从另一个角度来看，如果代替 η_μ，我们用坐标 $(\xi_1, \xi_2, \xi_3, \xi_4 \equiv \theta_1)$，它们定义为：

$$\begin{aligned}\lambda &= \tan\frac{\xi_4}{2},\\ \xi_1 &= \tan\frac{\theta_2}{2}\sin\theta_3\cos\theta_4,\\ \xi_2 &= \tan\frac{\theta_2}{2}\sin\theta_3\sin\theta_4,\\ \xi_3 &= \tan\frac{\theta_2}{2}\cos\theta_3.\end{aligned} \tag{41}$$

比较式(31)及(41)，有两组坐标之间的关系：

$$\begin{cases}\eta_i = \dfrac{2\lambda}{1+\rho^2}\xi_i,\\ \eta_4 = \lambda\dfrac{1-\rho^2}{1+\rho^2},\end{cases} \tag{42}$$

$$\rho^2 = \xi_i\xi^i, \qquad i = 1, 2, 3.$$

或即

$$\begin{cases}\xi_i = \dfrac{\eta_i}{\lambda+\eta_4},\\ \xi_4 = 2\tan^{-1}\lambda.\end{cases} \tag{43}$$

在作坐标变换 $(\eta_\mu \to \xi_\mu)$ 时，联络经受如下的变换：

$$\Gamma^A_{\mu B}(\eta) \to \Gamma^A_{\mu B}(\xi) = \frac{\partial\eta^\nu}{\partial\xi^\mu} S^A_C \Gamma^C_{\nu D}(\eta)(S^{-1})^D_B + S^A_C\frac{\partial}{\partial\xi^\mu}(S^{-1})^C_B, \tag{44}$$

其中

$$\begin{aligned}S^A_B &= e^A_\mu(\xi)\frac{\partial\xi^\mu}{\partial\eta^\nu} e^\nu_B(\eta),\\ (S^{-1})^A_B &= e^A_\mu(\eta)\frac{\partial\eta^\mu}{\partial\xi^\nu} e^\nu_B(\xi),\end{aligned} \tag{45}$$

式中 $e^A_\mu(\xi)$ 及 $e^\nu_B(\xi)$ 是采用坐标 ξ_μ 时球面上的正交归一标架向量及其逆．由定义有

$$e_\mu^A(\xi) = \begin{pmatrix} \frac{2r\sin\xi_4}{1+\rho^2} & & & \\ & \frac{2r\sin\xi_4}{1+\rho^2} & & \\ & & \frac{2r\sin\xi_4}{1+\rho^2} & \\ & & & r \end{pmatrix}, \tag{46a}$$

$$e_B^\nu(\xi) = \begin{pmatrix} \frac{1+\rho^2}{2r\sin\xi_4} & & & \\ & \frac{1+\rho^2}{2r\sin\xi_4} & & \\ & & \frac{1+\rho^2}{2r\sin\xi_4} & \\ & & & \frac{1}{r} \end{pmatrix}. \tag{46b}$$

把式(37),(43),(46)代入(45)算得

$$\begin{aligned} S_j^i &= \delta_j^i - \frac{2\xi^i\xi_j}{1+\rho^2}, \\ S_4^i &= \frac{-2\xi^i}{1+\rho^2}, \\ S_i^4 &= \frac{2\xi_i}{1+\rho^2}, \\ S_4^4 &= \frac{1-\rho^2}{1+\rho^2}, \\ j &= 1, 2, 3. \end{aligned} \tag{47}$$

以及

$$(S^{-1})_\nu^\mu = S_\mu^\nu \tag{48}$$

由式(47),(48),(39)及(44),可以得到在坐标-规范联合变换下的联络变换的表式:

$$\begin{aligned} \Gamma_i(\xi) = \frac{-2\cos\xi_4}{1+\rho^2} X_i^{(+)} + \frac{2}{1+\rho^2} \varepsilon_{ijk}\xi_j X_k^{(+)} \\ + \frac{2\cos\xi_4}{1+\rho^2} X_i^{(-)} + \frac{2}{1+\rho^2} \varepsilon_{ijk}\xi_j X_k^{(-)}, \\ \Gamma_4(\xi) = 0. \end{aligned} \tag{49}$$

$X_i^{(+)}$ 及 $X_i^{(-)}$ 的定义见式(12). 不难看出,式(49)就是式(26),与之对应的规范势见式(30),亦即杨振宁给出的单弦奇异解[3]. 显然是有单弦奇异性的.

7. 要得到无奇异弦的 $SU(2)$ 规范势,必须变回五维欧氏空间的坐标 x_a,即作坐标变换 $(\eta_2, \eta_3, \eta_4, \eta_5, r) \to (x_1, \cdots, x_5)$,在作坐标变换时,标架同时相应地转动[作 $O(5)$ 转动],这时联络 Γ_μ 经受如下的变换:

$$\Gamma_\mu \to \Gamma_a(x),$$

$$\Gamma^a_{cb}(x) = \frac{\partial\eta^\mu}{\partial x^c} S^a_A \Gamma^A_{\mu B}(S^{-1})^B_b + S^a_A \frac{\partial}{\partial x^c}(S^{-1})^A_b, \tag{50}$$

其中

$$S^a_A = e^\mu_A(\eta)\frac{\partial x^a}{\partial \eta^\mu} = \frac{1+\lambda^2}{2r}\frac{\partial x^a}{\partial \eta^A}, \tag{51a}$$

$$S^a_5 = \frac{\partial x^a}{\partial \eta^5} = \frac{x^a}{r}, \tag{51b}$$

$$(S^{-1})^B_b = e^B_\mu(\eta)\frac{\partial \eta^\mu}{\partial x^b} = \frac{2r}{1+\lambda^2}\frac{\partial \eta^B}{\partial x^b}, \tag{51c}$$

$$(S^{-1})^5_b = \frac{\partial \eta^5}{\partial x^b} = \frac{x_b}{r}. \tag{51d}$$

把式 (33), (39) 及 (51) 代入 (50), 可得

$$\begin{aligned} \Gamma^\mu_{\sigma\nu}(x) &= \frac{1}{r^2}(x^\mu\delta_{\nu\sigma} - x_\nu\delta^\mu_\sigma), \\ \Gamma^5_{\sigma\nu}(x) &= \frac{x^5}{r^2}\delta_{\sigma\nu}, \\ \Gamma^5_{5\nu}(x) &= -\frac{x_\nu}{r^2}. \end{aligned} \tag{52}$$

亦即

$$\Gamma^a_{cb}(x) = -\frac{1}{r^2}(x^a\delta_{bc} - x_b\delta^a_c), \tag{53}$$

此式显然是除原点外到处解析，且为 $O(5)$ 对称的.

由规范势与球面联络的对应关系[2]:

$$\Gamma^a_{cb}(x) = \left(\frac{1}{2}\, g W^{de}_c(x) X_{de}\right)^a_b, \tag{54}$$

其中 $W^{de}_c(x)$ 是规范势，g 是规范场的自耦合常数，X_{de} 是 $O(5)$ 群的生成元，便有

$$W^{ab}_c(x) = -\frac{1}{gr^2}(x^a\delta^b_c - x^b\delta^a_c), \tag{55}$$

这是以直角坐标表出的，作为定域 $O(5)$ 群的子群 $O(4)$ 磁单极规范势，它具有 $O(5)$ 对称，无奇异弦. $SU(2)$ 磁单极规范势也可由此籍 $O(4) \approx SU(2) \times SU(2)$ 如前文[1] 那样分出，也是无奇异弦的.

参　考　资　料

[1] 李华钟、冼鼎昌、郭硕鸿，中山大学学报(自然科学版)，1975，**3**，1. 物理学报 **25** (1976)，507.
[2] C. N. Yang, *Phys. Rev. Lett.*, **33** (1974), 445.
[3] 杨振宁，$O(5)$ 对称的 $SU(2)$ 磁单极，杨振宁在北京的学术报告之二 (1976. 4).

ON THE PROBLEM OF THE DUAL CHARGE (MAGNETIC MONOPOLE) OF NON-ABELIAN GROUPS (II)

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Abstract

The method of relating the gauge potential to the connection on a spherical surface developed in a previous paper[1] is generalized to the discussion of the potential of an $SU(2)$ magnetic monopole with $O(5)$ symmetry. Expressions for the potential of the monopole with double string singularity or single string singularity are obtained. Finally, by a combined coordinate-gauge transformation, we obtain a string free expression for an $SU(2)$ magnetic monopole with $O(5)$ symmetry.